In the case of a sample (or of a finite population) of $2n + 1$ members, it is easy to show by an argument similar to the one used in the proof of the theorem, that the measure S lies between $-n^{\frac{1}{2}}/(n + 1)^{\frac{1}{2}}$ and $n^{\frac{1}{2}}/(n + 1)^{\frac{1}{2}}$.

Finally, we note that $|S| < 1$ can be obtained in a different manner, [1], problem 5, p. 256.

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USE OF WILCOXON TEST THEORY IN ESTIMATING THE DISTRIBUTION OF A RATIO BY MONTE CARLO METHODS[1]

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1. Introduction and summary. If $r = x/y$ is the ratio of two independent continuous positive random variables, its distribution can be estimated by generating random samples from the distribution of x and y, and then proceeding in various ways. It is shown, using well-known results in the theory of Wilcoxon's test that the uniformly minimum variance unbiased estimate of $H(A) = P(r \leqq A)$ is obtained by computing Wilcoxon's statistic for the random variables $u_i = x_i$, $v_i = Ay_i (i = 1, \cdots, N)$. The variance of the estimate of $H(A)$ is readily estimated. The computations required by this approach are more arduous than those needed to estimate $H(A)$ from the quantities $r_i = x_i/y_i$, but may be worthwhile where the major part of the computations lies in generating the x_i and y_i. Extension of the reasoning leads to choosing different numbers of x's and y's if they are of different complexity to generate. Further, if the distribution of one of the quantities x or y is known then an effectivity infinite sample from that population is already available and the distribution of r can be estimated by sampling only the variable with unknown distribution, which may (or may not) result in economy of effort.

2. Results. Let x and y have continuous c.d.f.s F and G respectively, with $F(0) = G(0) = 0$. Let it be desired to estimate by Monte Carlo methods

$$H(A) = P((x/y) \leqq A), \tag{1}$$

where x and y are independently distributed.

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[1] This work was done while the author was a Fellow of the John Simon Guggenheim Memorial Foundation, and a visitor in the Department of Social Medicine, University of Oxford.

A straightforward method of estimating $H(A)$ is to construct N independent pairs (x_i, y_i), $i = 1, \cdots, N$, and for each pair to define $Z_i(A) = 1$ if $x_i/y_i = r_i \leqq A$ and $Z_i(A) = 0$ otherwise. Then $\tilde{H}(A) = N^{-1} \sum Z_i$ is an unbiased estimate of $H(A)$ with variance $N^{-1}H(A)[1 - H(A)]$, which can be estimated from the results in an obvious way.

Use of a well-known multiplicative device leads to estimates of smaller variance, as follows:

$$H(A) = P((x/y) \leqq A) = P(x \leqq yA).$$

Define

$$u_i = x_i \qquad v_i = Ay_i .$$

Then we define Wilcoxon's test statistic in the Mann-Whitney form,

$$U = N \sum_{i=1}^{N} F_N^*(v_i),$$

where by F_N^* we denote the sample c.d.f. of the u_i.

Lehmann [3] has shown that the uniformly minimum variance unbiased estimate of $H(A) = P(u \leqq v)$ is $U/N^2 = \hat{H}(A)$.

Further, the variance of this estimate is, except for obvious constant factors, the variance of Wilcoxon's statistic under the alternative hypothesis, given by Van Dantzig [4] and cited by Birnbaum and Klose [2]. This variance in our notation as as follows:

$$\sigma^2[\hat{H}(A)] = (1/N^2)[(N - 1)\varphi^2 + (N - 1)\gamma^2 + pq],$$

where

$$\varphi^2 = \int_0^\infty F^2(t)\, dG\left(\frac{t}{a}\right) - q^2,$$

$$\gamma^2 = \int_0^\infty G^2\left(\frac{t}{A}\right) dF(t) - p^2,$$

$$q = 1 - p = \int_0^\infty F(t)\, dG\left(\frac{t}{a}\right) = H(A).$$

To estimate $\sigma^2[\hat{H}(A)]$ the integrals above can be replaced by the corresponding sums from the sample data.

In general, if m observations are taken on x, and n observations are taken on y (rather than N on each, as above) the estimate and its variance are

$$\hat{H}(A) = (1/mn)U = (1/n) \sum_{j=1}^{n} F_m^*(v_j) \tag{2}$$

$$\sigma^2[\hat{H}(A)] = (1/mn)[(m - 1)\varphi^2 + (n - 1)\gamma^2 + p(1 - p)]. \tag{3}$$

If one of the two random variables were very much more difficult to sample than the other, m and n could be chosen to balance cost against precision in the light of this expression for the variance.

If one of the two random variables has an explicitly known distribution, say $F(t)$ is known, the distribution of r can be more precisely estimated (and perhaps more economically estimated) by sampling y and using numerical integration. In particular

$$\hat{\hat{H}}(A) = (1/n) \sum_{j=1}^{n} F(v_j) = (1/n) \sum_{j=1}^{n} F(Ay_j)$$

is an average of independent identically distributed random variables. It has mean

$$E(\hat{\hat{H}}(A)) = E\{F(Ay)\} = \int_0^t F(t)\, dG\left(\frac{t}{A}\right) = H(A)$$

and variance

$$\begin{aligned} \sigma^2[\hat{\hat{H}}(A)] &= (1/n) \text{ var } F(Ay) = \frac{1}{n}\{E(F^2(Ay)) - (EF(Ay))^2\} \\ &= (1/n)\left[\int_0^\infty F^2(t)\, dG\left(\frac{t}{A}\right) - \left(\int_0^\infty F(t)\, dG\left(\frac{t}{A}\right)\right)^2\right] \\ &= (1/n)\, \varphi^2, \end{aligned}$$

and this is strictly less than $\sigma^2[\hat{H}(A)]$ which uses m observations on x rather than knowledge of its distribution.

A few further matters deserve comment. First, the optimum properties of $\hat{H}(A)$ (and $\hat{\hat{H}}(A)$) would seem to commend the use of this procedure where the distribution of the ratio is to be estimated at not merely one point, A, but at a set of points $A_1, \cdots, A_k$. However, no justification beyond intuitive appeal is offered here. Second, the entire exposition has assumed both x and y to be strictly positive random variables. Actually, if y is positive, then no modification of the text is necessary when x is allowed to take both positive and negative values. Further, if either x or y is always of one sign, the situation is essentially unchanged. If both x and y take values of both signs, a somewhat more complicated adaptation of the procedure is necessary, and it is not immediately evident what optimum properties the estimate enjoys. Finally, something can be said about the amount of savings in sample available by this estimation procedure. Birnbaum and Klose [2] give sharp upper and lower bounds on the variance of U. Their sharp upper bound (Theorem 3.2) shows that zero savings can occur in certain extreme situations. Their sharp lower bound for $p \leqq \frac{1}{2}$ and $m = n$ (Theorem 3.5) shows that maximum possible savings, as expressed by that bound, grows without bound as p tends to zero. Some numerical examples are given in [1].

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ON THE EXACT DISTRIBUTION OF A CLASS OF MULTIVARIATE TEST CRITERIA

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1. Introduction and summary. It is known that the moments of several test criteria in multivariate analysis can be expressed in terms of Gamma functions. Various attempts have been made to determine the exact distributions from the known expressions for the moments. Wilks [11] identified such a distribution with that of a product of independent Beta variables, and he gave explicit results in some particular cases. Later, Nair [5], using the theory of Mellin transforms, expressed the distributions as solutions of certain differential equations, which he solved in some special cases. Kabe [3] expressed them in terms of Meijer G-functions. Asymptotic expansions of the distribution functons have been given by several authors; notably by Box [2], Tukey and Wilks [8], and Banerjee [1]. In this paper we give exact results in some important cases, in which it is found that the distribution is identical with that of a linear function of Gamma variates.

2. The distribution of a linear function of gamma variates. Let $X_1, X_2, \cdots, X_n$ be n independent random variables having the frequency functions

$$f(x_j) = \{x_j^{p_j-1}/2^{p_j}\Gamma(p_j)\}e^{-\frac{1}{2}x_j}, \qquad 0 < x_j < \infty, j = 1, 2, \cdots, n. \tag{2.1}$$

Then the characteristic function (c.f.) $\phi(t)$ of the variable $Z = \sum(X_j/\alpha_j)$, αs being given positive constants, is

$$\phi(t) = \prod_{j=1}^{n} \alpha_j^{p_j}(\alpha_j - 2it)^{-p_j}. \tag{2.2}$$

By inverting this c.f. we find that the frequency function (fr.f.) of Z is given by the expression

$$\begin{aligned} f(z) = \{\alpha_1^{p_1}\alpha_2^{p_2} \cdots \alpha_n^{p_n}/2^{\Sigma p_j}\Gamma(\textstyle\sum p_j)\}e^{-\frac{1}{2}\alpha_1 z}z^{\Sigma p_j-1}F_1^{(n-1)}[p_2, p_3, \cdots, p_n; \\ \textstyle\sum p_j; \frac{1}{2}(\alpha_1-\alpha_2)z, \frac{1}{2}(\alpha_1-\alpha_3)z, \cdots, \frac{1}{2}(\alpha_1-\alpha_n)z], \end{aligned} \tag{2.3}$$

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[1] On leave from Karnatak University, India. The work was completed at Karnatak University.