LARGE EXCURSIONS OF GAUSSIAN PROCESSES

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1. Introduction. It is known that the problem of determining the distribution of spacings between consecutive a-values of an ergodic Gaussian process, $x(t)$, $(Ex(t) = 0, Ex^2(t) = 1)$ is very difficult. Recently Palmer [1] and Rice [2] treated some limiting cases of this problem. In one limit they determine, for $a \to \infty$, the conditional probability

$$\Pr\{x(\tau) > a, \quad 0 \leqq \tau \leqq t\theta(a) \mid x(0) = a, \quad x'(0) > 0\} \tag{1.1}$$

where $\theta(a)$ is the average length of the times spent by $x(t)$ above the level a. Apart from some differences concerning the meaning of the conditional probability (1.1) both authors use the following heuristic device.

Since for large a, $\theta(a)$ is small, they write

$$x(\tau) = a + x'(0)\,\tau + \frac{x''(0)}{2}\,\tau^2 \tag{1.2}$$

and take for the time of the first downward crossing of the a-level

$$\tau = -2\,\frac{x'(0)}{x''(0)}. \tag{1.3}$$

It would thus seem that this procedure is limited to processes for which x'' exists. This would exclude, for example, the displacement of a harmonic oscillator in Brownian motion. It is precisely this point that led us to undertake the present investigation.

We have found an alternative derivation of the Palmer-Rice results which does not depend on the approximation (1.2) and hence is applicable to all cases of physical interest. We have also attempted to elucidate the ambiguity of (1.1) (see §2) and we have in §3 shown in what sense the sample functions $x(\tau)$ are approximated by parabolas as suggested in (1.2).

2. Conditional probability densities. It is well known that conditional probabilities and conditional probability densities must frequently be treated with some care. Since the material to follow contains some excellent examples of the subtle nature of these quantities, a few words on the subject are in order here.

Let $x(t)$ be a continuous ergodic Gaussian process possessing a derivative almost everywhere. Consider the "conditional probability density for the slope $\xi = x'(0)$ given that $x(0) = a$." From the ensemble point of view, the phrase in quotation marks has no meaning, since the set of sample functions satisfying the condition $x(0) = a$ is of probability zero. Yet, given a sample function of

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the process, one can imagine observing the slope of $x(t)$ at each value of t for which $x(t) = a$ and one can thus obtain an "empirical or time derived probability density for $x'(t)$ given that $x(t) = a$." This probability density will be the same for almost all sample functions. How do we reconcile these two points of view?

From the ensemble point of view, one can, of course, give meaning to the "probability density for $\xi = x'(0)$ given that $x(0) = a$" by means of limiting procedures. The condition $x(0) = a$ is replaced by some condition, A, of positive probability depending on parameters. The condition is chosen so that as the parameters assume limiting values, A becomes the condition $x(0) = a$. The conditional density of ξ given A, $p(\xi \mid A)$, is computed; the limit of this quantity as the parameters assume their limiting values can then be taken as a definition of $p(\xi \mid x(0) = a)$, the "density for ξ given that $x(0) = a$."

Unfortunately this limit depends in general on the manner in which A approaches the condition $x(0) = a$. We illustrate with a few examples.

(i) Let A be $a \leqq x(0) \leqq a + \delta$. Then

$$p(\xi|x(0) = a)_{\text{v.w.}} = \lim_{\delta\to 0} \frac{\int_a^{a+\delta} p(\xi, x)\, dx}{\int_a^{a+\delta} p(x)\, dx} = \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}}$$

where the subscript v.w. stands for "vertical window," $p(\xi, x)$ is the joint density for $\xi = x'(0)$ and $x(0)$, and $p(x)$ is the density for $x(0)$. We have made use of the independence of $x(0)$ and ξ and have assumed that $Ex(t) = 0$ and $E\xi^2 = \alpha$. This vertical window definition of the conditional density of ξ given that $x(0) = a$ thus reduces to the conventional one $p(\xi|x(0) = a) = [p(\xi, x)/p(x)]_{x=a}$.

(ii) Let A be the "horizontal window condition" $x(t) = a$ for some t such that $0 \leqq t \leqq \delta$. Then if $\xi \geqq 0$,

$$p(\xi \mid x(0) = a)_{\text{h.w.}}$$

$$= \lim_{\delta\to 0} \frac{\int_{a-\xi\delta}^{a} p(\xi, x)\, dx}{\int_0^\infty d\xi' \int_{a-\xi'\delta}^{a} dx\, p(\xi', x) + \int_{-\infty}^0 d\xi' \int_a^{a-\xi'\delta} dx\, p(\xi', x)} = \frac{\xi}{2\alpha} e^{-\frac{\xi^2}{2\alpha}} \tag{2.1}$$

since the condition A can be satisfied (to first order in small quantities) for a given value of slope, say $\xi' > 0$, only if $a - \xi'\delta \leqq x(0) \leqq a$. A similar calculation for $\xi < 0$, gives the final result

$$p(\xi \mid x(0) = a)_{\text{h.w.}} = \frac{|\xi|}{2\alpha} e^{-\frac{\xi^2}{2\alpha}}.$$

(iii) More generally, let A be the condition that $x(t)$ pass through a line segment of length δ and slope m having one end-point at $x = a$, $t = 0$. Then one finds by straightforward computation

$$p(\xi \mid x(0) = a)_m = \frac{\mid \xi - m \mid e^{-\frac{\xi^2}{2\alpha}}/\sqrt{2\pi\alpha}}{\sqrt{\frac{2\alpha}{\pi}}\, e^{-\frac{m^2}{2\alpha}} + m \int_{-m}^{m} \frac{e^{-\frac{x^2}{2\alpha}}}{\sqrt{2\pi\alpha}}\, dx}.$$

(iv) If A is the condition that $x(t)$ pass through a circle of radius δ with center at the point $x = a$, $t = 0$, then

$$p(\xi \mid x(0) = a)_0 = \frac{\sqrt{1 + \xi^2}\, e^{-\frac{\xi^2}{2\alpha}}}{\int_{-\infty}^{\infty} \sqrt{1 + x^2}\, e^{-\frac{x^2}{2\alpha}}\, dx}.$$

Which, if any, of these several versions of $p(\xi|x(0) = a)$ is equal to the empirical density obtained from a single sample function? The question can be answered readily in the following heuristic manner. Let ν be the expected number of zeros per unit time of $x(t) - a$. Let $S_b(y) = 1$ if $y \leqq b$ and be zero otherwise. The empirical cumulative distribution for ξ can be written,

$$\begin{aligned} \Pr\,(\xi \leqq b \mid x(0) = a)_{\text{emp}} &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{1}{\nu}\, \delta[x(t) - a] \mid x'(t) \mid S_b[x'(t)]\, dt \\ &= E\, \frac{1}{\nu}\, \delta[x(t) - a] \mid x'(t) \mid S_b[x'(t)] \\ &= \frac{1}{\nu} \int_{-\infty}^{\infty} dx \int_{-\infty}^{b} d\xi \delta(x - a) \mid \xi \mid p(\xi, x) \\ &= \frac{1}{\nu} \int_{-\infty}^{b} d\xi \mid \xi \mid p(\xi, a). \end{aligned}$$

Here we have appealed to the ergodic theorem. The empirical density for ξ then follows by differentiating with respect to b,

$$p(\xi \mid x(0) = a)_{\text{emp}} = \frac{1}{\nu} \mid \xi \mid p(\xi, a). \tag{2.2}$$

Now the denominator of (2.1) is the probability that $x(t) - a$ have a zero in the small time interval δ. Evaluating the integrals, one finds this probability to be $\sqrt{2\alpha/\pi}p(a)\delta$ to first order in δ. It follows then that $\nu = \sqrt{2\alpha/\pi}p(a)$. Inserting this value in (2.2) yields

$$p(\xi \mid x(0) = a)_{\text{emp}} = \frac{\mid \xi \mid}{2\alpha}\, e^{-\frac{\xi^2}{2\alpha}} = p(\xi \mid x(0) = a)_{\text{h.w.}}.$$

It might be mentioned that the interpretation of conditional probabilities in the h.w. sense is intimately connected with the definition of the mean recurrence time as introduced into statistical physics by Smoluchowski.

Let $x(t)$ be an ergodic process and consider the discrete observations $x(0)$, $x(\tau)$, $x(2\tau)$, $\cdots$. For these observations the mean recurrence time of the state defined by $| x(t) | < \delta$ is given by Smoluchowski's formula

$$\theta_{\delta,\tau} = \tau \frac{1 - \{\Pr \{| x(0) | < \delta\}}{\Pr \{| x(0) | < \delta, | x(\tau) | > \delta\}}$$

which he derived by direct time average considerations. (For a derivation of this formula as well as a discussion of its connection with the ergodic theorem see [3].) Denoting by $W(x)$ the probability density of $x(t)$ and by $W_\tau(x, y)$ the joint probability density of $x(t)$ and $x(t + \tau)$, we get

$$\theta_{\delta,\tau} = \tau \frac{1 - \int_{-\delta}^{\delta} W(x)\, dx}{\displaystyle\iint_{\substack{|x|<\delta \\ |y|>\delta}} W_\tau(x, y)\, dx\, dy}.$$

For a Gaussian ergodic process for which $x'(t)$ is defined, we can go further. Since

$$x(\tau) \sim x(0) + \tau x'(0)$$

and $x(0)$ and $x'(0)$ are independent, we have

$$\iint_{\substack{|x|<\delta \\ |y|>\delta}} W_\tau(x, y)\, dx\, dy \sim \frac{1}{2\pi\sqrt{\alpha}} \iint_{\substack{|x|<\delta \\ |x+\tau\xi|>\delta}} e^{-\left(\frac{x^2}{2}+\frac{\xi^2}{2\alpha}\right)} dx\, d\xi$$

where $\alpha = E[x'(0)]^2$. Now

$$\iint_{\substack{|x|<\delta \\ |x+\tau\xi|>\delta}} e^{-\left(\frac{x^2}{2}+\frac{\xi^2}{2\alpha}\right)} dx\, d\xi = \int_{-\delta}^{\delta} dx e^{-\frac{x^2}{2}} \int_{\frac{\delta-x}{\tau}}^{\infty} d\xi e^{-\frac{\xi^2}{2\alpha}} + \int_{-\delta}^{\delta} dx e^{-\frac{x^2}{2}} \int_{-\infty}^{\frac{-\delta-x}{\tau}} d\xi e^{-\frac{\xi^2}{2\alpha}}.$$

In the first of these integrals set $x = \delta - y\tau$. In the second, put $x = -\delta - y\tau$. There results

$$\tau \int_0^{\frac{2\delta}{\tau}} dy e^{-\frac{(\delta-y\tau)^2}{2}} \int_y^{\infty} d\xi e^{-\frac{\xi^2}{2\alpha}} + \tau \int_{-\frac{2\delta}{\tau}}^{0} dy e^{-\frac{(\delta+y\tau)^2}{2}} \int_{-\infty}^{y} d\xi e^{-\frac{\xi^2}{2\alpha}}$$

and hence

$$\lim_{\delta\to 0} \lim_{\tau\to 0} \theta_{\delta,\tau} = \frac{1}{\frac{1}{\pi\sqrt{\alpha}} \int_0^{\infty} dy \int_y^{\infty} d\xi e^{-\frac{\xi^2}{2\alpha}}} = \frac{\pi\sqrt{\alpha}}{\int_0^{\infty} y e^{-\frac{y^2}{2\alpha}} dy}$$

which agrees with the known result of Rice for the mean distance between zeros.

3. Joint distribution for large positive excursions. Let $x(t)$ be a continuous parameter ergodic differentiable Gaussian process with mean zero and covariance function $\rho(\tau)$. For convenience we choose $\rho(0) = 1$ and assume that in some interval about $\tau = 0$,

$$\rho(\tau) = 1 - \frac{\alpha}{2}\tau^2 + o(\tau^2), \qquad \alpha > 0. \tag{3.1}$$

Let $\theta = \theta(a)$ denote the expected length of the intervals during which $x(t) \geqq a$. Then [2]

$$\theta = \frac{\pi e^{\frac{a^2}{2}}}{\sqrt{\alpha}}\left[1 - \sqrt{\frac{2}{\pi}}\int_0^a e^{-\frac{t^2}{2}}\,dt\right] \tag{3.2}$$

and

$$\theta \sim \sqrt{\frac{2\pi}{\alpha}}\,\frac{1}{a} \tag{3.3}$$

for large positive a.

In this and the next several sections, we study some limiting properties of the related process

$$\Delta(t, \theta) = \frac{x(\theta t) - a}{\theta} \tag{3.4}$$

as $a \to +\infty$, (or, equivalently, as $\theta \to 0$ through positive values). We shall generally be concerned only with properties of $\Delta(t, \theta)$ conditioned by

$$\Delta'(0, \theta) = \left.\frac{\partial \Delta}{\partial t}\right|_{t=0} \geqq 0 \quad \text{and} \quad \Delta(0, \theta) = 0$$

in either the h.w. or v.w. sense of Section 2.

The main result of this section is that, as $a \to \infty$, the n-dimensional joint distribution function of $\Delta(t_1, \theta), \Delta(t_2, \theta), \cdots, \Delta(t_n, \theta)$ conditioned in the v.w. sense by $\Delta'(0, \theta) \geqq 0$, $\Delta(0, \theta) = 0$ approaches the singular n-dimensional half-normal distribution function of the random variables

$$\Delta_i = -\sqrt{\frac{\alpha\pi}{2}}\,t_i^2 + \sqrt{\alpha}\,t_i\,\xi, \qquad i = 1, \cdots, n, \tag{3.5}$$

where ξ is a random variable with probability density

$$p(\xi) = \begin{cases} 0, & \xi < 0 \\ \sqrt{\frac{2}{\pi}}\,e^{-\frac{\xi^2}{2}}, & \xi \geqq 0. \end{cases} \tag{3.6}$$

If the conditioning is done in the h.w. sense, the result remains the same except that ξ now has the Rayleigh density

$$p(\xi) = \begin{cases} 0, & \xi < 0 \\ \xi e^{-\frac{\xi^2}{2}}, & \xi \geqq 0. \end{cases} \tag{3.7}$$

In one sense, then, as $a \to +\infty$, the sample functions of the conditioned $\Delta(t, \theta)$ process become a family of random parabolas,

$$\Delta = -\sqrt{\frac{\alpha\pi}{2}}\,t^2 + \sqrt{\alpha}\,t\xi, \tag{3.8}$$

where ξ has either a half-normal or Rayleigh distribution according as $\Delta(0, \theta)$ is conditioned to be zero in the v.w. or h.w. sense. In terms of the original $x(t)$ process, one can say that, when properly scaled and normalized, the excursions of $x(t)$ above the level a approach parabolas as $a \to +\infty$.

It is worth noting that these results require only the existence of the first derivative of $x(t)$. Processes with pathologies only in higher order derivatives, such as the harmonic oscillator of physics, are sufficiently "tamed" by the normalization and scaling indicated in (3.4) to give the limits mentioned.

We obtain the limiting conditional distribution function for $\Delta(t_1, \theta), \cdots, \Delta(t_n, \theta)$ by computing the characteristic function, $\varphi_a(\eta_1, \eta_2, \cdots, \eta_n)$, for these quantities and determining the limiting function

$$\varphi(\eta_1, \cdots, \eta_n) = \lim_{a\to+\infty} \varphi_a(\eta_1, \cdots, \eta_n).$$

By a well-known theorem ([4], p. 102), $\varphi(\eta_1, \cdots, \eta_n)$, if continuous at $\eta_1 = \eta_2 = \cdots = \eta_n = 0$, is the characteristic function of the limiting distribution function for $\Delta(t_1, \theta), \cdots, \Delta(t_n, \theta)$.

Let $\xi = x'(0)$, $x_i = x(\theta t_i)$, $i = 0, 1, 2, \cdots, n$, and $t_0 = 0$. Then

$$\begin{aligned} p(x_1, \cdots, x_n \mid \xi \geqq 0, x_0 = a)_{\text{v.w.}} \\ &= \frac{\int_0^\infty d\xi p(\xi, x_0, x_1, \cdots, x_n)\,|_{x_0=a}}{\frac{1}{2}p(x_0)\,|_{x_0=a}} \\ &= \frac{\int_0^\infty d\xi p(x_1, \cdots, x_n \mid \xi, x_0 = a)p(\xi, x_0)\,|_{x_0=a}}{\frac{1}{2}p(x_0)\,|_{x_0=a}} \\ &= 2\int_0^\infty d\xi p(x_1, \cdots, x_n \mid \xi, x_0 = a)p(\xi). \end{aligned} \tag{3.9}$$

Now $p(x_1, \cdots, x_n|\xi, x_0 = a)$ is an n-variate Gaussian density (see Appendix). The conditional means and covariances are readily computed:

$$m_i = E(x_i \mid \xi, x_0 = a) = \rho(\theta t_i)a - \frac{1}{\alpha}\rho'(\theta t_i)\xi \tag{3.10}$$

and

$$\begin{aligned} \lambda_{ij} &= E[(x_i - m_i)(x_j - m_j) \mid \xi, x_0 = a] \\ &= \rho[\theta(t_i - t_j)] - \rho(\theta t_i)\rho(\theta t_j) - \frac{1}{\alpha}\rho'(\theta t_i)\rho'(\theta t_j), \\ &\qquad\qquad i, j, = 1, 2, \cdots, n. \end{aligned} \tag{3.11}$$

One can write, therefore,

$$p(x_1, \cdots, x_n \mid \xi, x_0 = a) = (2\pi)^{-n} \int_{-\infty}^{\infty} d\eta_1 \cdots \int_{-\infty}^{\infty} d\eta_n \, e^{-i \Sigma \eta_j (x_j - m_j) - \frac{1}{2} \Sigma \lambda_{jk} \eta_j \eta_k}.$$

Introduce this expression into (3.9), substitute $x_i = a + \theta \Delta_i$ and multiply the entire expression by θ^n to obtain the conditional density function for $\Delta(t_1, \theta), \cdots, \Delta(t_n, \theta)$ in the form

$$p[\Delta_1, \cdots, \Delta_n \mid \Delta'(0, \theta) \geqq 0, \Delta(0, \theta) = 0]_{\text{v.w.}}$$

$$= \frac{2\theta^n}{(2\pi)^n} \int_0^{\infty} d\xi \int_{-\infty}^{\infty} d\eta_1 \cdots \int_{-\infty}^{\infty} d\eta_n \, e^{-i \Sigma \eta_j (a + \theta \Delta_j - m_j) - \frac{1}{2} \Sigma \lambda_{jk} \eta_j \eta_k} \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}}.$$

Let $\xi = \sqrt{\alpha}\xi'$, $\theta\eta_i = \eta_i'$, $i = 1, \cdots, n$. Introduce the value of m_i given in (3.10), interchange the order of integration which is a step easily justified, and omit the primes. There results

$$p[\Delta_1, \cdots, \Delta_n \mid \Delta'(0, \theta) \geqq 0, \Delta(0, \theta) = 0]_{\text{v.w.}}$$

$$= \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} d\eta_1 \cdots \int_{-\infty}^{\infty} d\eta_n \, e^{-i \Sigma \eta_j \Delta_j} \varphi_a(\eta_1, \cdots, \eta_n)$$

where

$$\varphi_a(\eta_1, \cdots, \eta_n) = \exp\left[-i \sum \eta_j \frac{a}{\theta} [1 - \rho(\theta t_j)] - \tfrac{1}{2} \sum \frac{\lambda_{jk}}{\theta^2} \eta_j \eta_k \right] \cdot \int_0^{\infty} d\xi \sqrt{\frac{2}{\pi}} \, e^{-i \frac{\xi}{\alpha} \Sigma \eta_j \frac{\rho'(\theta t_j)}{\theta} - \frac{1}{2}\xi^2}.$$

On using (3.1), (3.3) and (3.11), one finds,

$$\varphi(\eta_1, \cdots, \eta_n) = \lim_{a \to \infty} \varphi_a(\eta_1, \cdots, \eta_n) = e^{-i\sqrt{\frac{\pi\alpha}{2}} \Sigma \eta_j t_j^2} \int_0^{\infty} d\xi \sqrt{\frac{2}{\pi}} \, e^{i\sqrt{\alpha}\xi \Sigma \eta_j t_j - \frac{1}{2}\xi^2}.$$

But this expression, which is continuous at $\eta_1 = \cdots = \eta_n = 0$, is just the characteristic function associated with the random variables (3.5), (3.6), as a trivial computation shows.

The determination of the limiting form of the joint distribution function for $\Delta(t_1, \theta), \ldots, \Delta(t_n, \theta)$ conditioned in the h.w. sense by $\Delta'(0, \theta) \geqq 0$, $\Delta(0, \theta) = 0$ proceeds in a similar manner. Here (3.9) is replaced by

$$p(x_1, \cdots, x_n \mid \xi \geqq 0, x_0 = a)_{\text{h.w.}}$$

$$= \frac{\int_0^{\infty} d\xi \xi p(\xi, x_0, x_1, \cdots, x_n) \mid_{x_0 = a}}{\int_0^{\infty} d\xi \xi p(\xi, x_0) \mid_{x_0 = a}}$$

$$= \sqrt{\frac{2\pi}{\alpha}} \int_0^{\infty} d\xi \xi p(x_1, \cdots, x_n \mid \xi, x_0 = a) p(\xi).$$

The remaining steps are as in the previous demonstration.

4. Asymptotic distribution of first return time to positive level. We now assume that

$$\rho(\tau) = 1 - \frac{\alpha}{2}\tau^2 + \frac{c_3}{3!}|\tau|^3 + \frac{c_4}{4!}\tau^4 + o(\tau^4). \tag{4.1}$$

Let $P_a(T)$ be the probability that $\Delta(t, \theta)$ be non-negative for $0 \leqq t \leqq T$ given that $\Delta'(0, \theta) \geqq 0$ and $\Delta(0, \theta) = 0$ in the h.w. sense. Then $Q_a(T) = -(dP_a/dT)$ is the probability density for the duration of the positive excursions of the $\Delta(t, \theta)$ process conditioned in the h.w. sense. In this section we show that

$$Q(T) \equiv \lim_{a\to\infty} Q_a(T) = \begin{cases} \frac{\pi T}{2} e^{-\frac{\pi}{4}T^2}, & T \geqq 0 \\ 0, & T < 0 \end{cases} \tag{4.2}$$

and

$$P(T) \equiv \lim_{a\to\infty} P_a(T) = \begin{cases} e^{-\frac{\pi}{4}T^2}, & T \geqq 0 \\ 0 \quad , & T < 0. \end{cases} \tag{4.3}$$

If the conditioning is done in the v.w. sense, the corresponding results are

$$Q(T) = \begin{cases} e^{-\frac{\pi}{4}T^2}, & T \geqq 0 \\ 0 \quad , & T < 0 \end{cases} \tag{4.4}$$

$$P(T) = \begin{cases} 1 - \int_0^T e^{-\frac{\pi}{4}x^2}\, dx, & T \geqq 0 \\ 0 \quad , & T < 0. \end{cases} \tag{4.5}$$

These results are consistent with the interpretation of the sample functions of the limiting Δ process as the family of random parabolas (3.8) with ξ distributed according to (3.6) or (3.7). Note that the results (4.2)–(4.5) are independent of the parameters defining $\rho(\tau)$. All differentiable ergodic Gaussian processes, when scaled as here, have the same asymptotic distribution for the duration of excursions above a level.

To compute $P_a(T)$, we make use of the method of "inclusion and exclusion" [5], p. 89, in a manner analogous to that of Rice [6], p. 70. Let A_i be the event "$x(t)$ assumes the value a for some value of t such that

$$i(T/n) \leqq t < (i + 1)(T/n)$$

given that $x'(0) \geqq 0$ and $x(0) = a$ in the h.w. sense." Then the probability, $W_a(T)$, that $x(t)$ be not less than a for $0 \leqq t \leqq T$ given that $x'(0) \geqq 0$ and $x(0) = a$ in the h.w. sense is

$$W_a(T) = 1 - \sum_i \Pr[A_i] + \sum_{i<j} \Pr[A_i \cap A_j] \sum_{i<j<k} \Pr[A_i \cap A_j \cap A_k] + \cdots$$

In the limit as $n \to \infty$, this becomes

$$W_a(T) = 1 - \frac{1}{1!}\int_0^T dt_1\, p_1(t_1) + \frac{1}{2!}\int_0^T dt_1 \int_0^T dt_2\, p_2(t_1, t_2) - \frac{1}{3!}\int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3\, p_3(t_1, t_2, t_3) + \cdots$$

where

$$p_j(t_1, \cdots, t_j)\, dt_1 \cdots dt_j$$

is the probability that $x(t)$ assumes the value a in each of the intervals $(t_1, t_1 + dt_1), \cdots, (t_j, t_j + dt_j)$ given that $x'(0) \geqq 0$ and $x(0) = a$ in the h.w. sense.

One has then, since $P_a(T) = W_a(\theta T)$,

$$P_a(T) = 1 - \frac{\theta}{1!}\int_0^T dt_1\, p_1(\theta t_1) + \frac{\theta^2}{2!}\int_0^T dt_1 \int_0^T dt_2\, p_2(\theta t_1, \theta t_2) - \cdots \tag{4.6}$$

and by differentiation

$$Q_a(T) = \theta p_1(\theta T) - \frac{\theta^2}{1!}\int_0^T dt_1\, p_2(\theta t_1, \theta T) + \cdots. \tag{4.7}$$

Here

$$p_n(\theta t_1, \cdots, \theta t_n) = \frac{\int_0^\infty d\xi_0 \int_{-\infty}^\infty d\xi_1 \cdots \int_{-\infty}^\infty d\xi_n\, \xi_0 \,|\xi_1| \cdots |\xi_n|\, p(\xi_0, \cdots, \xi_n, x_0, \cdots, x_n)\,|_{x\text{'s}=a}}{\sqrt{\frac{\alpha}{2\pi}}\, p(x_0)\,|_{x_0=a}} \tag{4.8}$$

where

$$x_i = x(\theta t_i), \qquad \xi_i = x'(\theta t_i), \qquad t_0 = 0, \qquad i = 0, 1, \cdots, n$$

and p denotes the joint density of the random variables indicated.

From the derivation of the method of inclusion and exclusion, the successive partial sums of (4.6) and (4.7) alternately over-estimate and under-estimate the limit sum. Therefore

$$0 \leqq P_a(T) - 1 + \theta \int_0^T dt_1\, p_1(\theta t_1) \leqq \frac{\theta^2}{2}\int_0^T dt_1 \int_0^T dt_2\, p_2(\theta t_1, \theta t_2), \tag{4.9}$$

$$0 \leqq Q_a(T) - \theta p_1(\theta T) \leqq \frac{\theta^2}{2}\int_0^T dt_1\, p_2(\theta t_1, \theta T). \tag{4.10}$$

We establish (4.2) and (4.3) by evaluating $\lim_{a\to\infty} \theta p_1(\theta t_1)$ and by showing the right members of (4.9) and (4.10) approach zero as $a \to \infty$. A completely analogous procedure gives (4.4) and (4.5).

To investigate the behavior of $\theta p_1(\theta t_1)$ for large a, we write (4.8) for $n = 1$ as

$$\theta\, p_1(\theta t_1) = \sqrt{\frac{2\pi}{\alpha}}\, \theta\, p(x_1 \mid x_0 = a)\,|_{x_1=a}\, I_a\,, \tag{4.11}$$

$$I_a = \int_0^\infty d\xi_0 \int_{-\infty}^\infty d\xi_1\, \xi_0 \mid \xi_1 \mid p(\xi_0\,, \xi_1 \mid x_0 = a, x_1 = a). \tag{4.12}$$

An elementary calculation shows that

$$\theta \sqrt{\frac{2\pi}{\alpha}}\, p(x_1 \mid x_0 = a)_{x_1=a} = \left[\frac{\theta^2}{\alpha(1-\rho^2)}\right]^{\frac{1}{2}} e^{-\frac{a^2}{2}\frac{1-\rho}{1+\rho}}$$

where $\rho = \rho(\theta t_1)$. Using (4.1) and (3.3), there results

$$\lim_{a\to\infty} \theta \sqrt{\frac{2\pi}{\alpha}}\, p(x_1 \mid x_0 = a)\,|_{x_1=a} = \frac{e^{-\frac{\pi}{4}t_1^2}}{\alpha t_1}\,. \tag{4.13}$$

The factor $p(\xi_0\,, \xi_1|x_0 = a, x_1 = a)$ in (4.12) is a bivariate Gaussian density. The conditional means are found to be (see Appendix)

$$m_0 = E(\xi_0 \mid x_0 = x_1 = a) = -\frac{a\,\rho'(\theta t_1)}{1+\rho(\theta t_1)} = -E(\xi_1 \mid x_0 = x_1 = a) = -m_1\,.$$

The conditional covariances are

$$\lambda_{00} = E[(\xi_0 - m_0)^2 \mid x_0 = x_1 = a] = \frac{\alpha - [\rho'(\theta t_1)]^2 - \alpha\rho^2(\theta t_1)}{1-\rho^2(\theta t_1)}$$

$$= \lambda_{11} = E[(\xi_1 - m_1)^2 \mid x_0 = x_1 = a],$$

$$\lambda_{01} = \frac{\rho^2(\theta t_1)\rho''(\theta t_1) - \rho''(\theta t_1) - \rho(\theta t_1)[\rho'(\theta t_1)]^2}{1-\rho^2(\theta t_1)}\,.$$

As $a \to \infty$, $m_0 \to \sqrt{\alpha\pi/2}t_1$, $m_1 \to -\sqrt{\alpha\pi/2}t_1$; the covariances are $O(\theta)$ if $c_3 \neq 0$ and $o(\theta)$ if $c_3 = 0$. By standard arguments, then, as $a \to \infty$ the contribution to I_a comes entirely from the neighborhood of the point $(m_0\,, m_1)$ and

$$\lim_{a\to\infty} I_a = \begin{cases} \frac{\alpha\pi}{2}\, t_1^2, & t_1 \geqq 0 \\ 0\;, & t_1 < 0. \end{cases}$$

Combining this result with (4.13), we find

$$\lim_{a\to\infty} \theta p_1(\theta t_1) = \begin{cases} \frac{\pi}{2}\, t_1\, e^{-\frac{\pi}{4}t_1^2}, & t_1 \geqq 0 \\ 0\;, & t_1 < 0. \end{cases} \tag{4.14}$$

To show that the right member of the (4.10) approaches zero, we write (4.8) for $n = 2$ in the form

$$\theta^2 p_2(\theta t_1\,, \theta t_2) = p(x_1\,, x_2 \mid x_0 = a)\,|_{\substack{x_1=a\\x_2=a}} \cdot J_a \tag{4.15}$$

with

$$J_a = \theta^2 \sqrt{\frac{2\pi}{\alpha}} \cdot \int_0^\infty d\xi_0 \int_{-\infty}^\infty d\xi_1 \int_{-\infty}^\infty d\xi_2\, \xi_0 \,|\, \xi_1 \,||\, \xi_2 \,|\, p(\xi_0, \xi_1, \xi_2 \,|\, x_0 = x_1 = x_2 = a). \tag{4.16}$$

The first factor of (4.15) is of the form

$$p(x_1, x_2 \mid x_0 = a) \big|_{\substack{x_1=a \\ x_2=a}} = \frac{e^{-\frac{a^2}{2}h(\theta)}}{2\pi\sqrt{d}}$$

where

$$d = 1 + 2\rho(\theta t_1)\rho(\theta t_2)\rho[\theta(t_2 - t_1)] - \rho^2(\theta t_1) - \rho^2(\theta t_2) - \rho^2[\theta(t_2 - t_1)]$$

and

$$h(\theta) = 2\,\frac{[1 - \rho(\theta t_1)][1 - \rho(\theta t_2)]\,\{1 - \rho[\theta(t_2 - t_1)]\}}{d}\,.$$

A lengthy calculation shows that as $a \to \infty$,

$$d = \begin{cases} \frac{2}{3}\,\alpha c_3\, t_1^2 t_2 (t_2 - t_1)^2 \theta^5 + o(\theta^5)\,, & c_3 \neq 0 \\ \frac{1}{4}\,\alpha(c_4 - \alpha^2) t_1^2 t_2^2 (t_2 - t_1)^2 \theta^6 + o(\theta^6)\,, & c_3 = 0, \end{cases} \tag{4.17}$$

so that

$$\frac{a^2}{2}\,h(\theta) = \begin{cases} \frac{3\pi\alpha t_2}{8c_3}\,\frac{1}{\theta} + o\left(\frac{1}{\theta}\right), & c_3 \neq 0 \\ \frac{\pi\alpha}{c_4 - \alpha^2}\,\frac{1}{\theta^2} + o\left(\frac{1}{\theta^2}\right), & c_3 = 0. \end{cases} \tag{4.18}$$

The first factor of (4.15) therefore approaches zero at least as fast as $A\theta^{3/2}e^{-B/\theta}$.

The proof is completed by showing that J_a is $O(1/\theta^r)$ for some finite r so that from (4.15) $\theta^2 p_2(\theta t_1, \theta t_2) \to 0$ as $a \to \infty$. Now

$$\begin{aligned} J_a/\theta^2 \sqrt{\frac{2\pi}{\alpha}} &\leqq \int_{-\infty}^\infty d\xi_0 \int_{-\infty}^\infty d\xi_1 \int_{-\infty}^\infty d\xi_2 \,|\, \xi_0 \,||\, \xi_1 \,||\, \xi_2 \,|\, p(\xi_0, \xi_1, \xi_2 \,|\, x_0 = x_1 = x_2 = a) \\ &\leqq 8 + E(\xi_0^2 \xi_1^2 \xi_2^2 \mid x_0 = x_1 = x_2 = a)\,. \end{aligned} \tag{4.19}$$

This conditional expectation, however, is a multi-nomial in the conditional means and variances of the ξ's. These latter quantities in turn are rational functions of a, $\rho(\theta t_1)$, $\rho(\theta t_2)$, $p[\theta(t_2 - t_1)]$, $\rho'(\theta t_1)$, $\cdots$, $\rho''[\theta(t_2 - t_1)]$. It follows then that the right side of (4.19), and hence J_a also, is $O(1/\theta^r)$.

We note in passing that in the case $c_3 = 0$ the factor $c_4 - \alpha^2$ in (4.17) and (4.18) is non-negative and vanishes only if $\rho(\tau) = \cos\beta\tau$. This can be established as follows. From (4.1) it is easily seen that when $c_3 = 0$, $\rho''(0)$ and $\rho^{(4)}(0)$ exist and are given by

$$(4.20) \qquad -\alpha = \rho''(0) \qquad c_4 = \rho^{(4)}(0).$$

Now since $\rho(\tau)$ is a covariance function, we can write

$$\rho(\tau) = \int_{-\infty}^{\infty} e^{2\pi i\lambda\tau}\, dF(\lambda)$$

where $F(\lambda)$ is non-decreasing. It follows then (see [4] p. 90 for a similar argument involving a distribution function and its characteristic function) that the second and fourth moments of F exist and that

$$\rho''(0) = -\alpha = -4\pi^2 \int_{-\infty}^{\infty} \lambda^2\, dF(\lambda),$$

$$\rho^{(4)}(0) = c_4 = 16\pi^4 \int_{-\infty}^{\infty} \lambda^4\, dF(\lambda).$$

The Schwartz inequality then gives

$$c_4 \geqq \alpha^2$$

with equality only if $\rho(\tau)$ is of the form $\cos\beta\tau$. Our derivation of (4.2)–(4.5) fails in this case. Indeed, we have already excluded this process with covariance $\cos\beta\tau$ from consideration since it is not ergodic. The results (4.2)–(4.5) are still valid for this process, however, as a separate calculation, omitted here, shows.

5. Asymptotic distribution of first return time to negative levels. As in the preceding section, let $Q_a(T)$ be the probability density for the duration of the excursions above the value a of the $\Delta(t, \theta)$ process conditioned in the h.w. sense. If in addition to (4.1), $\rho(\tau)$ and its first two derivatives approach zero as $\tau \to \infty$, then

$$\lim_{a\to-\infty} Q_a(T) = 2e^{-2T}.$$

This result follows readily from (4.7) and (4.8) and the asymptotic formula for θ for large negative values of a,

$$\theta \sim \frac{2\pi}{\sqrt{\alpha}}\, e^{\frac{a^2}{2}},$$

obtained from (3.2). For large negative a, the random variables $x_0 = x(\theta t_1), \cdots, x_n = x(\theta t_n)$, $\xi_0 = x'(\theta t_0), \cdots, \xi_n = n'(\theta t_n)$ tend toward independence and the density in the numerator of (4.8) approaches

$$\frac{e^{-\frac{1}{2\alpha}\Sigma \xi_i^2 - \frac{(n+1)}{2}a^2}}{(2\pi)^{n+1}\dfrac{n+1}{\alpha^2}}$$

in a uniformly continuous way. One finds, then, $\theta^n p_n(\theta t_1, \cdots, \theta t_n) \rightarrow 2^n$ and the series (4.7) sums to $2e^{-2T}$.

If the conditioning is done in the v.w. sense, the same result is found. It is to be noted that this limiting distribution, like those obtained for large positive a, (4.2)–(4.5), is independent of the covariance $\rho(\tau)$.

APPENDIX

The detailed calculations of this paper make frequent use of the multivariate conditional densities for Gaussian variables. Since these densities do not appear to be readily available in the literature, we present them here for the reader's convenience. They can be derived with a little effort from material given in many texts, e.g. [4] or [7], pp. 27–30.

Let $\xi_1, \cdots, \xi_n$ be jointly Gaussian with $E\xi_i = 0$, $E\xi_i\xi_j = \lambda_{ij}$, $i, j = 1, 2, \cdots, n$. Then

$$p(\xi_{p+1}, \cdots, \xi_n \mid \xi_1, \cdots, \xi_p) = \frac{e^{-\frac{1}{2}\sum_{p+1}^{n} \mu_{ij}^{-1}(\xi_i - m_i)(\xi_j - m_j)}}{(2\pi)^{\frac{n-p}{2}} \mid \mu \mid^{\frac{1}{2}}}$$

where

$$m_i = E(\xi_i \mid \xi_1, \cdots, \xi_p) = \sum_{j=1}^{p} \beta_{ij}\xi_j, \qquad i = p+1, \cdots, n$$

$$\mu_{ij} = E[(\xi_j - m_i)(\xi_j - m_j) \mid \xi_1, \cdots, \xi_p], \quad i, j = p+1, \cdots, n$$

and

$$\beta_{ij} = \frac{1}{d}\begin{vmatrix} \lambda_{11} & \cdots & \lambda_{1(j-1)} & \lambda_{1i} & \lambda_{1(j+1)} & \cdots & \lambda_{1p} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_{p1} & \cdots & \lambda_{p(j-1)} & \lambda_{pi} & \lambda_{p(j+1)} & \cdots & \lambda_{pp} \end{vmatrix}$$

$$\mu_{ij} = \frac{1}{d}\begin{vmatrix} \lambda_{ij} & \lambda_{i1} & \cdots & \lambda_{ip} \\ \lambda_{1j} & \lambda_{11} & \cdots & \lambda_{1p} \\ \vdots & \vdots & & \vdots \\ \lambda_{pj} & \lambda_{p1} & \cdots & \lambda_{pp} \end{vmatrix}$$

$$d = \begin{vmatrix} \lambda_{11} & \cdots & \lambda_{1p} \\ \vdots & & \vdots \\ \lambda_{p1} & \cdots & \lambda_{pp} \end{vmatrix}.$$

REFERENCES

[1] D. S. Palmer, "Properties of random functions," *Proc. Camb. Phil. Soc.*, Vol. 52 (1956), pp. 672–686.

[2] S. O. Rice, "Duration of fades in radio transmission," *Bell System Tech. J.*, Vol. 37 (1958), pp. 581–635.

[3] M. Kac, "On the notion of recurrence in discrete stochastic processes," *Bull. Am. Math. Soc.*, Vol. 53 (1947), pp. 1002–1010.

[4] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, N. J., 1946.
[5] W. Feller, *Probability Theory*, John Wiley & Sons, New York, 1957.
[6] S. O. Rice, "Mathematical analysis of random noise," Reprinted in *Noise and Stochastic Processes*, Dover Publications, New York, (1954). Original *Bell System Tech. J.*, Vol. 24 (1945), pp. 46–156.
[7] T. W. Anderson, *Introduction to Multivariate Statistical Analysis*, John Wiley & Sons, New York, 1958.