NOTES

A PROOF OF WALD'S THEOREM ON CUMULATIVE SUMS

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1. Introduction. In the theory of sequential analysis developed by Wald [1], there occurs a theorem, one form of which can be expressed as follows:

THEOREM 1. *If*

(i) $z_1, z_2, z_3 \cdots$ *are independent random variables with common expected value* $\mathcal{E}(z) = \mu$,

(ii) $\mathcal{E}(|z_i|) \leqq A < \infty$ *for all* i, *and some finite* A,

(iii) n *is a random variable taking values* $1, 2, 3, \cdots$ *with probabilities* $P_1, P_2, P_3 \cdots$ *respectively, and*

(iv) *the event* $\{n \geqq i\}$ *depends only on* $z_1, z_2, \cdots z_{i-1}$,

then, setting $Z_n = \sum_{i=1}^{n} z_i$,

$$\mathcal{E}(Z_n) = \mu\mathcal{E}(n).$$

This note presents a simple proof of this theorem. It appears to be an abbreviated form of an argument due to Wolfowitz [2].

In Sections 3 and 4 of this note an extension of the method to the evaluation of the variance of n is discussed.

2. Proof of Theorem 1. Let $y_i = 1$ if z_i is observed (i.e. if the event $\{n \geqq i\}$ occurs) and $y_i = 0$ if $\{n \geqq i\}$ is not observed, so that

$$\Pr\{y_i = 1\} = \Pr\{n \geqq i\} = \sum_{j=i}^{\infty} P_j .$$

Then $Z_n = \sum_{i=1}^{\infty} y_i z_i$ and $\mathcal{E}(Z_n) = \mathcal{E}(\sum_{i=1}^{\infty} y_i z_i) = \sum_{i=1}^{\infty} \mathcal{E}(y_i z_i)$ since $\sum_{i=1}^{\infty} |\mathcal{E}(y_i z_i)| < A\mathcal{E}(n) < \infty$. By reason of (iv),

$$\mathcal{E}(y_i z_i) = \mathcal{E}(y_i)\mathcal{E}(z_i),$$

so

$$\mathcal{E}(Z_n) = \sum_{i=1}^{\infty} \mathcal{E}(y_i)\mathcal{E}(z_i) = \mu \sum_{i=1}^{\infty} \mathcal{E}(y_i),$$

$$= \mu \sum_{i=1}^{\infty} (P_i + P_{i+1} + \cdots) = \mu \sum_{i=1}^{\infty} iP_i = \mu\mathcal{E}(n).$$

3. An analogous second moment theorem.

THEOREM 2. *If we assume, in addition to* (i)-(iv), *that*

(v) $\text{var}(z_i) = \mathcal{E}(z_i^2) - \mu^2 = \sigma^2 < \infty$, *with the same value for all* i,

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(vi) $\mathcal{E}[(z_j - \mu)^2 \mid n \geqq i] \leqq B < \infty$ *for all* $j < i$, *with* B *independent of* i,

(vii) $\mathcal{E}(n^2) = \sum_{i=1}^{\infty} i^2 P_i < \infty$, *then, setting* $Z_n' = Z_n - n\mu$

$$\mathcal{E}(Z_n'^2) = \sigma^2 \mathcal{E}(n).$$

PROOF. Let $z_i' = z_i - \mu$, so $\mathcal{E}(z_i'^2) = \sigma^2$. Then $Z_n' = \sum_{i=1}^{n} z_i'$ and $\mathcal{E}(Z_n'^2) = \mathcal{E}(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} y_i z_i' y_j z_j')$. Since

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathcal{E}(\,|\,y_i z_i' y_j z_j'\,|\,) &= \sum_{i=1}^{\infty} \mathcal{E}(y_i z_i'^2) + 2\sum_{i=1}^{\infty}\sum_{j=1}^{i-1} \mathcal{E}(y_i \,|\, z_i' z_j' \,|\,) \\
&= \sigma^2\, \mathcal{E}(n) + 2\sum_{i=1}^{\infty}\sum_{j=1}^{i-1} \mathcal{E}(y_i\, \mathcal{E}(\,|\,z_i' z_j'\,|\,\,|\, n \geqq i)) \\
&\leqq \sigma^2 \mathcal{E}(n) + 2\sum_{i=1}^{\infty}\sum_{j=1}^{i-1} \mathcal{E}(y_i)[\mathcal{E}(z_i'^2)\mathcal{E}(z_j'^2 \mid n \geqq i)]^{\frac{1}{2}} \\
&\leqq \sigma^2 \mathcal{E}(n)\; 2\sigma B^{\frac{1}{2}} \sum_{i=1}^{\infty}\sum_{j=1}^{i-1} \mathcal{E}(y_i) \\
&= \sigma^2 \mathcal{E}(n) + 2\sigma B^{\frac{1}{2}} \sum_{i=1}^{\infty} \frac{1}{2} i(i-1)P_i \\
&\leqq \sigma^2 \mathcal{E}(n) + \sigma B^{\frac{1}{2}}\, [\mathcal{E}(n^2) - \mathcal{E}(n)] \\
&< \infty,
\end{aligned}$$

we can invert the order of summation and expectation, giving

$$\begin{aligned}
\mathcal{E}(Z_n'^2) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathcal{E}(y_i z_i' y_j z_j') \\
&= \sum_{i=1}^{\infty} \mathcal{E}(y_i z_i'^2) + 2\sum_{i=1}^{\infty}\sum_{j=1}^{i-1} \mathcal{E}(y_i z_i' z_j') \\
&= \sum_{i=1}^{\infty} \mathcal{E}(y_i)\mathcal{E}(z_i'^2) + 2\sum_{i=1}^{\infty}\sum_{j=1}^{i-1} \mathcal{E}(y_i z_j')\mathcal{E}(z_i') \\
&= \sigma^2 \mathcal{E}(n),
\end{aligned}$$

since $\mathcal{E}(z_i') = 0$, and by reason of (iv).

4. The variance of n. If, now, we make the assumption (viii) $\mathcal{E}(Z_n \mid n)$ is independent of n, we have, using Theorem 2,

$$\begin{aligned}
\sigma^2 \mathcal{E}(n) = \mathcal{E}(Z_n'^2) &= \mathcal{E}[(Z_n - n\mu)^2] \\
&= \mathcal{E}(Z_n^2) - 2\mathcal{E}(n)\mu\mathcal{E}(Z_n) + \mathcal{E}(n^2)\mu^2 \\
&= \mathcal{E}(Z_n^2) - 2[\mu\mathcal{E}(n)]^2 + \mathcal{E}(n^2)\mu^2
\end{aligned}$$

(using Theorem 1).

Hence

$$\mathcal{E}(n^2) = [\sigma^2\mathcal{E}(n) - \mathcal{E}(Z_n^2)]\mu^{-2} + 2[\mathcal{E}(n)]^2$$

or

$$\begin{aligned}\text{var }(n) &= [\sigma^2\mathcal{E}(n) - \mathcal{E}(Z_n^2)]\mu^{-2} + [\mathcal{E}(n)]^2 \\ &= [\sigma^2\mathcal{E}(n) - \text{var }(Z_n)]\mu^{-2}\end{aligned}$$

5. Concluding remarks. Theorem 2 has been stated in [3] with the weaker condition

$$\text{(vi)}' \qquad \mathcal{E}(n^{\frac{3}{2}}) < \infty$$

in place of conditions (vi) and (vii), but an error in the proof was pointed out in [4].

Conditions (vi) and (vii) may be replaced by *either*

$$\text{(vi)}'' \qquad \mathcal{E}(n^{2+\delta}) < \infty \qquad (\delta > 0)$$

or

$$\text{(vi)}''' \qquad \textstyle\sum_{i=1}^{\infty}\sqrt{P_i} < \infty.$$

Condition (vi) is certainly satisfied if the event $\{n \geqq i\}$ is equivalent to $a < Z_j < b$ for all $j < i$. For then we must have $| z_j - \mu | < b - a + | \mu |$ and so $\mathcal{E}[(z_j - \mu)^2 \mid n \geqq i] < (b - a + | \mu |)^2$. This condition is therefore satisfied in standard sequential procedures, which have continuation regions of form $a < Z_j < b$. Condition (vii) is also satisfied by such procedures when (v) is satisfied (see [5]).

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