GAMES ASSOCIATED WITH A RENEWAL PROCESS

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1. Introduction. Consider a sequence of occurrences of a recurrent event $\mathcal{E}$ for which the intervals, $X_1, X_2, \cdots$, are independent identically distributed non-negative random variables (a renewal process) with common cdf (cumulative distribution function) $F(x)$. Robbins [3] considered games when X_1 is an integer-valued random variable. It seems of interest to extend his results to games when X_1 is not necessarily integer-valued. Thus, for example, $\{X_i\}$ may denote the lifetimes of similar articles, or the times between accidents of automobiles insured by a certain company. We will also consider games associated with *a general renewal process* where $\{X_i\}$ is preceded by another random variable X_0 which is independent of $\{X_i\}$ and may have a different distribution. Since the discrete case has been fully dealt with by Feller [1] and Robbins [3], the emphasis in this paper will be on the continuous case. However, the results will be presented in a general form which will include all such $F(x)$ *which do not have a jump at zero.*

2. Fixed-time games. Let us consider the following game called $\mathcal{G}$. The game starts at $t = 0$ when an event has just occurred. At the kth occurrence of the event $\mathcal{E}$, player A *receives* an amount $c(X_k)$ and *pays* an amount a_k. Here $c(t)$ is a given function which vanishes for $t < 0$, and a_i is a sequence of constants. For example, A may be a buyer of certain articles. The benefit which A derives from the kth article is a function $c(X_k)$ of its lifetime, X_k, and he pays the price a_k for the purchase. An insurance company A pays an amount a_k at the occurrence of kth death (life insurance) or kth accident (automobile or other accident insurance) and receives premiums and interests which are a function of time between such occurrences.

Let

$$\begin{aligned} T(t) &= \text{total amount received by } A \text{ in } (0, t]; \\ T_i(t) &= \text{total amount received by } A \text{ in } (X_i, X_i + t]; \qquad (1) \\ U(t) &= \text{total amount paid by } A \text{ in } (0, t]. \end{aligned}$$

Obviously the $T_i(t)$ have the same distribution as $T(t)$. If $ET(t) = EU(t)$ for all $0 \leqq t \leqq t_0$, we shall say that $\mathcal{G}$ is fair for $[0, t_0]$; if $t_0 = \infty$, we shall say that $\mathcal{G}$ is *fair*.

To avoid triviality we shall assume that $c(t)$ is not a null function. $c(t)$ will be called *realistic* if it is non-negative and finite for finite t. The game $\mathcal{G}$ will be called realizable if, given a realistic $c(t)$, a sequence $\{a_i\}$ of non-negative constants exists for which $\mathcal{G}$ is fair.

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If $f(t)$ is a function of the real variable t defined on the interval $0 \leqq t < \infty$, and of bounded variation in the interval $a \leqq t \leqq b$ for every positive a and every positive b, write

$$(2) \qquad f^*(s) = \int_{0+}^{\infty} e^{-st}\, df(t) = \lim_{\substack{a \to 0 \\ b \to \infty}} \int_a^b e^{-st}\, df(t)$$

when this limit exists. $f^*(s)$ will be called the Laplace-Stieltjes transform (LST) [5, ch: II] of $f(t)$.

Define

$$G(t) = \int_0^t c(x)\, dF(x), \qquad \text{if } t \geqq 0; \quad 0, \quad \text{if } t < 0;$$

and write $\bar{T}(t) = ET(t)$, and so on. Assuming that $G^*(s)$ exists for some $s = s_0$, where $s_0 = \sigma_0 + i\tau_0$, by arguments similar to those of Robbins [3, p. 190], we find

$$(3) \qquad \bar{T}^*(s) = G^*(s)/[1 - F^*(s)].$$

$\bar{T}^*(s)$ is analytic in the half plane $\sigma > c = \max(0, \sigma_0)$.

Similarly, formally introducing the series

$$(4) \qquad A(y) = \sum_{n=1}^{\infty} a_n y^n,$$

we obtain

$$(5) \qquad \bar{U}^*(s) = A(F^*(s)).$$

Hence the following theorem.

THEOREM 1. *$\mathcal{G}$ is fair if $G^*(s_0)$ exists, $A(y)$ converges for $|y| < F^*(c)$, and*

$$(6) \qquad A(F^*(s)) = G^*(s)/[1 - F^*(s)].$$

Further, if $\mathcal{G}$ is fair and $G^*(s_0)$ exists, then $A(y)$ converges for $|y| < F^*(c)$ and (6) holds.

Since, by assumption, $F(x)$ does not have a jump at zero, for real s, $F^*(s)$ is a monotone decreasing function on $[0, \infty]$ to $[0, 1]$. It therefore has a unique inverse $F^{*^{-1}}$ on $[0, 1]$ to $[0, \infty]$ which is a monotone decreasing function. Let

$$(7) \qquad K(y) = G^*(F^{*^{-1}}(y)), \qquad 0 \leqq y \leqq F^*(c).$$

The condition (6) then implies that

$$(8) \qquad A(y) = K(y)(1 - y)^{-1}.$$

If the Maclaurin's series expansion of $K(y)$ is given by

$$(9) \qquad K(y) = \sum_{n=0}^{\infty} b_n y^n,$$

We must have $b_0 = 0$, as $K(0) = G^*(\infty) = 0$, and the series converges for $0 \leqq y < F^*(c)$, as $G^*(s)$ is analytic for $\sigma > c$. Thus from (1)–(9) we have the following corollary.

COROLLARY 1. *For any reward function $c(t)$ for which the* LST *of $G(t)$ exist, there is a unique sequence of fees $\{a_i\}$ which makes $\mathcal{G}$ fair. The $\{a_i\}$ are given by*

$$a_n = \sum_{j=1}^{n} b_j, \qquad n = 1, 2, \cdots.$$

Furthermore, if $c(t)$ is realistic, and $\sum_{j=1}^{n} b_j \geqq 0$, for $n = 1, 2, \cdots$, the game is realizable.

REMARK 1. If the LST of $G(t)$ does not exist for any finite s, we can still find a sequence $\{a_i\}$ which will make $\mathcal{G}$ fair for $[0, t_0]$, where t_0 is an arbitrary positive number. We simply set $dT(t) = 0$ if $t \geqq t_0$, and the theorem applies with $G^*(s)$ replaced by

$$G_0^*(s) = \int_{0+}^{t_0} e^{-st} c(t)\, dF(t).$$

The analogue of Robbins' Corollary 2 [3, p. 192] is

COROLLARY 3. *$\mathcal{G}$ is fair if and only if*

$$\int_0^t c(x)\, dF(x) = \sum_{n=1}^{\infty} a_n H_n(t), \qquad 0 \leqq t \leqq \infty,$$

where $H_n(t) = F_n(t) - F_{n+1}(t)$ is the probability of exactly n occurrences in the interval $(0, t]$.

3. A continuous version of the Petersburg game. A continuous analogue of the Petersburg game [1, ch. 10] may be described as follows. We consider a Poisson process where the events are occurring at the average rate of λ, $0 < \lambda < \infty$, per unit time, and at the kth occurrence A receives an amount $\exp(\nu X_k)$. We have

$$dF(t) = \lambda e^{-\lambda t}\, dt, \qquad c(t) = e^{\nu t}, \qquad \text{if } t \geqq 0, \tag{10}$$

and zero if $t < 0$. We note that $Ec(X_k) = \infty$ if $\nu \geqq \lambda$. A straightforward calculation gives $A(y) = y(1 - \nu y/\lambda)^{-1}(1 - y)^{-1}$, i.e.,

$$a_n = \sum_{j=0}^{n-1} (\nu/\lambda)^j = \begin{cases} n, & \text{if } \nu = \lambda, \\ \dfrac{(\nu/\lambda)^n - 1}{(\nu/\lambda) - 1}, & \text{if } \nu \neq \lambda. \end{cases} \tag{11}$$

The game is realizable if $\nu \geqq -\lambda$. Also,

$$\begin{aligned} \bar{T}(t) = \bar{U}(t) &= \lambda t + \tfrac{1}{2}\lambda^2 t^2, & &\text{if } \nu = \lambda, \\ &= \lambda\nu(\nu - \lambda)^{-2}[e^{(\nu-\lambda)t} - 1] - \lambda^2(\nu - \lambda)^{-1} t, & &\text{if } \nu \neq \lambda. \end{aligned} \tag{12}$$

It is interesting to compare this game with the classical type of game associated with a Poisson process. Here, $F(t)$ and $c(t)$ are given by (10) and the game is played until the event occurs n times. The interesting case, of course, is when $Ec(X_k) = \infty$, i.e., when $\nu \geqq \lambda$. The problem is to find the total fee, e_n, to be paid by A, so that

$$\text{plim}_{n\to\infty} \{[c(X_1) + \cdots + c(X_n)]/e_n\} = 1. \tag{13}$$

Writing $a = \lambda/\nu$, and assuming $0 < a \leqq 1$, we have the common distribution of $c(X_i)$

$$dP(t) = at^{-a-1}\,dt, \qquad \text{if } 1 \leqq t \leqq \infty;\, 0, \text{ if } t < 1. \tag{14}$$

Let S_n denote the numerator in (13). Applying Theorem 3 of Gnedenko and Kolmogorov [2, p. 57], we have (13), if and only if,

$$\lim_{n\to\infty} |\phi_n(t)| = 1 \tag{15}$$

uniformly in every finite interval of t. Here $\phi_n(t)$ is the characteristic function of $S_n/e_n - 1$ and is given by $\phi_n(t) = \{P^*(-it/e_n)\}^n e^{-it}$.

A series expansion of $\phi_n(t)$ shows that no solution exists for $0 < a < 1$; and for $a = 1$

$$e_n = n \log_e n$$

satisfies the condition (15). This compares with Feller's [1, p. 237] solution of the Petersburg paradox, where he obtains the solution $e_n = n \log_2 n$, when X_1 is integer valued, Pr $(X_1 = n) = 2^{-n}$, $n = 1, 2, \cdots$, and $c(n) = 2^n$.

4. Fixed-time games with a general renewal process. We consider a general renewal process [4, p. 245] $\{X_0, X_1, X_2, \cdots\}$, where $\{X_i\}_{i=1}^{\infty}$ is a renewal process which is preceded by another non-negative random variable X_0 which is independent of $\{X_i\}$, and has cdf $B(x)$. For example, the game may be started at some arbitrary time origin which does not necessarily coincide with the occurrence of an initial event.

Let $F_n(x)$ denote the cdf of $X_1 + \cdots + X_n$ and $K_n(x)$ that of

$$X_0 + X_1 + \cdots + X_{n-1}.$$

Then

$$K_n(x) = B * F_{n-1}(x) \tag{16}$$

where $*$ denotes the convolution of two cdfs.

In addition to the random variables introduced in (1), introduce $T_0(t)$ obtained from $T_i(t)$ by setting $i = 0$. Then $T_0(t)$, $T_1(t), \cdots$, have identical distributions, but $T(t)$ may have a different one.

In this case, we obtain

$$\begin{aligned} d\bar{T}(t) &= \int_0^t d\bar{T}_0(t-u)\, dB(u) + c(t)\, dB(t), \\ d\bar{T}_0(t) &= \int_0^t d\bar{T}_0(t-u)\, dF(u) + c(t)\, dF(t). \end{aligned} \tag{17}$$

Thus writing

$$dG_1(t) = c(t)\, dB(t), \tag{18}$$

and assuming that $G_1^*(s)$ and $G^*(s)$ both exist for some $s = s_0$, we find

$$\bar{T}^*(s) = G^*(s)B^*(s)/[1 - F^*(s)] + G_1^*(s). \tag{19}$$

Also

$$\bar{U}^*(s) = B^*(s)A(F^*(s))/F^*(s). \tag{20}$$

Thus, if $\mathcal{G}$ is associated with this process, we have the following theorem.

THEOREM 2. *$\mathcal{G}$ is fair if $G^*(s_0)$ and $G_1^*(s_0)$ exist, $A(y)$ converges for*

$$|y| < F^*(c),$$

where $c = \max(0, \sigma_0)$, *and if*

$$A(F^*(s)) = G^*(s)F^*(s)/[1 - F^*(s)] + G_1^*(s)F^*(s)/B^*(s). \tag{21}$$

Further, if $\mathcal{G}$ is fair and $G^*(s_0)$ and $G_1^*(s_0)$ exist, then $A(y)$ converges for

$$|y| < F^*(c)$$

and (21) *holds.*

If we set $B^* = F^*$ so that $G_1^* = G^*$, we have Theorem 1. Hence, Theorem 1 may be considered a corollary of Theorem 2.

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