ASYMPTOTICALLY EFFICIENT TESTS BASED ON THE SUMS OF OBSERVATIONS[1]

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Summary. For tests, $\boldsymbol{\phi} = \{\phi_k\}$, of composite hypotheses, ω_1 and ω_2, asymptotic efficiency is defined in terms of the behavior as $\alpha \to 0$ of the sample size N_ϕ required to reduce the maximum risk to α. For problems where the ω_i contain elements θ_i whose relative densities satisfy

$$\sup_{\omega_1} \inf_{t>0} E_\theta(f_2/f_1)^t = \inf_t E_i(f_2/f_1)^t = \sup_{\omega_2} \inf_{t<0} E_\theta(f_2/f_1)^t,$$

Chernoff's Theorem 1 [2] is applied to the non-randomized test $\boldsymbol{\phi}^*$, with $\phi_k^* = 1$ or 0 according as $\sum \log (f_2/f_1) > 0$ or not, and proves $\boldsymbol{\phi}^*$ asymptotically efficient (Theorem 2.1).

The principal results of the paper are applications of Theorem 2.1 to tests of the difference $(\xi - \eta)$ of binomial probabilities with samples of relative size m/n. For $\omega_1 = \{\xi - \eta \leqq -\delta\}$, $\omega_2 = \{\xi - \eta \geqq \delta\}$, certain tests of the form $\phi_k^* = 1$ if and only if $\lambda(\hat{\xi} - \frac{1}{2}) > (\hat{\eta} - \frac{1}{2})$, with λ increasing in m/n, turn out to be asymptotically efficient, while all tests of the form $\psi_k = 1$, a_k, 0 according as $(\hat{\xi} - \hat{\eta})$ is greater than, equal to, or less than c_k are asymptotically inefficient when $m \neq n$. For given relative sampling costs, the ratio m/n may be chosen so that the asymptotic cost of observations is minimized.

1. Introduction. Our results concerning asymptotic efficiency depend heavily on the work of Cramér and Chernoff ([1], [2]). In order to use these results in connection with the binomial problem mentioned above, we find a test of the composite hypothesis which depends on the sum of observations X, each of which is the likelihood ratio of distributions indexed by $\theta_i \varepsilon \omega_i$. If $M_\theta(t)$ is the moment generating function of X and $\rho_\theta = \inf M_\theta(t)$, we try to choose the θ_i so that ρ_θ attains its maximum in ω_i at the point θ_i. We then employ a Bayes risk to establish a lower bound for the minimum sample size required to reduce the maximum risk to α, and use Chernoff's Theorem 1 to show that the corresponding sample size for our test is asymptotically equal to this lower bound.

Let $\theta \varepsilon \Omega$ be a 1–1 index on a class of distributions on a probability space with elements Y and let ω_1, ω_2 be disjoint subsets of Ω. Let $\mathbf{Y} = (Y_1, Y_2, \cdots)$ be a sequence of independent random elements with a common distribution indexed by $\theta \varepsilon \Omega$. A test (sequence of tests) $\boldsymbol{\phi} = \{\phi_k\}$ with ϕ_k depending only on $Y_1, \cdots, Y_k$ will be described by the probabilities $\phi_k(\mathbf{Y})$, assigned to the decision "$\theta \varepsilon \omega_2$." The loss of the decision "$\theta \varepsilon \omega_i$" is denoted by $w_i(\theta)$ and it is assumed that

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(1.1) $0 \leqq w_i(\theta) \leqq \bar{w} < \infty, \qquad 0 < w_i(\theta)$, if and only if $\theta \varepsilon \omega_j, \quad j \neq i = 1, 2.$

The k-observation expected loss for the test ϕ is designated $r_k(\theta, \phi)$,

$$r_k(\theta, \phi) = w_2(\theta)E_\theta\phi_k + w_1(\theta)E_\theta(1 - \phi_k).$$

Definitions 1.1. For any test ϕ, any distribution P on Ω, and any $\alpha > 0$,

$$r_k(P, \phi) = E_P r_k(\theta, \phi), \qquad r_k(\phi) = \sup_\Omega r_k(\theta, \phi), \qquad r_k = \inf_\phi r_k(\phi),$$

and, with LI abbreviating "the least integer k such that,"

$$N_P = LI\,[\inf_\phi r_k(P, \phi) \leqq \alpha], \qquad N = LI[r_k \leqq \alpha], \qquad N_\phi = LI[r_k(\phi) \leqq \alpha].$$

We note that $\inf_{\phi'} r_k(P, \phi') \leqq r_k \leqq r_k(\phi)$ for all k, P and ϕ and hence

(1.2) $$N_P \leqq N \leqq N_\phi \qquad \text{for all } \alpha, P, \phi.$$

Thus, for each α, N/N_ϕ defines an index of efficiency of the test ϕ, and ϕ will be called asymptotically efficient as $\alpha \to 0$ if and only if

(1.3) $$N \sim N_\phi \qquad \text{as } \alpha \to 0.$$

2. Asymptotic efficiency of tests based on sums. Let $X_1, X_2 \cdots$ denote the values of a real function at $Y_1, Y_2 \cdots$ respectively.

Definitions 2.1.

$$M_\theta(t) = E_\theta e^{tX}, \qquad -\infty < t < \infty, \qquad \rho_\theta = \inf_t M_\theta(t),$$

$$S_k = X_1 + \cdots + X_k, \qquad \phi_k^*(\mathbf{Y}) = \begin{Bmatrix} 1 \text{ if } S_k > 0 \\ 0 \text{ otherwise} \end{Bmatrix}.$$

We shall use Chernoff's Theorem 1, a variant of his remark (3.11) and part of the general version of his lemma 8 [2].

THEOREM (*Chernoff*). *If* $-\infty \leqq EX < 0$ *and* $0 < \epsilon \leqq \rho$, *where* $\rho = \inf M(t)$,

(T) $$(\rho - \epsilon)^k = o[\Pr\{S_k \geqq 0\}], \qquad \Pr\{S_k \geqq 0\} \leqq \rho^k.$$

(R) Remark (Chernoff). If $M(t) < 1$ for some $t > 0$, $\quad EX < 0$.
(A direct proof consists in first noting that the existence of EX is implied, and then that M is non-decreasing on $t \geqq 0$ if $EX \geqq 0$.)

LEMMA (*Chernoff*). *If f_1 and f_2 are probability densities with respect to μ of distinct distributions and* $X(Y) = \log f_2(Y) - \log f_1(Y)$, *then*

$$M_2(t) = M_1(t + 1) \qquad 0 < t < 1,$$

(L) *the ρ_i are attained for t_i with* $t_2 = t_1 - 1 < 0 < t_1$ *and*

$$\rho_1 = \rho_2 = \inf_{0 < t < 1} \int f_2^t f_1^{1-t}\, d\mu < 1.$$

The following theorem is implicit in [2] for the case where the ω_i are simple.

THEOREM 2.1. *If*

(a) $X(Y) = \log f_2(Y) - \log f_1(Y)$ *with* f_2/f_1 *the likelihood ratio of distributions indexed by* $\theta_2 \varepsilon \omega_2$ *and* $\theta_1 \varepsilon \omega_1$,

(b) $\rho_\theta \leqq \rho_i$ *on* ω_i $(i = 1, 2)$,

(c) $E_\theta X < 0$ *on* ω_1, $E_\theta X > 0$ *on* ω_2, *then*

(i) *the test* ϕ^* *is asymptotically efficient as* $\alpha \to 0$ *and,*

(ii) *except in the trivial case where* $\rho_1 = 0$ *and* $N = N_{\phi^*} = 1$,

$$N \sim N_{\phi^*} \sim \frac{\log \alpha}{\log \rho_1}.$$

PROOF. By (c) it follows from (L) that $\rho_1 = \rho_2 < 1$. Thus if $\rho_1 = 0$, $\rho_\theta \equiv 0$ on $\omega_1 \cup \omega_2$ and it follows from (c) and the definition of ρ_θ that the distributions of ω_1 concentrate on $X < 0$ while those of ω_2 concentrate on $X > 0$ (Lemma 1 of [2]).

By (a) ϕ_k^* is Bayes with respect to P^* concentrating on θ_1 and θ_2 and assigning to θ_i probabilities proportional to $w_i(\theta_j)$, $j \neq i = 1, 2$. Letting

$$w^* = w_1(\theta_2)w_2(\theta_1)/[w_1(\theta_2) + w_2(\theta_1)],$$

$$r_k(P^*, \phi^*) = w^*[E_1\phi_k^* + E_2(1 - \phi_k^*)],$$

(T) applied to X at θ_1, $-X$ at θ_2 yields for any $0 < \epsilon \leqq \rho_1$:

$$r_k(P^*, \phi^*) \geqq 2w^*[(\rho_1 - \epsilon)^k] \qquad \text{for all } k \geqq k(\epsilon). \tag{2.1}$$

Hence $\alpha \geqq r_{N_{P^*}}(P^*, \phi^*) \geqq 2w^*(\rho_1 - \epsilon)^{N_{P^*}}$ for all $\alpha < 2w^*(\rho_1 - \epsilon)^{k(\epsilon)}$ and therefore

$$N_{p^*} \geqq \frac{\log \alpha - \log 2w^*}{\log(\rho_1 - \epsilon)} \text{ for all } \alpha < 2w^*(\rho_1 - \epsilon)^{k(\epsilon)}. \tag{2.2}$$

To complete the proof in the case $\rho_1 > 0$ we obtain an upper bound for N_{ϕ^*} through one for $r_k(\phi^*)$. Using (1.1), (T) for each θ and (b):

$$\begin{aligned} r_k(\phi^*) &= \max\,[\sup_{\omega_1} w_2(\theta)E_\theta\phi_k^*, \sup_{\omega_2} w_1(\theta)E_\theta(1 - \phi_k^*)], \\ &\leqq \bar{w} \max\,[\rho_2^k, \rho_1^k] = \bar{w}\rho_1^k. \end{aligned} \tag{2.3}$$

From (2.3) with $k = N_{\phi^*} - 1$

$$N_{\phi^*} < 1 + \frac{\log \alpha - \log \bar{w}}{\log \rho_1} \tag{2.4}$$

which, with (1.2) and (2.2), completes the proof.

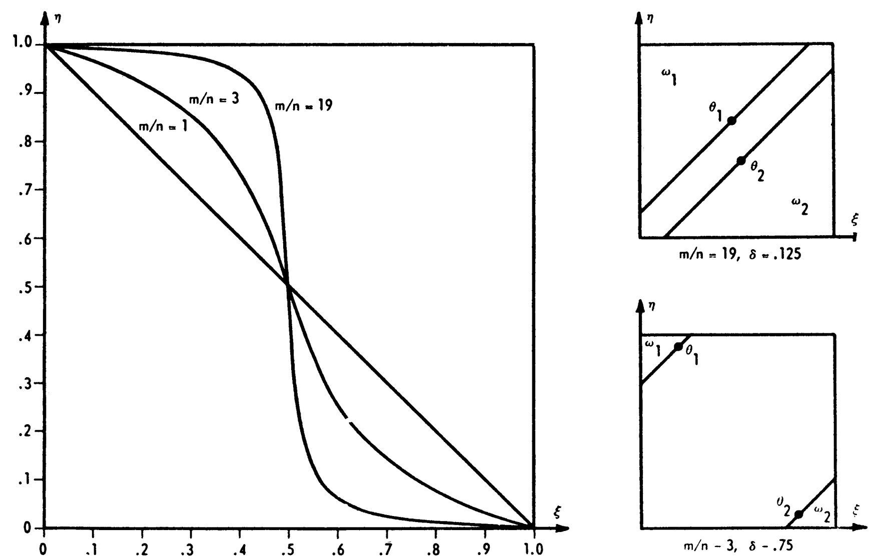

FIG. 1. Loci of θ_i for various ratios m/n, and the sets ω_i for $\delta = .125, .75$.

3. Applications to tests on the difference of binomial probabilities.

EXAMPLE 1. (See Figure 1.) Let $\theta = (\xi, \eta)$, $\Omega = [0, 1] \times [0, 1]$, $0 < \delta < 1$, $\omega_1 = \{\xi - \eta \leqq -\delta\}$, $\omega_2 = \{\xi - \eta \geqq \delta\}$, and $Y = (U, V)$ with density

$$f_\theta(y) = \binom{m}{u} \xi^u (1 - \xi)^{m-u} \binom{n}{v} \eta^v (1 - \eta)^{n-v} \text{ on } \{0, 1 \cdots, m\} \times \{0, 1 \cdots, n\}.$$

Then

$$\begin{aligned} X(Y) &= X(Y \mid \theta_1, \theta_2) \\ &= U \log \frac{\xi_2}{\xi_1} + (m - U) \log \frac{1 - \xi_2}{1 - \xi_1} + V \log \frac{\eta_2}{\eta_1} + (n - V) \log \frac{1 - \eta_2}{1 - \eta_1}, \end{aligned}$$

and

$$\begin{aligned} M_\theta(t) &= M_\theta(t \mid \theta_1, \theta_2) \\ &= \left[\xi \left(\frac{\xi_2}{\xi_1}\right)^t + (1 - \xi) \left(\frac{1 - \xi_2}{1 - \xi_1}\right)^t\right]^m \left[\eta \left(\frac{\eta_2}{\eta_1}\right)^t + (1 - \eta) \left(\frac{1 - \eta_2}{1 - \eta_1}\right)^t\right]^n. \end{aligned}$$

Since (with (1, 1) abbreviated to 1) $\omega_2 = 1 - \omega_1$, $M_{1-\theta}(t \mid \theta_1, \theta_2) = M_\theta(-t \mid 1 - \theta_2, 1 - \theta_1)$, we will seek θ_1, θ_2 in $\{\theta_1 + \theta_2 = 1\}$ satisfying (b) and (c) of Theorem 2.1 on ω_1. Here

$$\begin{aligned} M_\theta(t) &= M_\theta(t \mid \theta_1) \\ &= [\xi(\xi_1^{-1} - 1)^t + (1 - \xi)(\xi_1^{-1} - 1)^{-t}]^m \\ &\quad \cdot [\eta(\eta_1^{-1} - 1)^t + (1 - \eta)(\eta_1^{-1} - 1)^{-t}]^n \end{aligned} \tag{3.1}$$

is the moment generating function of

$$[(2U - m) \log (\xi_1^{-1} - 1) + (2V - n) \log (\eta_1^{-1} - 1)]$$

and [since this variable is degenerate if and only if $\theta_1 = (\frac{1}{2}, \frac{1}{2}) \varepsilon \omega_1$] is strictly convex. The component powers are either identically one or have unique infima at

$$(3.2) \qquad t_\xi = \frac{1}{2}\frac{\log (\xi^{-1} - 1)}{\log (\xi_1^{-1} - 1)}, \qquad t_\eta = \frac{1}{2}\frac{\log (\eta^{-1} - 1)}{\log (\eta_1^{-1} - 1)}.$$

In any event

$$(3.3) \quad L(\theta) = [4\xi(1 - \xi)]^{m/2}[4\eta(1 - \eta)]^{n/2} \leqq \inf_t M_\theta(t \mid \theta_1) = \rho_\theta(\theta_1).$$

Since for $\theta = \theta_1$ equality holds in (3.3), θ_1 can satisfy (b) only if θ_1 maximizes $L(\theta)$.

We first obtain an explicit characterization of the maximizer of $L(\theta)$ on ω_1. Let $I(\delta)$ be the open interval of ξ,

$$I(\delta) = (\max [0, \tfrac{1}{2} - \delta], \min [\tfrac{1}{2}, 1 - \delta]) = \begin{cases} (\frac{1}{2} - \delta, \frac{1}{2}) \text{ if } \delta \leqq \frac{1}{2} \\ (0, 1 - \delta) \text{ if } \delta > \frac{1}{2} \end{cases}.$$

It follows from

$$L(\xi, \eta) \leqq \begin{cases} L(\xi, \frac{1}{2}) < L(\frac{1}{2} - \delta, \frac{1}{2}) \text{ if } \xi \varepsilon (0, \frac{1}{2} - \delta), \\ L(\xi, \xi + \delta) \text{ if } \xi \varepsilon \mathrm{I}(\delta), \\ L(\xi, \xi + \delta) < L(\frac{1}{2}, \frac{1}{2} + \delta) \text{ if } \xi \varepsilon (\frac{1}{2}, 1 - \delta), \end{cases}$$

that the maximum can only occur in $\{\theta \mid \eta = \xi + \delta, \xi \varepsilon I(\delta)\}$. Since

$$(3.4) \quad \frac{d}{d\xi} \log L(\xi, \xi + \delta) = \frac{m}{2}\left[\frac{1}{\xi} - \frac{1}{1 - \xi}\right] + \frac{n}{2}\left[\frac{1}{\xi + \delta} - \frac{1}{1 - \xi - \delta}\right]$$

is decreasing with respect to ξ on $(0, 1 - \delta)$ and changes sign from positive to negative as ξ traverses $I(\delta)$, $L(\theta)$ has the unique maximizer

$$(3.5) \qquad \theta_1 = (\xi_1, \eta_1), \qquad \eta_1 = \xi_1 + \delta, \ \xi_1 \text{ the unique zero of (3.4) in } I(\delta).$$

Since by (3.3) $\rho_\theta(\theta_1) \leqq M_\theta(\frac{1}{2} \mid \theta_1)$ and $M_{\theta_1}(\frac{1}{2} \mid \theta_1) = \rho_1$, (b) will be satisfied if θ_1 maximizes $M_\theta(\frac{1}{2}, \theta_1)$. Because $\rho_1 < 1$ the remark (R) will then show that (c) is also satisfied.

To dispose of this maximization, note that $\xi_1 < \frac{1}{2} < \eta_1$ and hence that $M_\theta(\frac{1}{2} \mid \theta_1)$ can be maximal only on $\{\eta = \xi + \delta\}$. For such θ

$$(3.6) \quad \begin{aligned} &\frac{d}{d\xi} \log M_{(\xi, \xi+\delta)}(\tfrac{1}{2} \mid \theta_1) \\ &= m\frac{(1 - \xi_1) - \xi_1}{\xi(1 - \xi_1) + (1 - \xi)\xi_1} + n\frac{(1 - \eta_1) - \eta_1}{(\xi + \delta)(1 - \eta_1) + (1 - \xi - \delta)\eta_1}, \end{aligned}$$

which is decreasing with respect to ξ on $(0, 1 - \delta)$ and, by (3.5), vanishes at ξ_1. Thus θ_1 is the unique maximizer of $M_\theta(\frac{1}{2} \mid \theta_1)$ and (b), (c) of Theorem 2.1 are satisfied.

We summarize this application of Theorem 2.1 to Example 1 in terms of the maximum likelihood estimates (sample proportions), $\hat{\xi}$ and $\hat{\eta}$.

THEOREM 3.1. *For testing* $\{\xi - \eta \leqq -\delta\}$ *against* $\{\xi - \eta \geqq \delta\}$ *with bounded positive losses for wrong decisions, the nonrandomized test* ϕ^* *with* $\phi_k^* = 1$ *if and only if* $(\hat{\xi} - \frac{1}{2})m \log (\xi_1^{-1} - 1) + (\hat{\eta} - \frac{1}{2})\, n \log (\eta_1^{-1} - 1) > 0$, *where* $\xi_1 < \frac{1}{2} < \eta_1 = \xi_1 + \delta$ *and* ξ_1 *is the unique root of* (3.4) *in* $I(\delta)$, *is asymptotically efficient.*

To characterize the behavior of this test with respect to (m, n) variation, first note that by (3.5) m/n increases from 0 to ∞ as ξ_1 increases across $I(\delta)$ and hence that

$$\xi_1 \text{ increases across } I(\delta) \text{ as } m/n \text{ increases from 0 to } \infty. \tag{3.7}$$

To find the ratio m/n of maximum efficiency put $m + n = 2M$, $m(1 - z) = n(1 + z)$, and minimize $\rho_1 = L(\xi_1, \xi_1 + \delta)$ by choice of z. By (3.4), ξ_1 is a monotone increasing function of z and we have

$$\begin{aligned} \frac{d}{dz} \log L(\xi_1, \xi_1 + \delta) &= \frac{d\xi_1}{dz}\left(\frac{\partial}{\partial \xi} \log L(\xi, \xi + \delta)\,\Big|_{\xi=\xi_1}\right) \\ &\quad + \frac{M}{2}\,[\log \xi_1(1 - \xi_1) - \log(\xi_1 + \delta)(1 - \xi_1 - \delta)] \\ &= \frac{M}{2}\,[\log \xi_1(1 - \xi_1) - \log (\xi_1 + \delta)(1 - \xi_1 - \delta)], \end{aligned} \tag{3.8}$$

which increases from $-$ to $+$ as ξ_1 crosses $I(\delta)$, and vanishes for $\xi_1 = (1 - \delta)/2$. Thus ρ_1 has a unique minimum for $z = 0$ and $m/n = 1$.

If the relative costs of sampling are c and $1 - c(0 < c < 1)$, the total sampling cost is $N[cm + (1 - c)n]$, which is asymptotically $K(z) \log \alpha$ where

$$K(z) = \frac{M[1 + z(2c - 1)]}{\log L(\xi_1, \xi_1 + \delta)}.$$

Thus asymptotically minimum cost occurs when z is chosen to maximize $K(z)$. Using (3.8),

$$\begin{aligned} \frac{dK}{dz} &= \frac{M}{(\log L)^2}\left\{(2c - 1) \log L - \frac{M}{2}\,[1 + z(2c - 1)] \log \frac{4\xi_1(1 - \xi_1)}{4\eta_1(1 - \eta_1)}\right\} \\ &= \frac{M^2}{(\log L)^2} \log \{[4\xi_1(1 - \xi_1)]^{c-1}[4(\xi_1 + \delta)(1 - \xi_1 - \delta)]^c\}. \end{aligned} \tag{3.9}$$

The log in (3.9) decreases as ξ_1 crosses $I(\delta)$ and vanishes for ξ_1 in $I(\delta)$ satisfying

$$c = \left[1 + \frac{\log 4(\xi_1 + \delta)(1 - \xi_1 - \delta)}{\log 4\xi_1(1 - \xi_1)}\right]^{-1}. \tag{3.10}$$

Hence the asymptotic cost of sampling is minimized for ξ_1 in $I(\delta)$ satisfying (3.10). c decreases monotonically from 1 to 0 as ξ_1 crosses $I(\delta)$, and ξ_1 decreases across $I(\delta)$ as c increases from 0 to 1. Therefore the most economical ratio m/n decreases from ∞ to 0 as c increases from 0 to 1.

Representing the set of Y where $\phi_k^* = 1$ in the form $\lambda(\hat{\xi} - \frac{1}{2}) > (\hat{\eta} - \frac{1}{2})$,

$$\lambda = \frac{-m \log (\xi_1^{-1} - 1)}{n \log (\eta_1^{-1} - 1)} = \frac{R(\xi_1)}{R(1 - \eta_1)}, \tag{3.11}$$

where

$$R(p) = \frac{\log (1 - p) - \log p}{p^{-1} - (1 - p)^{-1}} \quad \text{on} \quad 0 < p < \tfrac{1}{2}.$$

From $1 - v^{-1} < \log v < v - 1$ for $v > 1$, $p < R(p) < 1 - p$. Thus $R(p)$ is positive and increasing from $R(0+) = 0$ to $R(\frac{1}{2} -) = \frac{1}{2}$ and from this, (3.7), and (3.11), as m/n increases from 0 to ∞, λ increases

$$\begin{cases} \text{from } 2R(\frac{1}{2} - \delta) \text{ to } 1/[2R(\frac{1}{2} - \delta)] \text{ if } \delta < \frac{1}{2}, \\ \text{from } 0 \text{ to } \infty \text{ if } \delta \geqq \frac{1}{2}. \end{cases} \tag{3.12}$$

If $m = n (c = \frac{1}{2})$, it is noteworthy that $\xi_1 = (1 - \delta)/2$, $\eta_1 = (1 + \delta)/2$, $\lambda = 1$ and $\phi_k^* = 1$ if and only if $\hat{\xi} > \hat{\eta}$. If $m \neq n$ the following theorem shows that this test is asymptotically inefficient.

THEOREM 3.2. *If $m \neq n$ and $\psi = \psi^{c,a}$ is a test with $\psi_k = 1, a_k, 0$ according as $(\hat{\xi} - \hat{\eta})$ is $> c_k$, $= c_k$, $< c_k$, then ψ is asymptotically inefficient as $\alpha \to 0$.*

PROOF. Let $w = \min[w_2(\theta_1), w_1(\theta_2)]$ and abbreviate $\psi^{0,0}$ to ψ^*. It follows from the definition of $r_k(\phi)$ and the relations $E_2(1 - \psi_k^{c,a}) = E_1\psi_k^{-c,1-a}$, $\psi_k^{-|c|,0} \geqq \psi_k^*$, that $r_k(\psi) \geqq w \max\{E_1\psi_k^{c,a}, E_1\psi_k^{-c,1-a}\} \geqq wE_1\psi_k^*$. Now ψ^* is a test based on $(2U - m)/m - (2V - n)/n$ which [c.f. (3.1), (3.2)] has moment generating function (at θ_1), $M_0(t) = [\xi_1 e^{t/m} + (1 - \xi_1)e^{-t/m}]^m[\eta_1 e^{-t/n} + (1 - \eta_1)e^{t/n}]^n$. As in (3.2)–(3.3), $\rho_0 = \inf M_0(t) \geqq \rho_1$ but equality would imply $\lambda = 1$, hence is impossible by (3.12). As in the development of (2.2) it follows from (T) that

$$N_\psi \geqq \frac{\log \alpha - \log w}{\log (\rho_0 - \epsilon)} \quad \text{for all} \quad \alpha < w(\rho_0 - \epsilon)^{k(\epsilon)}.$$

Thus, asymptotically, $N_\psi \gtrsim \log \alpha / \log \rho_0$, and $N/N_\psi \lesssim N_{\phi^*}/N_\psi \lesssim \log \rho_0 / \log \rho_1 < 1$, which completes the proof.

It should be added that for several binomial two population problems, including the one of this example, tests of the form $\psi = 1$ if and only if $(\hat{\xi} - \hat{\eta}) > c$ turn out to be asymptotically efficient as $\delta \to 0$, [5].

EXAMPLE 2. Consider the problem of Example 1 modified only by taking $\omega_1 = \{\xi - \eta \leqq 0\}$ and specialized to the case $m = n = 1$.

We will be content to show that (b) and (c) of Theorem 2.1 are satisfied for the choice,

$$\theta_1 = (\tfrac{1}{2}, \tfrac{1}{2}), \qquad \theta_2 = \left(\frac{1 + \delta}{2}, \frac{1 - \delta}{2}\right).$$

For this choice we have (by specialization in Example 1)

$$X(Y) = (U - V) \log \frac{1 + \delta}{1 - \delta} + \log (1 - \delta^2),$$

$$M_\theta(t) = [\xi(1 + \delta)^t + (1 - \xi)(1 - \delta)^t][\eta(1 - \delta)^t + (1 - \eta)(1 + \delta)^t].$$

Since

$$M_\theta(t) \leqq \begin{Bmatrix} M_{(\xi,\xi)}(t) \leqq M_1(t) \text{ for } \theta \,\varepsilon\, \omega_1, \, t > 0 \\ M_{(\xi,\xi-\delta)}(t) \leqq M_2(t) \text{ for } \theta \,\varepsilon\, \omega_2, \, t < 0 \end{Bmatrix},$$

(b) and (c) follow from (L) and (R), and the non-randomized test ϕ^*, with $\phi_k^* = 1$ if, and only if,

$$\hat{\xi} - \hat{\eta} > \frac{-\log (1 - \delta^2)}{\log (1 + \delta) - \log (1 - \delta)}$$

is asymptotically efficient.

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