in which the s_i are positive with $s_r \geqq s_{r+1}$, $r = 1, \cdots, t-1$, and $\phi = \min(j, \gamma)$.

Cases in which the a_i are not distinct can be treated as above except that (36) must be replaced by the corresponding limit formulae.

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IMPROVED BOUNDS ON A MEASURE OF SKEWNESS

By Kulendra N. Majindar

Delhi University, India

In 1932, Hotelling and Solomons [2] proved that the absolute value of a certain measure of skewness for a population can not exceed 1. This result has been used by Madow [3] in his study of systematic sampling. The proof given by Hotelling and Solomons covers the case of a discrete random variable. In this note we extend and strengthen the inequality for any random variable with a positive standard deviation. Let X be a random variable with a positive standard deviation, M its median and $F(x)$ its cumulative distribution function. If the median is not uniquely defined, we will define it by $M = \frac{1}{2}\sup\{x : F(x) < \frac{1}{2}\} + \frac{1}{2}\inf\{x : F(x) > \frac{1}{2}\}$. The measure of skewness, S, considered here is the ratio of the difference between the mean and median to the standard deviation of X. With this definition we establish the following theorem.

Theorem. *The measure of skewness S of a random variable X with a finite positive standard deviation satisfies the inequality*

$$|S| < 2(pq)^{\frac{1}{2}}/(p+q)^{\frac{1}{2}},$$

where $p = \Pr\{X > E(X)\}$ *and* $q = \Pr\{X < E(X)\}$.

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PROOF. Without loss of generality for our purpose we assume that the mean is 0. So we have $\int x\, dF = 0$. Then

$$(1) \qquad \int_{0+}^{\infty} x\, dF = \int_{-\infty}^{0-} |x|\, dF = a \quad \text{(say)}.$$

By the Cauchy-Schwarz inequality we have

$$(2) \qquad a^2 \leqq p \int_{0+}^{\infty} x^2\, dF,$$

$$(3) \qquad a^2 \leqq q \int_{-\infty}^{0-} x^2\, dF.$$

If $M > 0$ we have

$$(4) \qquad a = \int_{0+}^{\infty} x\, dF \geqq \int_{M-0}^{\infty} x\, dF \geqq M \int_{M-0}^{\infty} dF,$$

and so

$$(5) \qquad 2a \geqq M.$$

Similarly, if $M < 0$ we get

$$(6) \qquad 2a \geqq -M.$$

Therefore for both cases we have

$$(7) \qquad 2a \geqq |M|\,.$$

By (2) and (3), $a(1/p + 1/q)^{\frac{1}{2}}$ is less than or equal to the standard deviation. Now using (7) we see that $|S| \leqq 2(pq)^{\frac{1}{2}}/(p + q)^{\frac{1}{2}}$.

Let $S = 2(pq)^{\frac{1}{2}}/(p + q)^{\frac{1}{2}}$, if possible. Then $M \neq 0$ and the equality sign must hold in (2), (3) and (7). In view of (1), the equality sign in (2) and (3) can hold if and only if X assumes only one positive value, α say, with probability p and only one negative value, $-\beta$ say, with probability q. Then $a = \alpha p = \beta q$. Suppose $\Pr\{X = 0\} = r; 0 \leqq r = 1 - p - q < 1$. If $q > \frac{1}{2}$, then M, by our definition, is $-\beta$ and (7) gives $2a = 2\beta q = \beta$, i.e., $q = \frac{1}{2}$—a contradiction. Likewise if $p > \frac{1}{2}$ we obtain the contradiction $p = \frac{1}{2}$. If $p < \frac{1}{2}$, $q < \frac{1}{2}$ then M is 0, which is impossible. If $p = \frac{1}{2}$, $q = \frac{1}{2}$, then $r = 0$, $\alpha = \beta$, and M is 0—a contradiction. Next, if $p = \frac{1}{2}$, $q < \frac{1}{2}$, then $M = \frac{1}{2}(0 + \alpha) = \frac{1}{2}\alpha$, and now (7) gives $2\,\alpha p = \alpha = \frac{1}{2}\alpha$—an absurdity. Similarly $p < \frac{1}{2}$, $q = \frac{1}{2}$ is also impossible. So $|S| \neq 2(pq)^{\frac{1}{2}}/(p + q)^{\frac{1}{2}}$ and this completes the proof.

If only one of the numbers p, q and r is known, one may get bounds for S from the inequality

$$|S| < 2(pq)^{\frac{1}{2}}/(p + q)^{\frac{1}{2}} \leqq \min(2(p(1 - p))^{\frac{1}{2}}, 2(q(1 - q))^{\frac{1}{2}}, (1 - r)^{\frac{1}{2}}) \leqq 1.$$

We notice that the inequality $-1 < S < 1$ holds for any random variable with a positive standard deviation; this gives an improvement of the bounds of Hotelling and Solomons.

In the case of a sample (or of a finite population) of $2n + 1$ members, it is easy to show by an argument similar to the one used in the proof of the theorem, that the measure S lies between $-n^{\frac{1}{2}}/(n + 1)^{\frac{1}{2}}$ and $n^{\frac{1}{2}}/(n + 1)^{\frac{1}{2}}$.

Finally, we note that $|S| < 1$ can be obtained in a different manner, [1], problem 5, p. 256.

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USE OF WILCOXON TEST THEORY IN ESTIMATING THE DISTRIBUTION OF A RATIO BY MONTE CARLO METHODS[1]

By Lincoln E. Moses

Stanford University

1. Introduction and summary. If $r = x/y$ is the ratio of two independent continuous positive random variables, its distribution can be estimated by generating random samples from the distribution of x and y, and then proceeding in various ways. It is shown, using well-known results in the theory of Wilcoxon's test that the uniformly minimum variance unbiased estimate of $H(A) = P(r \leqq A)$ is obtained by computing Wilcoxon's statistic for the random variables $u_i = x_i$, $v_i = Ay_i(i = 1, \cdots, N)$. The variance of the estimate of $H(A)$ is readily estimated. The computations required by this approach are more arduous than those needed to estimate $H(A)$ from the quantities $r_i = x_i/y_i$, but may be worthwhile where the major part of the computations lies in generating the x_i and y_i. Extension of the reasoning leads to choosing different numbers of x's and y's if they are of different complexity to generate. Further, if the distribution of one of the quantities x or y is known then an effectivity infinite sample from that population is already available and the distribution of r can be estimated by sampling only the variable with unknown distribution, which may (or may not) result in economy of effort.

2. Results. Let x and y have continuous c.d.f.s F and G respectively, with $F(0) = G(0) = 0$. Let it be desired to estimate by Monte Carlo methods

$$H(A) = P((x/y) \leqq A), \tag{1}$$

where x and y are independently distributed.

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[1] This work was done while the author was a Fellow of the John Simon Guggenheim Memorial Foundation, and a visitor in the Department of Social Medicine, University of Oxford.