[2] Birnbaum, Z. W. and Klose, Orval M. (1957). Bounds for the variance of the Mann-Whitney statistic. *Ann. Math. Statist.* **28** 933–945.
[3] Lehmann, E. L. (1951). Consistency and unbiasedness of certain non-parametric tests *Ann. Math. Statist.* **22** 165–179.
[4] van Dantzig, D. (1951). On the consistency and the power of Wilcoxon's two-sample test. *Koninklijke Nederlandse Akademie Van Wetenschappen, Proceedings, Ser. A* **54** No. 1, 1–9.

ON THE EXACT DISTRIBUTION OF A CLASS OF MULTIVARIATE TEST CRITERIA

By D. G. Kabe[1]

Wayne State University and Karnatak University

1. Introduction and summary. It is known that the moments of several test criteria in multivariate analysis can be expressed in terms of Gamma functions. Various attempts have been made to determine the exact distributions from the known expressions for the moments. Wilks [11] identified such a distribution with that of a product of independent Beta variables, and he gave explicit results in some particular cases. Later, Nair [5], using the theory of Mellin transforms, expressed the distributions as solutions of certain differential equations, which he solved in some special cases. Kabe [3] expressed them in terms of Meijer G-functions. Asymptotic expansions of the distribution functons have been given by several authors; notably by Box [2], Tukey and Wilks [8], and Banerjee [1]. In this paper we give exact results in some important cases, in which it is found that the distribution is identical with that of a linear function of Gamma variates.

2. The distribution of a linear function of gamma variates. Let $X_1, X_2, \cdots, X_n$ be n independent random variables having the frequency functions

$$f(x_j) = \{x_j^{p_j-1}/2^{p_j}\Gamma(p_j)\}e^{-\frac{1}{2}x_j}, \qquad 0 < x_j < \infty, j = 1, 2, \cdots, n. \tag{2.1}$$

Then the characteristic function (c.f.) $\phi(t)$ of the variable $Z = \sum(X_j/\alpha_j)$, αs being given positive constants, is

$$\phi(t) = \prod_{j=1}^{n} \alpha_j^{p_j}(\alpha_j - 2it)^{-p_j}. \tag{2.2}$$

By inverting this c.f. we find that the frequency function (fr.f.) of Z is given by the expression

$$\begin{aligned} f(z) = \{\alpha_1^{p_1}\alpha_2^{p_2} \cdots \alpha_n^{p_n}/2^{\Sigma p_j}\Gamma(\textstyle\sum p_j)\}e^{-\frac{1}{2}\alpha_1 z}z^{\Sigma p_j - 1}F_1^{(n-1)}[p_2, p_3, \cdots, p_n; \\ \textstyle\sum p_j; \frac{1}{2}(\alpha_1 - \alpha_2)z, \frac{1}{2}(\alpha_1 - \alpha_3)z, \cdots, \frac{1}{2}(\alpha_1 - \alpha_n)z], \end{aligned} \tag{2.3}$$

Received December 22, 1958; revised July 22, 1961.

[1] On leave from Karnatak University, India. The work was completed at Karnatak University.

where

$$F_1^{(n-1)}[p_2, p_3, \cdots, p_n; \textstyle\sum p_j; \frac{1}{2}(\alpha_1 - \alpha_2)z, \frac{1}{2}(\alpha_1 - \alpha_3)z, \cdots, \frac{1}{2}(\alpha_1 - \alpha_n)z] \tag{2.4}$$
$$= \sum_{q_2=0}^{\infty} \cdots \sum_{q_n=0}^{\infty} \frac{(p_2)_{q_2} \cdots (p_n)_{q_n}}{(\sum p_j)_{q_2+\cdots+q_n}} \frac{(\frac{1}{2}(\alpha_1 - \alpha_2)z)^{q_2}}{q_2\,!} \cdots \frac{(\frac{1}{2}(\alpha_1 - \alpha_n)z)^{q_n}}{q_n\,!}$$

is a hypergeometric series with $n - 1$ factorials in the numerator and one factorial in the denominator. In (2.4) we assume that $\alpha_1 - \alpha_j$, $j = 2, \cdots, n$, is positive.

The fr.f. of Z has also been obtained, as an $n$ple series, by Robbins [6] (when each $p_j = \frac{1}{2}$), and by Robbins and Pitman [7] (p_j's unequal). The advantage of the result (2.3) over those quoted is that (2.3) is an $(n - 1)$ple series and that when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ the series (2.3) yields the well-known result

$$f(z) = \{\alpha^{\Sigma p_j}/2^{\Sigma p_j}\Gamma(\textstyle\sum p_j)\}e^{-\frac{1}{2}\alpha z}z^{\Sigma p_j - 1}, \tag{2.5}$$

which, however, does not readily follow from the results in [6] and [7]. We give below a proof of (2.3) for the case $n = 2$. When $n > 2$ the proof follows on similar lines. For $n = 2$, the fr.f. of Z is given by

$$\begin{aligned} f(z) &= \alpha_1^{p_1}\alpha_2^{p_2}\int_{-\infty}^{\infty}\frac{(2\pi)^{-1}e^{-itz}}{(\alpha_1 - 2it)^{p_1}(\alpha_2 - 2it)^{p_2}}\,dt \\ &= \alpha_1^{p_1}\alpha_2^{p_2}\int_{-\infty}^{\infty}\frac{(2\pi)^{-1}e^{-itz}}{(\alpha_1 - 2it)^{p_1+p_2}\left[1 - \frac{(\alpha_1 - \alpha_2)}{(\alpha_1 - 2it)}\right]^{p_2}}\,dt \\ &= \alpha_1^{p_1}\alpha_2^{p_2}\int_{-\infty}^{\infty}\frac{2(\pi)^{-1}e^{-itz}}{(\alpha_1 - 2it)^{p_1+p_2}}\left[1 - \frac{(\alpha_1 - \alpha_2)}{(\alpha_1 - 2it)}\right]^{-p_2}dt. \end{aligned} \tag{2.6}$$

On expanding the expression in square brackets and integrating term by term, we find that

$$f(z) = \{\alpha_1^{p_1}\alpha_2^{p_2}/2^{p_1+p_2}\Gamma(p_1 + p_2)\}e^{-\frac{1}{2}\alpha_1 z}z^{p_1+p_2-1} \cdot {}_1F_1[p_2; p_1 + p_2; \tfrac{1}{2}(\alpha_1 - \alpha_2)z]. \tag{2.7}$$

Here we assume $\alpha_1 > \alpha_2$, however, if $\alpha_1 \leqq \alpha_2$ then it follows that

$$f(z) = \{\alpha_1^{p_1}\alpha_2^{p_2}/2^{p_1+p_2}\Gamma(p_1 + p_2)\}e^{-\frac{1}{2}\alpha_2 z}z^{p_1+p_2-1} \cdot {}_1F_1[p_1; p_1 + p_2; \tfrac{1}{2}(\alpha_2 - \alpha_1)z]. \tag{2.8}$$

3. Exact distribution of L. There is a wide class of test criteria in Multivariate Analysis, whose moments can be expressed in terms of Gamma functions. Consider, e.g., the criterion L for the simultaneous testing of the sample means. The hth moment of L ([4], p. 340) is

$$\mu_h'(L) = \prod_{m=1}^{p}\frac{\Gamma(\frac{1}{2}[n - m])}{\Gamma(\frac{1}{2}[n - m] - \frac{1}{2}[k - 1])}\prod_{m=1}^{p}\frac{\Gamma(\frac{1}{2}[n - m] - \frac{1}{2}[k - 1] + h)}{\Gamma(\frac{1}{2}[n - m] + h)}. \tag{3.1}$$

The c.f. $\phi(t)$ of the variable $Y = -2 \log L$ is easily found to be

$$(3.2)\quad \phi(t) = \prod_{m=1}^{p} \frac{\Gamma(\frac{1}{2}[n-m])}{\Gamma(\frac{1}{2}[n-m] - \frac{1}{2}[k-1])} \prod_{m=1}^{p} \frac{\Gamma(\frac{1}{2}[n-m] - \frac{1}{2}[k-1] - 2it)}{\Gamma(\frac{1}{2}[n-m] - 2it)}.$$

When the integers p, k take some special values, the second factor on the right-hand side of (3.2) may be split up into several factors as shown by Wald and Brookner [9], and Banerjee [1]. We consider the following values of p and k.

CASE 1. Let k be odd $(=2s+1)$ and p odd or even. Take p odd $(=2s-3$, say). Now, if r be any positive integer then we know that

$$(3.3)\quad [\Gamma(x+r)/\Gamma(x)] = x(x+1)\cdots(x+r-1).$$

Applying (3.3) to the second member on the right-hand side of (3.2), we have in the notation of Banerjee [1]

$$(3.4)\quad \prod_{m=1}^{p} \frac{\Gamma(\frac{1}{2}[n-m] - \frac{1}{2}[k-1] - 2it)}{\Gamma(\frac{1}{2}[n-m] - 2it)} = \Lambda_p^{-1},$$

where

$$\begin{aligned}
\Lambda_p = {} & (\tfrac{1}{2}[n-3] - 2it)(\tfrac{1}{2}[n-5] - 2it)^2 \cdots (\tfrac{1}{2}[n-p-2] - 2it)^{\frac{1}{2}(p+1)} \\
& \cdot (\tfrac{1}{2}[n-2s-1] - 2it)^{\frac{1}{2}(p+1)}(\tfrac{1}{2}[n-2s-3] - 2it)^{\frac{1}{2}(p-1)} \cdots \\
& \cdot (\tfrac{1}{2}[n-2s-p] - 2it) \\
(3.5)\quad & \cdot (\tfrac{1}{2}[n-4] - 2it)(\tfrac{1}{2}[n-6] - 2it)^2 \cdots (\tfrac{1}{2}[n-p-1] - 2it)^{\frac{1}{2}(p-1)} \\
& \cdot (\tfrac{1}{2}[n-2s-2] - 2it)^{\frac{1}{2}(p-1)} \\
& \cdot (\tfrac{1}{2}[n-2s-2] - 2it)^{\frac{1}{2}(p-1)}(\tfrac{1}{2}[n-2s-4] - 2it)^{\frac{1}{2}(p-3)} \cdots \\
& \cdot (\tfrac{1}{2}[n-p+1-2s] - 2it).
\end{aligned}$$

The total number of terms in Λ_p is $\frac{1}{2}p(k-1)$ and the number of different factors if $4s-5$.

The c.f. of the variable Y may now be written as

$$(3.6)\quad \phi(t) = \prod_{m=1}^{p} \frac{\Gamma(\frac{1}{2}[n-m])}{\Gamma(\frac{1}{2}[n-m] - \frac{1}{2}[k-1])} \Lambda_p^{-1}$$

It is obvious from (3.6) that Y has the c.f. of a linear function of Gamma variates; using (2.3) we have for the fr.f. of Y the expression

$$\begin{aligned}
f(y) = {} & \prod_{m=1}^{p} \frac{\Gamma(\frac{1}{2}[n-m])2^{-\frac{1}{2}p(k-1)}}{\Gamma(\frac{1}{2}[n-m] - \frac{1}{2}[k-1])\Gamma(\frac{1}{2}p[k-1])} e^{-\frac{1}{4}(n-3)y} y^{\frac{1}{2}p(k-1)-1} \\
(3.7)\quad & \cdot F_1^{(4s-5)}[2, 3, \cdots, \tfrac{1}{2}(p+1), \tfrac{1}{2}(p+1), \tfrac{1}{2}(p+1), \\
& \tfrac{1}{2}(p-1), 3, 2, 1, 1, 2, 3, \cdots, \tfrac{1}{2}(p-1), \tfrac{1}{2}(p-1), 3, 2, 1; \\
& \tfrac{1}{2}p(k-1); \tfrac{1}{4}y, y, \tfrac{3}{4}y, \cdots, \tfrac{1}{2}(\tfrac{1}{2}p+s-2)y].
\end{aligned}$$

Further, we notice from (3.4) that for odd p there is a single factor in the central position of the factors of Λ_p. In case p is even there will be two factors in this central position and the result (3.7) will be slightly changed.

CASE 2. For k and p both even the terms of the second factor of (3.2) may be grouped two by two by applying the duplication formula for the Gamma functions and the case reduces to Case 1.

CASE 3. For k even and p odd the terms may be grouped two by two as in Case 2 and the last term may either be expanded by a known formula (see, e.g., [10], p. 260) for the expansion of $\Gamma(x + r)/\Gamma(x)$ when r is not an integer, or else it may be approximated by the formula

$$\Gamma(x + r)/\Gamma(x) \doteqdot x^r \tag{3.8}$$

and the case reduces to Case 1.

We wish to consider the computational aspects of the result (3.7) in a future communication.

REFERENCES

[1] BANERJEE, D. P. (1958). On the exact distribution of a test in multivariate analysis. *J. Roy. Statist. Soc., Ser. B* **20** 108–110.

[2] BOX, G. E. P. (1949). A general distribution theory for a class of likelihood criteria. *Biometrika* **36** 317–346.

[3] KABE, D. G. (1958). Some applications of Meijer *G*-functions to distribution problems in statistics. *Biometrika* **45** 578–580.

[4] KENDALL, M. G. (1951). *Advanced Theory of Statistics.* **2** Charles Griffin, London.

[5] NAIR, U. S. (1939). The application of the moment functions in the study of distribution laws in statistics. *Biometrika* **30** 274–294.

[6] ROBBINS, HERBERT (1948). The distribution of a definite quadratic form. *Ann. Math. Statist.* **19** 266–270.

[7] ROBBINS, HERBERT and PITMAN, E. J. G. (1949). Application of the method of mixture to quadratic forms in normal variables. *Ann. Math. Statist.* **20** 552–560.

[8] TUKEY, J. W. and WILKS, S. S. (1946). Approximation of the distribution of the product of Beta variables by a single Beta variable. *Ann. Math. Statist.* **17** 318–324.

[9] WALD, A. and BROOKNER, R. J. (1941). On the distribution of Wilks' statistic for testing the independence of several groups of variates. *Ann. Math. Statist.* **12** 137–152.

[10] WHITTAKER, E. T. AND WATSON, G. N. (1927). *A Course of Modern Analysis.* Cambridge Univ. Press.

[11] WILKS, S. S. (1932). Certain generalizations in the analysis of variance *Biometrika* **24** 471–494.

ON THE PARAMETERS AND INTERSECTION OF BLOCKS OF BALANCED INCOMPLETE BLOCK DESIGNS

BY KULENDRA N. MAJINDAR

Delhi University, India

1. Summary. In this investigation we derive a few properties of the intersection of blocks in a balanced incomplete block (b.i.b. for conciseness) design with

Received June 26, 1961; revised February 13, 1962.