SOME CONVERGENCE THEOREMS FOR STATIONARY STOCHASTIC PROCESSES[1]

By T. Kawata

Tokyo Institute of Technology and Princeton University

1. Introduction. Let $\mathcal{E}(t)$ $(-\infty < t < \infty)$ be a continuous stationary process of the second order (in the wide sense) with mean zero; that is,

$$E\mathcal{E}(t+u)\mathcal{E}(t) = \rho(u) \tag{1.1}$$

is a continuous function of u only, and

$$E\mathcal{E}(t) = 0, \qquad -\infty < t < \infty. \tag{1.2}$$

Here E means the expectation of a random variable.

We have, then,

$$\mathcal{E}(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ(\lambda), \tag{1.3}$$

and

$$\rho(u) = \int_{-\infty}^{\infty} e^{iu\lambda}\, dF(\lambda), \tag{1.4}$$

where $F(\lambda)$ is a bounded non-decreasing function such that

$$F(+\infty) - F(-\infty) = \rho(0) = E\,|\,\mathcal{E}(t)|^2,$$

and $Z(\lambda)$ is an orthogonal process such that

$$E\,|\,Z(\lambda') - Z(\lambda)|^2 = F(\lambda' - 0) - F(\lambda - 0). \tag{1.5}$$

$F(u)$ and $Z(\lambda)$ are called the spectral function and the random spectral function of $\mathcal{E}(t)$ respectively. (See, e.g., Doob [5], Chapter XI). Let

$$X(t) = f(t) + \mathcal{E}(t), \qquad -\infty < t < \infty, \tag{1.6}$$

where $f(t)$ is a numerical valued function, and consider

$$\int_{-\infty}^{\infty} x(t-s)K(s,n)\, ds, \tag{1.7}$$

$K(s, n)$ being also a numerical valued function depending on a parameter n.

Integrals of the type (1.7) appear in many fields in the theory of probability and statistics. For instance, we often encounter (1.6) in the problem of smoothing data of observed values, in the problem of predicting future values of $x(t)$, and in the problem of estimating the spectral density of a stationary process.

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But here we shall consider the behavior of (1.6) from the analytical point of view along the lines of the classical theory of the general Fourier integral, and we shall show convergence theorems, some of which may be already known implicitly.

Next we shall consider the special case of (1.7)

$$\frac{1}{T}\int_{-T}^{T} X(s)K(s)\,ds = J(T). \tag{1.7a}$$

If $K(s) = e^{-i\xi s}$, then $|J(T)|^2$ may be considered as a function similar to the periodogram, in which $f(t)$ is a trigonometric polynomial and $\mathcal{E}(t) \equiv 0$. It is known that if $F(\lambda)$ is absolutely continuous and $p(\lambda) = F'(\lambda)$ (the spectral density of $\mathcal{E}(t)$), then $E\,|J(T)|^2$ converges to $p(\xi)$ provided $p(\xi)$ is continuous at ξ. We shall treat the convergence of $|J(T)|^2$ itself.

2. Preliminaries. Let the spectral function of the continuous (weakly) stationary process $\mathcal{E}(t)$ be $F(\lambda)$ as in the preceding section. Then the necessary and sufficient condition for the existence of

$$\eta(t) = \int_{-\infty}^{\infty} \mathcal{E}(t-s)\,dL(a)^{2} \tag{2.1}$$

for every s is that there exists a function $k(x)$ such that

$$\int_{-\infty}^{\infty} |\,k(x)\,|^2\,dF(x) < \infty$$

and

$$\lim_{\substack{A\to-\infty\\ B\to\infty}} \int_{-\infty}^{\infty} \left| \int_{A}^{B} e^{ixs}\,dL(s) - k(x) \right|^2 dF(x) = 0,$$

where we assume that $L(s)$ is a function of bounded variation in every finite interval. $k(x)$ is called the Fourier-Stieltjes transform of $L(s)$ with respect to $F(x)$. In particular if $K(x)\ \varepsilon\ L_1(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} \mathcal{E}(t-s)K(s)\,ds \tag{2.2}$$

exists.

We frequently use the following lemmas which are very well known.

LEMMA 2.1:

(i) *The stochastic process* (2.1) *is also a stationary process in the wide sense and we have* $E\eta(t) = 0$ *and*

$$E\eta(t+u)\overline{\eta(t)} = \int_{-\infty}^{\infty} |\,k(x)\,|^2\cdot e^{iux}\,dF(x), \tag{2.3}$$

where $F(x)$ *is the spectral function of* $\mathcal{E}(t)$.

[2] The integral is taken here as $\text{l.i.m.}_{A\to-\infty,\,B\to\infty} \int_A^B \mathcal{E}(t-s)dL(s)$, where l.i.m. means the limit in the mean of order 2 and the finite integral in the definition is also defined as a Riemann-Stieltjes integral, the limit process being taken as l.i.m. See M. Loève [10] or J. L. Doob [5].

(ii) *If we are given another process*

$$\eta_1(t) = \int_{-\infty}^{\infty} \mathcal{E}(t-s)\, dL_1(s), \tag{2.4}$$

where $L_1(s)$ *is of bounded variation in every finite interval and* (2.4) *is assumed to exist, then*

$$E\eta(t+u)\overline{\eta_1(t)} = \int_{-\infty}^{\infty} k(x)\overline{k_1(x)}e^{iux}\, dF(x), \tag{2.5}$$

$k_1(x)$ *being the Fourier-Stieltjes transform of* $L_1(s)$ *with respect to* $F(x)$.

LEMMA 2.2: *The stochastic process* $\eta(t)$ *in Lemma* 2.1 *can be represented as*

$$\eta(t) = \int_{-\infty}^{\infty} e^{itx}k(x)\, dZ(x), \tag{2.6}$$

$Z(x)$ *being the random spectral function of* $\mathcal{E}(t)$.

If $f(t)$ is a numerical function such that

$$\int_{-\infty}^{\infty} f(t-s)\, dL(s)$$

exists for every t as an absolutely convergent Riemann-Stieltjes integral, and $X(t) = f(t) + \mathcal{E}(t)$, then we define

$$\int_{-\infty}^{\infty} x(t-s)\, dL(s) = \int_{-\infty}^{\infty} \mathcal{E}(t-s)\, dL(s) + \int_{-\infty}^{\infty} f(t-s)\, dL(s).$$

3. Convergence theorems. In this section we shall consider processes of the type

$$Y_n(t) = n\int_{-\infty}^{\infty} X(t-s)K(ns)\, ds, X(t) = f(t) + \mathcal{E}(t) \tag{3.1}$$

and discuss the convergence (in the mean) of $Y_n(t)$ as $n \to \infty$. Similar discussions are classical when $\mathcal{E}(t) \equiv 0$ in the theory of the Fourier integral; for example we have the following fact which we shall state as

LEMMA 3.1:[3] *Suppose that*

(i) $\dfrac{f(s)}{1+|s|} \varepsilon L_1(-\infty, \infty)$,

(ii) $K(s) \varepsilon L_1(-\infty, \infty)$

and

(iii) $K(s) = o(|s|^{-1})$ *when* $|s| \to \infty$, *and* $K(s)$ *is bounded. Then one has*

$$\lim_{n\to\infty} n\int_{-\infty}^{\infty} f(t-s)K(ns)\, ds = f(t)\int_{-\infty}^{\infty} K(s)\, ds. \tag{3.2}$$

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[3] S. Bochner [1], S. Bochner–K. Chandresekharan [2]. More general theorems are known. See S. Bochner and S. Izumi [3].

Appealing to this lemma we have the following theorem.

THEOREM 3.1: *Let $f(t)$ and $K(u)$ satisfy the conditions of Lemma 3.1. Then we have*

$$\text{(3.3)} \qquad \underset{n\to\infty}{\text{l.i.m.}}\, n \int_{-\infty}^{\infty} X(t-s)K(ns)\,ds = X(t) \int_{-\infty}^{\infty} K(s)\,ds.$$

The proof of this theorem will be omitted since it is very similar to and easier than the one of Theorem 3.2 later.

If we want to estimate the error between both sides of (3.3) for instance as $o(1/n)$, it is necessary to prove a convergence theorem which contains an error term such as the following lemma:

LEMMA 3.2: *Suppose that*

$$\text{(i)} \qquad \frac{f(s)}{1+|s|^{3/2}} \,\varepsilon\, L_1(-\infty, \infty),$$

(ii) $f(t+u) - f(t) = O(u)$
for small u,

(iii) $(1+|s|)K(s) \,\varepsilon\, L_1$ *and*

(iv) $K(s)$ *is bounded and* $o(|s|^{-3/2})$ *as* $|s| \to \infty$.

Then one has

$$\text{(3.4)} \qquad n \int_{-\infty}^{\infty} f(t-s)K(s)\,ds = f(t) \int_{-\infty}^{\infty} K(s)\,ds + o(1/\sqrt{n}).$$

PROOF: Put

$$I = n \int_{-\infty}^{\infty} f(t-s)K(ns)\,ds - f(t) \int_{-\infty}^{\infty} nK(ns)\,ds.$$

We want to prove

$$\text{(3.5)} \qquad I = o(1/\sqrt{n}).$$

We have

$$\begin{aligned} I &= n \int_{-\infty}^{\infty} [f(t-s) - f(t)]K(ns)\,ds \\ &= n \int_{|s|<\frac{\alpha}{n^{1/2}}} [f(t-s) - f(t)]K(ns)\,ds \\ &\qquad + n \int_{|s|>\alpha/n^{1/2}} f(t-s)K(ns)\,ds - nf(t) \int_{|s|>\alpha/n^{1/2}} K(ns)\,ds \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say, where α is an arbitrary positive number fixed for the moment.

By (iii), we have

$$(3.5)\quad \begin{aligned} I_3 &= O\left(n \cdot \frac{n^{1/2}}{\alpha} \int_{|s|>\alpha/n^{1/2}} |sK(ns)|\, ds\right) \\ &= O\left(\frac{1}{\alpha n^{1/2}} \int_{|u|>\alpha n^{1/2}} |uK(u)|\, du\right) = o\left(\frac{1}{\alpha n^{1/2}}\right) \end{aligned}$$

as $n \to \infty$.

By (ii), we have

$$(3.6)\quad \begin{aligned} I_1 &= O\left(n \int_{|s|<\alpha/n^{1/2}} |sK(ns)|\, ds\right) = O\left(\alpha n^{1/2} \int_{|s|<\frac{\alpha}{n^{1/2}}} |K(ns)|\, ds\right) \\ &= O\left(\frac{\alpha}{n^{1/2}} \int_{-\infty}^{\infty} |K(u)|\, du\right) = O\left(\frac{\alpha}{n^{1/2}}\right). \end{aligned}$$

Lastly we have, by making use of (iv) and (i)

$$(3.7)\quad \begin{aligned} I_2 &= O\left(n \int_{|s|>\alpha/n^{1/2}} |f(t-s)|\,|K(ns)|\, ds\right) \\ &= o\left(\frac{1}{n^{1/2}} \int_{|s|>\alpha/n^{1/2}} |f(t-s)| \frac{ds}{(1+|s|)^{3/2}}\right) \\ &= o\left(\frac{1}{n^{1/2}} \int_{-\infty}^{\infty} \frac{|f(t-s)|}{(1+|s|)^{3/2}}\, ds\right) = o\left(\frac{1}{n^{1/2}}\right). \end{aligned}$$

Combining (3.5), (3.6) and (3.7), we get

$$\limsup_{n\to\infty} \sqrt{n} I = O(\alpha).$$

Since α is arbitrary, we must have

$$\lim_{n\to\infty} \sqrt{n} I = 0,$$

which proves the lemma.

We shall prove

THEOREM 3.2: *If the conditions* (i), (ii), (iii) and (iv) *of Lemma* 3.2 *are satisfied, and the spectral function* $F(x)$ *of a continuous stationary process* $\mathcal{E}(t)$ *satisfies*

$$(3.8)\quad \int_{-\infty}^{\infty} |x|\, dF(x) < \infty,$$

then

$$(3.9)\quad E\left| n \int_{-\infty}^{\infty} X(t-s)K(ns)\, ds - X(t) \int_{-\infty}^{\infty} K(s)\, ds \right|^2 = o\left(\frac{1}{n}\right),$$

where $X(t) = f(t) + \mathcal{E}(t)$.

PROOF: We have

$$
\begin{aligned}
I &= n \int_{-\infty}^{\infty} X(t-s)K(ns)\,ds - X(t)n \int_{-\infty}^{\infty} K(ns)\,ds \\
&= n \int_{-\infty}^{\infty} [X(t-s) - X(t)]K(ns)\,ds \\
&= n \int_{-\infty}^{\infty} [\mathcal{E}(t-s) - \mathcal{E}(t)]K(ns)\,ds + n \int_{-\infty}^{\infty} [f(t-s) - f(t)]K(ns)\,ds \\
&= I_1 + I_2,
\end{aligned}
$$

say. Since $EI_1 = 0$, we have

$$E\,|\,I\,|^2 = E\,|\,I_1\,|^2 + |\,I_2\,|^2.$$

Lemma 3.2 shows $|\,I_2\,|^2 = o(1/n)$. Hence it is sufficient to show that

$$E\,|\,I_1\,|^2 = o(1/n) \tag{3.10}$$

We may now write

$$I_1 = n \int_{-\infty}^{\infty} \mathcal{E}(t-s)K(ns)\,ds - \int_{-\infty}^{\infty} \mathcal{E}(t-s)\,du(s) \cdot \int_{-\infty}^{\infty} K(s)\,ds$$

where $u(s) = 0$ for $s < 0$, $= 1$ for $s > 0$. $u(s)$ has the Fourier-Stieltjes transform identically equal to 1. Hence by Lemma 2.1 (2.3), we have

$$I_1 = \int_{-\infty}^{\infty} \mathcal{E}(t-s)\,d\left(n \int_{-\infty}^{s} K(n\xi)\,d\xi - u(s) \int_{-\infty}^{\infty} K(\xi)\,d\xi\right),$$

$$E\,|\,I_1|^2 = \int_{-\infty}^{\infty} \left| n \int_{-\infty}^{\infty} (e^{-ixs} - 1)K(ns)\,ds \right|^2 dF(x).$$

Minkowski's inequality shows

$$E\,|\,I_1|^2 \leqq \left(n \int_{-\infty}^{\infty} |\,K(ns)\,|\,ds \left(\int_{-\infty}^{\infty} |\,e^{-ixs} - 1\,|^2\,dF(x)\right)^{1|2}\right)^2,$$

which we write as

$$\left(n \int_{-\infty}^{\infty} |\,K(ns)\,| \left\{s^2 \int_{-G}^{G} |\,x\,|^2\,dF(x) + 2 \int_{|x|>G} |\,xs\,|\,dF(x)\right\}^{1/2} ds\right)^2,$$

G being an arbitrary positive number. This does not exceed

$$\left[n \int_{-\infty}^{\infty} |s|\,|K(ns)|\,ds \left(\int_{-G}^{G} |x|^2\,dF(x)\right)^{1/2} + 2^{1/2}n \int_{-\infty}^{\infty} |s|^{1/2}\,|K(ns)|\,ds \left(\int_{|x|>G} |x|\,dF(x)\right)^{1/2}\right]^2$$

$$= \left[\frac{1}{n} \int_{-\infty}^{\infty} |uK(u)|\,du \left(\int_{-G}^{G} |x|^2\,dF(x)\right)^{1/2} + 2^{1/2}\frac{1}{n^{1/2}} \int_{-\infty}^{\infty} |uK(u)|\,du \left(\int_{|x|>G} |x|\,dF(x)\right)^{1/2}\right]^2.$$

Hence we have

$$\limsup_{n\to\infty} nE\,|I_1|^2 = O\left(\int_{|x|>G} |x|\,dF(x)\right)$$

which proves (3.10), since G may be arbitrarily large. Thus the theorem is proved.

If further conditions are imposed on $f(t)$ and $F(x)$, then we can go further and get the asymptotic expression of $\int_{-\infty}^{\infty} X(t - s)\,dK(s)$. We shall leave this until another occasion.

4. Wiener's formula. Wiener was concerned with the formula

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} f(t)aK(at)\,dt = \int_{-\infty}^{\infty} K(x)\,dx \cdot \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,dt,$$

under suitable conditions on $f(t)$ and $K(t)$. We shall consider the similar formula concerning a stationary process. Let $X(t) = f(t) + \mathcal{E}(t)$ as in the preceding sections. It seems convenient first of all to state a remark.

It is known as the law of large numbers that $(1/2T) \int_{-T}^{T} \mathcal{E}(t)\,dt$ is convergent in mean as $T \to \infty$ and actually

$$\underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2T} \int_{-T}^{T} \mathcal{E}(t)e^{-i\xi t}\,dt = Z(\xi + 0) - Z(\xi - 0),$$

where ξ is any number and $Z(x)$ is the random spectral function. This is also well known (Doob [5]). Hence if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\xi t}\,dt = M_\xi$$

exists for some ξ, then

$$\text{(4.1)} \qquad \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2T} \int_{-T}^{T} X(t)e^{-i\xi t}\,dt = Z(\xi + 0) - Z(\xi - 0) + M_\xi.$$

Now we consider

$$\int_{-\infty}^{\infty} \mathcal{E}(t)e^{-i\xi t}aK(at)\,dt. \tag{4.2}$$

Then it is easy to show

THEOREM 4.1: *If $K(t) \varepsilon L_1(-\infty, \infty)$, then*

$$\underset{a\to 0}{\text{l.i.m.}} \int_{-\infty}^{\infty} \mathcal{E}(t)e^{-i\xi t}aK(at)\,dt = [Z(\xi+0) - Z(\xi-0)] \int_{-\infty}^{\infty} K(t)\,dt. \tag{4.3}$$

For putting the representation (1.3) into (4.2), assuming $\xi = 0$ without loss of generality, we have

$$\int_{-\infty}^{\infty} \mathcal{E}(t)aK(at)\,dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{\frac{ixt}{a}} K(t)\,dt\right) dZ(x),$$

and $\int_{-\infty}^{\infty} e^{ixt/a}k(t)\,dt$ tends to zero boundedly as $a \to 0$ when $x \neq 0$ by the Riemann-Lebesgue lemma and is $\int_{-\infty}^{\infty} K(t)\,dt$ when $x = 0 (a \neq 0)$. Here we used, the fact that if $\int_{-\infty}^{\infty} | g_a(x) - g(x)|^2 \,dF(x) \to 0$, then

$$\int_{-\infty}^{\infty} g_a(x)\,dZ(x) \to \int_{-\infty}^{\infty} g(x)\,dZ(x).$$

Now a Wiener-type formula of S. Bochner's states [1]: if

(i) $K(x)$ is absolutely continuous in every finite interval,

(4.4) (ii) $| x^2K(x) | < H$, $K(x) \varepsilon L_1(-\infty, \infty)$, H being a constant,

(4.5) (iii) $\frac{1}{2T}\int_{-T}^{T} | f(t) |\,dt \leqq G$, G being a constant, and

(iv) $M = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\,dt$ exists,

then

$$\lim_{a\to 0} \int_{-\infty}^{\infty} f(t)aK(at)\,dt = M \int_{-\infty}^{\infty} K(t)\,dt. \tag{4.6}$$

This fact and Theorem 4.1 show immediately that

$$\underset{a\to 0}{\text{l.i.m.}} \int_{-\infty}^{\infty} X(t)e^{-i\xi t}aK(at)\,dt = [M_\xi + Z(\xi+0) - Z(\xi-0)] \int_{-\infty}^{\infty} K(t)\,dt. \tag{4.7}$$

From (4.7) and (4.1) the following theorem follows immediately

THEOREM 4.2: *If conditions* (i), (ii) *and* (iii) *above are satisfied and*

$$\frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\xi t}\,dt$$

exists for some ξ, then

$$\underset{a\to 0}{\text{l.i.m.}} \int_{-\infty}^{\infty} X(t)e^{-i\xi t}aK(at)\,dt = \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2T}\int_{-T}^{T} X(t)e^{i\xi t}\,dt \int_{-\infty}^{\infty} K(t)\,dt. \tag{4.8}$$

Formula (4.8) means that the both sides exist and are equal.

5. Periodogram. Let $X(t) = f(t) + \mathcal{E}(t)$ as before. We suppose in this section that the spectral function $F(x)$ of $\mathcal{E}(t)$ is absolutely continuous and we denote the spectral density as $p(x)$:

$$\int_{-\infty}^{x} p(x)\,dx = F(x).$$

It is known and easily proved that

$$\lim_{T\to\infty} E\frac{1}{4\pi T}\left|\int_{-T}^{T} \mathcal{E}(t)e^{ixt}\,dt\right|^2 = p(x), \tag{5.1}$$

provided $p(x)$ is continuous at x.

Now we suppose that

$$\lim_{T\to\infty} \frac{1}{4\pi T^{1-\alpha}} \int_{-T}^{T} f(t)e^{-ixt}\,dt = M_{x,\alpha} \tag{5.2}$$

exists for some x and for some $0 \leqq \alpha < 1$.

Then we have

$$E\frac{1}{4\pi T}\left|\int_{-T}^{T} X(t)e^{-ixt}\,dt\right|^2 = \frac{1}{4\pi T}E\left|\int_{-T}^{T} \varepsilon(t)e^{-ixt}\,dt\right|^2 + \frac{1}{4\pi T}\left|\int_{-T}^{T} f(t)e^{-ixt}\,dt\right|^2.$$

Hence we get, letting $T \to \infty$,

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{4\pi T} E\left|\int_{-T}^{T} X(t)e^{-ixt}\,dt\right|^2 \\ &\qquad = p(x), &&\text{if } \alpha > \tfrac{1}{2}, \\ &\qquad = p(x) + M_{x,\frac{1}{2}}, &&\text{if } \alpha = \tfrac{1}{2}, \\ &\qquad = \infty, &&\text{if } 0 \leqq \alpha < \tfrac{1}{2} \text{ and } M_{x,\alpha} \neq 0. \end{aligned} \tag{5.3}$$

Now we consider the mean convergence of

$$\frac{1}{4\pi T}\left|\int_{-T}^{T} X(t)e^{-ixt}\,dt\right|^2 \tag{5.4}$$

when $T \to \infty$. We shall call (5.4) the periodogram of $X(t)$, mentioning that this is exactly the periodogram of $f(t)$ if $f(t)$ is a trigonometric polynomial and $\mathcal{E}(t) \equiv 0$. Many authors suggest (U. Grenander [6], U. Grenander and M. Rosenblatt [7, 8], Z. A. Lomnicki and S. K. Zaremba [11]) that (5.4) or

$$J(T) = \frac{1}{4\pi T}\left|\int_{-T}^{T} \mathcal{E}(t)e^{-ixt}\,dt\right|^2 \tag{5.5}$$

does not converge to $p(x)$.

We shall discuss in the following sections the behavior of (5.5) and prove that $J(t)$ does not converge in mean to any random variable when $\mathcal{E}(t)$ behaves like a stationary Gaussian process in a certain sense. In the case where $\mathcal{E}(t)$ is stationary Gaussian process U. Grenander and M. Rosenblatt gave extensive discussions (e.g. U. Grenander and M. Rosenblatt [8]).

6. Theorems on the periodogram. We shall impose further conditions on $\mathcal{E}(t)$. We suppose hereafter that $\mathcal{E}(t)$ is real, $E \mid \mathcal{E}(t) \mid^4 < \infty$ for every t, and

$$E\mathcal{E}(s)\mathcal{E}(s + u)\mathcal{E}(s + v)\mathcal{E}(s + w) = P(u, v, w), \tag{6.1}$$

is a function of u, v and w alone and independent of s; that is $\mathcal{E}(t)$ is a stationary process of the fourth order. Further let $P(u, v, w)$ be a continuous function of u, v and w in the whole range R_3.

Put

$$P(u, v, w) = Q(u, v, w) + P_G(u, v, w), \tag{6.2}$$

where

$$P_G(u, v, w) = \rho(u)\rho(v - w) + \rho(v)\rho(w - u) + \rho(w)\rho(u - v), \tag{6.3}$$

$\rho(u)$ being the covariance function (1.4) of $\mathcal{E}(t)$ as before. If $\mathcal{E}(t)$ is a Gaussian process, then $Q(u, v, w) \equiv 0$. Thus $Q(u, v, w)$ will be considered as a measure of non-Gaussianess and was introduced by Magness (T. A. Magness [12], see also E. Parzen [13]). We also assume that $Q(u, v, w)$ is the Fourier transform of a function $q(\xi', \eta', \zeta')$ which is integrable in R_3, bounded, continuous and satisfies the Lipschitz condition at a point $(-\xi, -\xi, \xi)$,

$$Q(u, v, w) = (1/2\pi)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(\xi', \eta', \zeta') e^{-i(u\xi' + v\eta' + w\zeta')} \, d\xi' \, d\eta' \, d\zeta' \tag{6.4}$$

Let $\mathcal{E}(t)$ have the spectral density $p(x)$, assumed to be continuous at $x = \xi$ and bounded.

Under these conditions, we shall prove the following theorem.

THEOREM 6.1:

$$J(T) = \frac{1}{4\pi T} \left| \int_{-T}^{T} \mathcal{E}(t) e^{-i\xi t} \, dt \right|^2$$

satisfies the limit relation

$$\lim_{T' > T \to \infty} \left\{ E \mid J(T) - J(T') \mid^2 - \left(1 - \frac{T}{T'}\right) 2p^2(\xi) \right\} = 0 \tag{6.5}$$

if $\xi \neq 0$, *and*

$$\lim_{T' > T \to \infty} \left\{ E \mid J(T) - J(T') \mid^2 - \left(1 - \frac{T}{T'}\right) \cdot 4p^2(0) \right\} = 0. \tag{6.6}$$

The theorem implies that $E \mid J(T) - J(T') \mid^2$ never converges; in other words $J(T)$ never converges in mean except at a point ξ where $p(\xi) = 0$.

As a theorem for the covariance of $J(T)$ we get under our assumptions above

THEOREM 6.2: *We have*

$$\lim_{T' > T \to \infty} \left\{ \mathrm{Cov}\,(J(T), J(T')) - \left(1 + \frac{2T}{T'}\right) p^2(\xi) \right\} = 0 \tag{6.7}$$

if $\xi \neq 0$, and

$$\lim_{T'>T\to\infty} \left\{\text{Cov}\,(J(T), J(T')) - \frac{2T}{T'} p^2(0)\right\} = 0. \tag{6.8}$$

This follows immediately from the fact that

$$\lim_{T'>T\to\infty} \left\{EJ(T)J(T') - \left(1 + \frac{T}{T'}\right) p^2(\xi)\right\} = 0, \tag{6.9}$$

if $\xi \neq 0$ and

$$\lim_{T'>T\to\infty} \left\{EJ(T)J(T') - \left(1 + \frac{2T}{T'}\right) p^2(0)\right\} = 0. \tag{6.10}$$

The proofs of above theorems will be done in Section 10.

7. Lemmas. It seems convenient to state lemmas in advance which will be used in the courses of proofs of the theorems.

LEMMA 7.1: *Let $p(x) \in L_1(-\infty, \infty)$ and be continuous. Then we have*

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} p(x) \frac{\sin T(x+\xi) \sin T(x-\xi)}{T(x+\xi)(x-\xi)}\, dx &= p(0), \quad \text{if} \quad \xi = 0 \\ &= 0, \qquad \text{if} \quad \xi \neq 0. \end{aligned} \tag{7.1}$$

The integral when $\xi = 0$ is the Fejér integral and the case $\xi = 0$ is very well known. The case $\xi \neq 0$ was proved by U. Grenander [6]. Some Fourier integral theorems involving the integral (7.1) and having a close connection with estimation theory of the spectral density of a stationary process were discussed by the author recently (T. Kawata [8]).

LEMMA 7.2: *Let $p(x) \in L_1(-\infty, \infty)$ and be continuous. Then we have*

$$\lim_{T'\geqq T\to\infty} \left\{\frac{1}{\pi\sqrt{TT'}} \int_{-\infty}^{\infty} p(x) \frac{\sin T(x+\xi) \sin T'(x+\xi)}{(x+\xi)^2}\, dx - p(\xi) \sqrt{\frac{T}{T'}}\right\} = 0 \tag{7.2}$$

and

$$\lim_{T'\geqq T\to\infty} \frac{1}{\pi} \frac{1}{\sqrt{TT'}} \int_{-\infty}^{\infty} p(x) \frac{\sin T(x+\xi) \sin T'(x-\xi)}{(x+\xi)(x-\xi)}\, dx = 0, \quad \text{if} \quad \xi \neq 0. \tag{7.3}$$

LEMMA 7.3: *Let δ be any positive number and let $S(\delta)$ be the domain $|x| < \delta$, $|y| < \delta$, $|z| < \delta$ in R_3, the three dimensional Euclidean space. Then*

$$\iiint_{R_3 - S(\delta)} \left| \frac{\sin T(x+y+z)}{x+y+z} \frac{\sin Tx}{x} \frac{\sin T'y}{y} \frac{\sin T'z}{z} \right| dx\, dy\, dz = O(T \log^2 T') \tag{7.4}$$

as T and T' tend to infinity in any way.

LEMMA 7.4: *If $\varphi(x, y, z)$ is bounded and satisfies*

$$|\varphi(x, y, z)| \leqq C(|x| + |y| + |z|)$$

for some constant C, near the origin, then

$$\int\int\int_{-\infty}^{\infty} \left| \varphi(x, y, z) \frac{\sin T(x + y + z)}{x + y + z} \frac{\sin Tx}{x} \frac{\sin T'y}{y} \frac{\sin T'z}{z} \right| dx\, dy\, dz \tag{7.5}$$
$$= O(\log^2 T \log T' + T \log^2 T')$$

as T and T' tend to infinity in such a way that $T' > T$.

We should like to add a remark. Lemma 7.4 suggests that we should have a convergence theorem like

$$\lim_{T\to\infty} C \int\int\int_{-\infty}^{\infty} f(x, y, z) \frac{\sin T(x + y + z) \sin Tx \sin Ty \sin Tz}{T(x + y + z)xyz}\, dx\, dy\, dz \tag{7.6}$$
$$= f(0, 0, 0).$$

In fact, under some conditions, (7.5) is true, and a more general theorem was proved by Bochner and the author [4].

We shall prove Lemma 7.2. Since

$$\frac{1}{\pi\sqrt{TT'}} \int_{-\infty}^{\infty} \frac{\sin T(x + \xi) \sin T'(x + \xi)}{(x + \xi)^2}\, dx$$
$$= \frac{1}{\pi\sqrt{TT'}} \int_{-\infty}^{\infty} \frac{\sin T w \sin T'w}{w^2}\, dw = \sqrt{\frac{T}{T'}}, \qquad \text{if} \quad T' \geqq T,$$

we have

$$\frac{1}{\pi\sqrt{TT'}} \int_{-\infty}^{\infty} p(x) \frac{\sin T(x + \xi) \sin T'(x + \xi)}{(x + \xi)^2}\, dx - p(\xi) \sqrt{\frac{T}{T'}}$$
$$= \frac{1}{\pi\sqrt{TT'}} \int_{-\infty}^{\infty} [p(w - \xi) - p(\xi)] \frac{\sin Tw \sin T'w}{w^2}\, dw \tag{7.7}$$
$$= \frac{1}{\pi\sqrt{TT'}} \int_{|w|>\delta} + \frac{1}{\pi\sqrt{TT'}} \int_{|w|<\delta},$$

where δ is a positive number such that for a given $\epsilon > 0$,

$$|p(w - \xi) - p(\xi)| < \epsilon, \qquad \text{for } |w| < \delta, \tag{7.8}$$

because of the continuity and evenness of $p(x)$. The first term of (7.7) converges to zero as $T, T' \to \infty$, and the second term is less than

$$\frac{\epsilon}{\pi\sqrt{TT'}} \int_{-\infty}^{\infty} \left| \frac{\sin Tw}{w} \frac{\sin T'w}{w} \right| dw$$
$$\leqq \frac{\epsilon}{\pi\sqrt{TT'}} \left(\int_{-\infty}^{\infty} \frac{\sin^2 Tw}{w}\, dw \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \frac{\sin^2 T'w}{w}\, dw \right)^{\frac{1}{2}} = \epsilon.$$

We shall next prove (7.3). We can easily prove, by the Parseval relation,

$$\frac{1}{\pi\sqrt{TT'}}\int_{-\infty}^{\infty}\frac{\sin T(x+\xi)\sin T'(x-\xi)}{(x+\xi)(x-\xi)}\,dx=\frac{\sin 2T\xi}{\sqrt{TT'}\xi}$$

if $T' \geqq T$, $\xi \gtrless 0$. Hence the left hand side of (7.3) becomes

$$\frac{1}{\pi\sqrt{TT'}}\int_{-\infty}^{\infty}[p(x)-p(\xi)]\,\frac{\sin T(x+\xi)\sin T'(x-\xi)}{(x+\xi)(x-\xi)}\,dx+p(\xi)\,\frac{\sin 2T\xi}{\pi\sqrt{TT'}\xi}$$

Dividing the first integral into three parts as

$$\int_{|x-\xi|<\delta}+\int_{|x+\xi|<\delta}+\int_{|x-\xi|>\delta,\,|x+\xi|>\delta},$$

δ being chosen so that $|\delta| < \xi$, and proceeding as the proof of Lemma (7.2), we can prove (7.3).

8. Proof of Lemma 7.3. We shall change the notation for simplicity. We write $T_0 = T$, $T_1 = T$, $T_2 = T'$, $T_3 = T'$ and x_1, x_2, x_3 for x, y, z respectively. Denote $D_0 = [x_i > \delta,\ i = 1, 2, 3]$. The integral in (7.4) is written as

$$I=\iiint_{R3-S(\delta)}\left|\frac{\sin(T_0\sum x_i)}{\sum x_i}\prod_{i=1}^{3}\frac{\sin T_i x_i}{x_i}\right|dv,$$

dv being a volume element in R_3, which we divide as

$$I=\iiint_{D_0}+\sum_i\iiint_{D_i}+\sum_{i\neq j}\iiint_{D_{ij}}=I_1+I_2+I_3, \tag{8.1}$$

say, D_i being the domain $D_0 - [\,|x_i| > \delta]$, and D_{ij} being

$$D_0-[\,|x_i|>\delta,\ |x_j|>\delta].$$

The first integral of the right hand side of (8.1) will be further divided into integrals of four types such as

$$\iiint_{x_1,x_2,x_3>\delta} \tag{8.2}$$

$$\iiint_{x_i,x_j>\delta;x_k<-\delta} \tag{8.3}$$

$$\iiint_{x_i>\delta;x_j,x_k<-\delta} \tag{8.4}$$

$$\iiint_{x_1,x_2,x_3<-\delta}, \tag{8.5}$$

where i, j, k are distinct. We shall estimate each of the integrals successively. First (8.2) is not greater than

$$\iiint_{x_1,x_2,x_3>\delta} \frac{dv}{x_1\, x_2\, x_3(x_1+x_2+x_3)}$$

$$(8.6)\quad \begin{aligned} &= \int_\delta^\infty \frac{dx_3}{x_3}\int_\delta^\infty \frac{dx_2}{x_2}\int_\delta^\infty \frac{dx_1}{x_1(x_1+x_2+x_3)} = \int_\delta^\infty \frac{dx_3}{x_3}\int_\delta^\infty \frac{\log[(\delta+x_2+x_3)/\delta]}{x_2(x_2+x_3)}\, dx_2 \\ &\leqq C\int_\delta^\infty \frac{dx_3}{x_3}\int_\delta^\infty \frac{\log\,(x_2+x_3)/\delta}{x_2(x_2+x_3)}\, dx_2 \leqq C\int_\delta^\infty \frac{dx_3}{x_3}\int_\delta^\infty \frac{|\log x_2| + \log}{x_2(x_2+x_3)} \\ &= 2C\int_\delta^\infty \frac{1}{x^2}\, |\log x|\, \log\frac{x+\delta}{\delta}\, dx \end{aligned}$$

which is finite. Here C is a constant $C(\delta)$ which may differ on each occurrence. Considering the integral of type (8.3), we shall have, for instance

$$\iiint_{x_1,x_2>\delta,x_3<-\delta}$$

which is

$$(8.7)\qquad \iiint_{x_1,x_2,x_3>\delta} \left| \frac{\sin T_0(x_1+x_2-x_3)}{x_1+x_2-x_3} \prod_{i=1}^{3} \frac{\sin T_i\, x_i}{x_i} \right| dv_x$$

The integrand of (8.7) does not exceed $1/(x_1x_2x_3\,|\,x_1+x_2-x_3\,|)$. If we integrate this over $x_1, x_2, x_3 > \delta$, $|\,x_1+x_2-x_3\,| > \delta/2$, then we see that it is not greater than

$$\int_0^\infty \frac{dx_1}{x_1}\int_\delta^\infty \frac{dx_2}{x_2}\frac{1}{(x_1+x_2)}\log\frac{x_1+x_2+\delta/2}{\delta/2} + \int_0^\infty \frac{dx_1}{x_1}\int_\delta^\infty \frac{dx_2}{x_2}\frac{1}{(x_1+x_2)}\log\frac{x_1+x_2-\delta/2}{\delta/2}$$

and it can be easily shown, as in the estimation of (8.6), that this is finite.

On the other hand the integrand of (8.7) over the domain $x_1, x_2, x_3 > \delta$, $|\,x_1+x_2-x_3\,| < \delta/2$ does not exceed $T(1/x_1x_2x_3)$, and the integral over the domain can be proved to be $\leqq CT$. Hence it has been shown that (8.2) $= O(T)$.

The integrals of type (8.3) will be $O(T)$, which is also shown easily. Each of the integrals of type (8.4) is just the same as the corresponding integral in (8.3) and the integral (8.4) is the same one as (8.2). Hence we get

$$(8.8)\qquad I_1 = O(T)$$

in (8.1).

Next we shall consider I_2 in the right hand side of (8.1). For instance

$$(8.9)\quad \begin{aligned} \iiint_{D_1} &= \iiint_{|x_1|\leqq\delta,|x_2|,|x_3|>\delta} \left| \frac{\sin T_0 \sum x_i}{\sum x_i} \prod_{i=1}^{3} \frac{\sin T_i\, x_i}{x_i} \right| dv_x \\ &= \iiint_{D_{11}} + \iiint_{D_{12}} = I_{21} + I_{22}, \end{aligned}$$

say, where D_{11} is the domain $[\,|x_1| \leqq \delta, |x_2| > \delta, |x_3| > \delta, |x_1 + x_2 + x_3| > \delta]$ and $D_{12} = [\,|x_1| \leqq \delta, |x_2| > \delta, |x_3| > \delta, |x_1 + x_2 + x_3| < \delta]$.

If we write

$$I_{21} = \iiint_{D_{11}=[|x_1|<\delta/2]} + \iiint_{D_{11}[\delta/2<x_1<\delta]}, \tag{8.10}$$

then the latter is found to be bounded as in the arguments in the first step of the estimation of (8.7). The first integral does not exceed

$$\int_{|x_1|<\delta/2} \left|\frac{\sin Tx_1}{x_1}\right| dx_1 \int_{|x_2|>\delta} \frac{dx_2}{|x_2|} \int_{|x_3|>\delta, |x_1+x_2+x_3|>\delta} \frac{dx_3}{|x_3(x_1+x_2+x_3)|}$$
$$\leqq 2\int_{|x_1|<\delta/2} \left|\frac{\sin Tx_1}{x_1}\right| dx_1 \int_{|x_2|>\delta} \frac{dx_2}{|x_2|} \int_{|x_3|>\delta, |x_2+x_3|>\delta/2} \frac{dx_3}{|x_3(x_2+x_3)|}$$

since $|x_1 + x_2 + x_3| > \delta$, $|x_1| < \delta/2$ implies $|x_2 + x_3| > \delta/2$. The last integral is not greater than

$$2\int_{|x_2|>\delta} \frac{dx_2}{|x_2|} \int_{|x_3|>\delta, |x_2+x_3|>\delta/2} \frac{dx_3}{|x_3(x_2+x_3)|} \cdot \int_{|x_1|<\delta/2} \left|\frac{\sin Tx_1}{x_1}\right| dx_1$$

in which the first factor is $O(1)$ as in the evaluation of (8.7) and the second integral is $O(\log T)$ as is known since it is the Lebesgue constant. Thus

$$I_{21} = O(\log T). \tag{8.11}$$

Next I_{22} will be computed, being written as

$$I_{22} = \iiint_{D_{11}\cdot[|x_1+x_2+x_3|<1/T]} + \iiint_{D_{11}\cdot[\delta>|x_1+x_2+x_3|>1/T]} \tag{8.12}$$

We consider the integral over the domain D_{111}: $|x_1| < \delta/2$, $|x_2| > \delta$, $|x_3| > \delta$, $|x_1 + x_2 + x_3| < 1/T$,

$$\iiint_{D_{111}} \tag{8.13}$$

in place of the first integral in the right side of (8.12). The remaining part of the integral can be estimated in the same way as was done in I_1, to be $O(T)$.

Thus we have

$$\iiint_{D_{111}} \leqq T\iiint_{D_{111}} \left|\prod_{i=1}^{3} \frac{\sin T_i x_i}{x_i}\right| dv_x$$
$$\leqq T\int_{|x_1|<\delta/2} \left|\frac{\sin Tx_1}{x_1}\right| dx_1 \iint_{\substack{|x_2|,|x_3|>\delta \\ |x_1+x_2+x_3|<1/T}} \left|\frac{\sin T'x_2}{x_2}\frac{\sin T'x_3}{x_3}\right| dx_2\, dx_3$$
$$= 2T\int_{|x_1|<\delta/2} \left|\frac{\sin Tx_1}{x_1}\right| dx_1 \iint_{\substack{x_2>\delta, x_3<-\delta \\ |x_1+x_2+x_3|<1/T}} \left|\frac{\sin T'x_2}{x_2}\frac{\sin T'x_3}{x_3}\right| dx_2\, dx_3$$

since $x_2 > \delta$, $x_3 > \delta$, $| x_1 + x_2 + x_3 | < 1/T$ is impossible for large T. The last integral does not exceed

$$2T \int_{|x_1|<\delta/2} \left| \frac{\sin Tx_1}{x_1} \right| dx_1 \int_\delta^\infty \frac{dx_2}{x_2} \int_{-x_2-x_1-1/T} \frac{dx_3}{x_3}$$

$$= 2T \int_{|x_1|<\delta/2} \left| \frac{\sin Tx_1}{x_1} \right| dx_1 \int_\delta^\infty \frac{1}{x_1} \log \left(1 + \frac{2}{T} \frac{1}{x_2 + x_1 - 1/T}\right) dx_2$$

$$\leqq CT \int_{|x_1|<\delta/2} \left| \frac{\sin Tx_1}{x_1} \right| dx_1 \frac{1}{T} \int_\delta^\infty \frac{dx_2}{x_2(x_2 - \delta - 1/T)}$$

$$\leqq C \int_{|x_1|<\delta/2} \left| \frac{\sin Tx_1}{x_1} \right| dx_1 = O(\log T).$$

Hence we get that the first integral of the right hand side of I_{22} is $O(T)$.

We next consider the second integral of I_{22} in (8.12), which is not greater than

$$(8.14) \qquad 2 \int_{|x_1|<\delta/2} \left| \frac{\sin Tx_1}{x_1} \right| dx_1 \iint_{\substack{x_2>\delta, x_3<-\delta \\ 1/T<|x_1+x_2+x_3|<\delta}} \left| \frac{1}{(x_1 + x_2 + x_3)x_2\, x_3} \right| dx_2\, dx_3 .$$

The inner integral, by a change of a variable, becomes

$$\iint_{\substack{x_2, x_3>\delta \\ 1/T<|x_1+x_2-x_3|<\delta}} \frac{dx_2\, dx_3}{| x_1 + x_2 - x_3 |\, x_2\, x_3}$$

which is not greater than the sum

$$\int_\delta^\infty \frac{dx_2}{x_2} \int_{x_1+x_2+1/T}^\infty \frac{dx_3}{(x_3 - x_1 - x_2)x_3} + \int_\delta^\infty \frac{dx_2}{x_2} \int_{x_1+x_2-1/T>x_3} \frac{dx_3}{(x_1 + x_2 - x_3)x_3}$$

$$\leqq C \int_{x_1+x_2>\delta, x_2>\delta} \frac{1}{x_2(x_1 + x_2)} \log T(x_1 + x_2)\, dx_2 .$$

This is easily proved to be $O(\log T)$ and hence (8.12) is $O(\log^2 T)$, Lebesgue's constant being involved. Hence we get

$$I_{22} = O(T) + O(\log^2 T) = O(T).$$

Inserting this result and (8.10) into (8.9), we have shown $\iiint_{D_1} = O(T)$. Similar arguments show that

$$\iiint_{D_2} = O(T) + O(\log T \log T')$$

and

$$\iiint_{D_3} = O(T) + O(\log T \log T').$$

The domain D_2 and D_3 are defined analogously to D_1 and the above estimates are easily verified. Combining these results, we get

$$I_2 = O(T \log T'). \tag{8.14}$$

Lastly we shall consider I_3. We shall treat, for instance, the integral over D_{12}, that is,

$$\begin{aligned} \iiint_{D_{12}} &= \iiint_{|x_1|<\delta, |x_2|<\delta, |x_3|>\delta} \left| \frac{\sin T(\sum x_i)}{\sum x_i} \prod_{i=1}^{3} \frac{\sin T_i x_i}{x_i} \right| dx_1\, dx_2\, dx_3 \\ &= \iiint_{|x_1|<\delta, |x_2|<\delta, |x_3|>3\delta} + \iiint_{|x_1|<\delta, |x_2|<\delta, \delta<|x_3|<3\delta} \end{aligned} \tag{8.15}$$

Replacing $|x_1 + x_2 + x_3|$ by $\frac{1}{2}|x_3|$, because of

$$|x_1 + x_2 + x_3| > |x_3| - |x_1| - |x_2| > \tfrac{1}{2}|x_3|,$$

we see that the first integral does not exceed

$$2\int_{|x_1|<\delta} \left| \frac{\sin x_1 T}{x_1} \right| dx_1 \int_{|x_2|<\delta} \left| \frac{\sin x_2 T'}{x_2} \right| dx_2 \int_{|x_3|>3\delta} dx_3$$

This is clearly

$$O(\log T \log T'). \tag{8.16}$$

The latter integral of (8.15) is not greater than

$$\begin{aligned} &\frac{1}{\delta}\int_{|x_1|<\delta} \left| \frac{\sin Tx_1}{x_1} \right| dx_1 \int_{|x_2|<\delta} \left| \frac{\sin T'x_2}{x_2} \right| dx_2 \int_{\delta<|x_3|<3\delta} \left| \frac{\sin T(x_1+x_2+x_3)}{x_1+x_2+x_3} \right| dx_3 \\ &\qquad \leqq \frac{1}{\delta}\int_{|x_1|<\delta} \left| \frac{\sin Tx_1}{x_1} \right| dx_1 \int_{|x_2|<\delta} \left| \frac{\sin T'x_2}{x_2} \right| dx_2 \int_{|u|<5\delta} \left| \frac{\sin Tu}{u} \right| du \\ &\qquad = O(\log^2 T \cdot \log T'). \end{aligned}$$

We have thus reached $\iiint_{D_{12}} = O(\log^2 T \log T')$. The other integrals in I_3 may be shown by similar arguments to be $O(\log^2 T \log T')$ or $O(\log^2 T' \cdot \log T)$. Hence, combining these results, we get

$$I_3 = O(\log^2 T \log T' + \log^2 T' \cdot \log T). \tag{8.17}$$

By (8.8), (8.14) and (8.17), we finally get $I = O(T \log^2 T')$.

9. Proof of Lemma 7.4. Let

$$|\varphi(x, y, z)| \leqq C(|x| + |y| + |z|) \tag{9.1}$$

in $S(\delta): |x|, |y|, |z| < \delta$. Let M be the upper bound of $\varphi(x, y, z)$. We partition the integral as

$$\iiint_{-\infty}^{\infty} \left| \varphi(x, y, z) \frac{\sin T(x + y + z)}{x + y + z} \frac{\sin Tx}{x} \frac{\sin T'y}{y} \frac{\sin T'z}{z} \right| dx\, dy\, dz$$

$$= \iiint_{S(\delta)} + \iiint_{R_3 - S(\delta)} = J_1 + J_2,$$

say. Then

$$\begin{aligned} |J_2| &\leq M \iiint_{R_3 - S(\delta)} \left| \frac{\sin T(x + y + z)}{x + y + z} \frac{\sin Tx}{x} \frac{\sin T'y}{y} \frac{\sin T'z}{z} \right| dx\, dy\, dz \\ &= O(T \log^2 T') \end{aligned} \tag{9.2}$$

by Lemma 7.3.

Next, inserting the relation (9.1), we have

$$\begin{aligned} |J_1| \leq C \iiint_{S(\delta)} (|x| + |y| + |z|) &\left| \frac{\sin T(x + y + z)}{x + y + z} \right| \\ &\cdot \left| \frac{\sin Tx}{x} \frac{\sin T'y}{y} \frac{\sin T'z}{z} \right| dx\, dy\, dz \Bigg). \end{aligned} \tag{9.3}$$

We consider, for instance, the following part of this integral:

$$\begin{aligned} &\iiint_{S(\delta)} |y| \left| \frac{\sin T(x + y + z)}{x + y + z} \frac{\sin Tx}{x} \frac{\sin T'y}{y} \frac{\sin T'z}{z} \right| dx\, dy\, dz \\ &\leq \iiint_{S(\delta)} \left| \frac{\sin T(x + y + z)}{x + y + z} \frac{\sin Tx}{x} \frac{\sin T'z}{z} \right| dx\, dy\, dz \\ &= \int_{-\delta}^{\delta} \left| \frac{\sin Tx}{x} \right| dx \int_{-\delta}^{\delta} \left| \frac{\sin T'z}{z} \right| dz \int_{-\delta}^{\delta} \left| \frac{\sin T(x + y + z)}{x + y + z} \right| dy \\ &\leq \int_{-\delta}^{\delta} \left| \frac{\sin Tx}{x} \right| dx \int_{-\delta}^{\delta} \left| \frac{\sin T'z}{z} \right| dz \int_{-3\delta}^{3\delta} \left| \frac{\sin Tu}{u} \right| du \\ &= O\,(\log^2 T \log T'). \end{aligned}$$

Other parts in (9.3) may be estimated by similar arguments to be

$$O(\log T \log^2 T') \qquad \text{or} \qquad O(\log^2 T \log T').$$

Keeping (9.2) in mind, we get the proof of Lemma 7.4.

10. Proofs of theorems. We are now in a position to prove the theorems in Section 6. Throughout this section we of course assume all conditions stated in these.

First of all we shall evaluate $EJ^2(T)$.

$$EJ^2(T) = \frac{1}{16\pi^2} \frac{1}{T^2} E \iiiint_{-T}^{T} \mathcal{E}(s)\mathcal{E}(t)\mathcal{E}(u)\mathcal{E}(v) \cdot e^{-i(s-t)\xi + i(u-v)\xi}\, ds\, dt\, du\, dv.$$

Inserting (6.2) here, we have

$$EJ^2(T) = \frac{1}{16\pi^2}\frac{1}{T^2}\iiiint_{-T}^{T} Q(t-s, u-s, v-s) \cdot e^{-i\{(s-t)-(u-v)\}\xi}\, ds\, dt\, du\, dv \tag{10.1}$$

$$+ \frac{1}{16\pi^2}\frac{1}{T^2}\iiiint_{-T}^{T} P_G(t-s, u-s, v-s) e^{-i\{(s-t)-(u-v)\}\xi}\, ds\, dt\, du\, dv$$

say. Because of (6.4), we have

$$J_1 = \frac{1}{16\pi^2 T^2}\left(\frac{1}{2\pi}\right)^{3/2}\iiint_{-\infty}^{\infty} q(x, y, z)\, dx\, dy\, dz \cdot \iiiint_{-T}^{T} e^{-is(x+y+z+\xi)} \cdot e^{it(x+\xi)+iu(y+\xi)+iv(z-\xi)}\, ds\, dt\, du\, dv$$

$$= \left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{\pi^2 T^2}\iiiint_{-\infty}^{\infty} q(x, y, z)\frac{\sin T(x+y+z+\xi)}{x+y+z+\xi} \cdot \frac{\sin T(x+\xi)}{x+\xi} \cdot \frac{\sin T(y+\xi)}{y+\xi} \cdot \frac{\sin T(z-\xi)}{z-\xi}\, dx\, dy\, dz \tag{10.2}$$

$$= \left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{\pi^2 T^2}\iiint_{-\infty}^{\infty} q(x-\xi, y-\xi, z+\xi)\frac{\sin T(x+y+z)}{x+y+z} \cdot \frac{\sin Tx}{x}\frac{\sin Ty}{y}\frac{\sin Tz}{z}\, dx\, dy\, dz.$$

Now the integral (10.2) with $q(x, y, z) \equiv 1$ is easily shown to be $\pi^3 T$ by the repeated applications of Parseval's relation. Hence we have

$$J_1 = \left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{\pi^2 T^2}\iiint_{-\infty}^{\infty} \{q(x-\xi, y-\xi, z+\xi) - q(-\xi, -\xi, \xi)\} \cdot \frac{\sin T(x+y+z)}{x+y+z}\frac{\sin Tx}{x}\frac{\sin Ty}{y}\frac{\sin Tz}{z}\, dx\, dy\, dz + g(-\xi, -\xi, \xi)\left(\frac{1}{2\pi}\right)^{3/2}\frac{\pi}{T} \tag{10.3}$$

Since $q(x, y, z)$ satisfies the Lipschtz condition at the point $(-\xi, -\xi, \xi)$, we have, by Lemma 7.4 with $T = T'$,

$$J_1 = O\left(\frac{1}{T}\log^2 T\right) + O\left(\frac{1}{T}\right) = o(1). \tag{10.4}$$

Next

$$J_2 = \frac{1}{16\pi^2}\frac{1}{T^2}\iiint\int_{-T}^{T} \cdot\{\rho(t-s)\rho(u-v) + \rho(u-s)\rho(v-t) + \rho(v-s)\rho(t-u)\} \cdot e^{-i\{(s-t)-(u-v)\}}\, ds\, dt\, du\, dv. \tag{10.5}$$

Here we shall have, for instance,

$$\frac{1}{16\pi^2}\frac{1}{T^2}\iiint\int_{-T}^{T} \rho(t-s)\rho(u-v)e^{-i\{(s-t)-(u-v)\}\xi}\, ds\, dt\, du\, dv \to p^2(\xi), \quad \text{as} \quad T\to\infty, \tag{10.6}$$

because this will become, inserting $\rho(u) = \int_{-\infty}^{\infty} p(x)e^{iux}\,dx$, and making a change of orders of integration,

$$\frac{1}{16\pi^2}\frac{1}{T^2}\int_{-\infty}^{\infty} p(x)\,dx \int_{-\infty}^{\infty} p(y)\,dy \iiint\int_{-T}^{T} e^{i(t-s)(x+\xi)+i(u-v)(y+\xi)}\,dt\,ds\,du\,dv$$

$$= \int_{-\infty}^{\infty} p(x)\,\frac{\sin^2 T(x+\xi)}{\pi T(x+\xi)^2}\,dx \cdot \int_{-\infty}^{\infty} p(y)\,\frac{\sin^2 T(y+\xi)}{\pi T(y+\xi)^2}\,dy$$

which tends obviously to $p^2(-\xi) = p^2(\xi)$ as $T\to\infty$ by the well known property of Fejer's integral.

As for the second part of the integral of the right hand side of (10.5), we obtain

$$\frac{1}{16\pi^2 T^2}\iiint\int_{-T}^{T} \rho(u-s)\rho(v-t)e^{-i\{(s-t)-(u-v)\}}\, ds\, dt\, du\, dv \to p^2(\xi), \quad \text{as} \quad T\to\infty. \tag{10.7}$$

However we find a difference in considering the remaining part. In fact the integral

$$\frac{1}{16\pi^2 T^2}\iiint\int_{-\infty}^{\infty} \rho(v-s)\rho(t-u)e^{-i\{(s-t)-(u-v)\}}\, ds\, dt\, du\, dv$$

becomes, after similar treatments,

$$\left[\int_{-\infty}^{\infty} p(x)\,\frac{\sin T(x+\xi)\sin T(x-\xi)}{\pi T(x+\xi)(x-\xi)}\,dx\right]^2,$$

which converges to $p^2(0)$ if $\xi = 0$, and 0 if $\xi \neq 0$, by Lemma 7.1. Hence combining this result with (10.6), (10.7), we get

$$\lim_{T\to\infty} J_2 = 3\,p^2(0), \qquad \text{if } \xi = 0,$$

$$= 2\,p^2(\xi), \qquad \text{if } \xi \neq 0.$$

We finally get

$$(10.8)\qquad \lim_{T\to\infty} EJ^2(T) = 3\,p^2(0), \qquad \text{if } \xi = 0,$$
$$= 2\,p^2(\xi), \qquad \text{if } \xi \neq 0.$$

Moreover we shall find out here the limit value of $EJ(T)J(T')$. The same method as above leads us to

$$(10.9)\qquad \begin{aligned} &\frac{1}{16\pi^2 TT'} E \int\!\!\int_{-T}^{T} \mathcal{E}(s)\mathcal{E}(t)e^{-i(s-t)\xi}\,ds\,dt \int\!\!\int_{-T}^{T} \mathcal{E}(u)\mathcal{E}(v)e^{i(u-v)\xi}\,du\,dv \\ &= \frac{1}{16\pi^2 TT'} \int\!\!\int_{-T}^{T} ds\,dt \int\!\!\int_{-T'}^{T'} du\,dv\cdot Q(t-s, u-v, v-s)e^{-i[(s-t)-(u-v)]\xi} \\ &+ \frac{1}{16\pi^2 TT'} \int\!\!\int_{-T}^{T} ds\,dt \int\!\!\int_{-T'}^{T'} du\,dv P_G(t-s, u-s, v-s)e^{-i\{(s-t)-(u-v)\}\xi} \\ &= K_1(T, T') + K_2(T, T') \end{aligned}$$

say. By the same way we got (10.2), we shall have

$$K_1(T, T') = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\pi^2}\frac{1}{TT'} \int\!\!\int\!\!\int_{-\infty}^{\infty} q(x-\xi, y-\xi, z+\xi) \cdot \frac{\sin T(x+y+z)}{x+y+z}\,\frac{\sin Tx}{x}\,\frac{\sin T'y}{y}\,\frac{\sin T'z}{z}\,dx\,dy\,dz.$$

Now the Parseval relation proves $K_1(T, T')$ with $q(x, y, z) \equiv 1$, to be

$$(10.10)\qquad \int\!\!\int\!\!\int_{-\infty}^{\infty} \frac{\sin T(x, y, z)}{x+y+z}\,\frac{\sin Tx}{x}\,\frac{\sin T'y}{y}\,\frac{\sin T'z}{z}\,dx\,dy\,dz = \pi^3 T,$$

if $T' > T$. Hence we have

$$\begin{aligned} K_1(T, T') = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\pi^2}\frac{1}{TT'} \int\!\!\int\!\!\int_{-\infty}^{\infty} &\{q(x-\xi, y-\xi, z+\xi) - q(-\xi, -\xi, \xi)\} \\ &\cdot \frac{\sin T(x+y+z)}{x+y+z}\,\frac{\sin Tx}{x}\,\frac{\sin T'y}{y}\,\frac{\sin T'z}{z}\,dx\,dy\,dz \\ &+ \left(\frac{1}{2\pi}\right)^{3/2} \frac{\pi}{T'}\,q(-\xi, -\xi, \xi) \end{aligned}$$

which is, by Lemma 2,

$$O\left(\frac{1}{TT'}(\log^2 T \log T' + T \log^2 T') + \frac{1}{T'}\right).$$

Hence

$$(10.11)\qquad K_1(T, T') = o(1), \qquad \text{as } T' \geqq T \to \infty.$$

Next $K_2(T, T')$ will become after some arguments

$$\frac{1}{16\pi^2}\frac{1}{TT'}\iint_{-T}^{T} ds\, dt \iint_{-T'}^{T'} du\, dv \rho(t-s)\rho(u-v)e^{-i\{(s-t)-(u-v)\}\xi}$$

$$+ \iint_{-T}^{T} ds\, dt \iint_{-T'}^{T'} du\, dv \rho(u-s)\rho(v-t)e^{-i\{(s-t)-(u-v)\}\xi}$$

$$+ \iint_{-T}^{T} ds\, dt \iint_{-T'}^{T'} du\, dv \rho(v-s)\rho(t-u)e^{-i[(s-t)-(u-v)]\xi}$$

$$= K_{21} + K_{21} + K_{23},$$

say. It will be seen that the asymptotic behaviors are much different among K_{21}, K_{22} and K_{23}.

In fact K_{21} becomes, by an argument like that used in considering (10,6),

$$\int_{-\infty}^{\infty} \frac{\sin^2 T(x+\xi)}{\pi T(x+\xi)^2} p(x)\, dx \int_{-\infty}^{\infty} \frac{\sin^2 T'(y-\xi)}{\pi T'(y-\xi)^2} p(y)\, dy,$$

which tends to $p(\xi)p(-\xi) = p^2(\xi)$ as $T, T' \to \infty$. Thus

$$\lim_{T' \geqq T \to \infty} K_{21}(T, T') = p^2(\xi). \tag{10.12}$$

Next we see that $K_{22}(T, T')$ will become

$$\frac{1}{\pi^2}\frac{1}{TT'}\int_{-\infty}^{\infty} \frac{\sin T(x+\xi)\sin T'(y+\xi)}{(x+\xi)^2} p(x)\, dx$$
$$\cdot \int_{-\infty}^{\infty} \frac{\sin T(y-\xi)\sin T'(y-\xi)}{(y-\xi)^2} p(y)\, dy.$$

Then Lemma 7.2, (7.2) shows

$$\lim_{T' \geqq T \to \infty} \left\{K_{22}(T, T') - p^2(\xi)\frac{T}{T'}\right\} = 0. \tag{10.13}$$

Finally K_{23} becomes, after some arguments,

$$\frac{1}{\pi^2}\frac{1}{TT'}\left[\int_{-\infty}^{\infty} \frac{\sin (x+\xi)T \sin T'(x-\xi)}{(x+\xi)(x-\xi)} p(x)\, dx\right]^2$$

which converges to 0 if $\xi \neq 0$, by Lemma 7.2, (7.3). On the other hand if $\xi = 0$. then it reduces to K_{22} and

$$\lim_{T' \geqq T \to \infty} \left\{K_{22}(T, T') - p^2(0)\frac{T}{T'}\right\} = 0.$$

Combining (10.12), (10.13) and the last result, we have

$$\lim_{T' \geqq T \to \infty} \left\{ K_2(T, T') - \left(1 + \frac{T}{T'}\right) p^2(\xi) \right\} = 0, \qquad \text{if } \xi \neq 0, \tag{10.14}$$

$$\lim_{T' \geqq T \to \infty} \left\{ K_2(T, T^1) - \left(1 + \frac{2T}{T'}\right) p^2(0) \right\} = 0. \tag{10.15}$$

Hence putting (10.11), (10.14) and (10.15) into (10.9) we get (6.9) and (6.10).

After these preparations, the proofs of the theorems are now very easy. For

$$\begin{aligned} E\{J(T) - J(T')\}^2 - \left(1 - \frac{T}{T'}\right) 2p^2(\xi) \\ = (EJ^2(T) - 2p^2(\xi)) + (EJ^2(T') - 2p^2(\xi)) \\ - 2\left(EJ(T)J(T') - \left(1 - \frac{T}{T'}\right) p^2(\xi)\right), \end{aligned}$$

which tends to zero by virtue of (10.8) and (6.9). Formula (6.6) is also proved using (10.8) and (6.10).

The proof of Theorem 6.2 is also immediate.

REFERENCES

[1] S. BOCHNER, *Vorlesungen über Fourierintegrale*, Teubner, Leipzig, 1932.

[2] S. BOCHNER AND K. CHANDRASEKHARAN, *Fourier Transforms*, Princeton University Press, 1949.

[3] S. BOCHNER AND S. IZUMI, "Some general convergence theorems," *Tohoku Math. J.* Vol. 42 (1936), pp. 191–194.

[4] S. BOCHNER AND T. KAWATA, "A special integral transformation in Euclidean space," *Ann. Mathematics*, Vol. 68 (1958), pp. 150–158.

[5] J. L. DOOB, *Stochastic Processes*, John Wiley, New York, 1952.

[6] U. GRENANDER, "On empirical spectral analysis of stochastic processes," *Ark. Mat.* Vol. 1 (1951), pp. 503–531.

[7] U. GRENANDER AND M. ROSENBLATT, "Statistical spectral analysis of time series arising from stationary stochastic processes," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 537–558.

[8] U. GRENANDER AND M. ROSENBLATT, *Statistical Analysis of Stationary Time Series*, John Wiley, New York, 1957.

[9] T. KAWATA, "Some Fourier integral theorems," to be published in *Kodai Math. Sem. Rep.*

[10] M. LOÈVE, *Fonctions Aléatoires du Second Ordre*, Supplement to P. Levy, *Processus Stochastiques et Mouvement Brownien*, Gauthier, Paris, 1948.

[11] Z. A. LOMNICKI AND S. K. ZAREMBA, "On the estimation of autocorrelation," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 140–158.

[12] T. A. MAGNESS, "Spectral response of a quadratic device to non-Gaussian noise," *J. Appl. Physics*, Vol. 25 (1954), pp. 1357–1365.

[13] E. PARZEN, "On consistent estimates of the spectrum of a stationary time series," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 329–348.