CONSTRUCTION OF ROTATABLE DESIGNS THROUGH BALANCED INCOMPLETE BLOCK DESIGNS

By M. N. Das and V. L. Narasimham

Institute of Agricultural Research Statistics, New Delhi

1. Introduction and summary. Rotatable designs were introduced by Box and Hunter (1954, 1957) for the exploration of response surfaces. They constructed these designs through geometrical configurations and obtained several second order designs. Afterwards, Gardiner and others (1959) obtained some third order designs through the same technique for two and three factors and a third order design for four factors. Bose and Draper (1959) obtained some second order designs by using a different method. Draper (1960 a) gave a method of construction of an infinite series of second order designs in three and more factors. Recently, Box and Behnken (1960 a) have obtained a class of second order rotatable designs from those of first order. Draper (1960 b) has obtained some third order rotatable designs in three dimensions and a third order rotatable design in four dimensions. Das (1961) has obtained such designs, both second and third orders up to 8 factors as fractional replicates of factorial designs. The method of construction of the designs presented in this paper is essentially based on that presented by Das (1961). After the manuscript of this paper was submitted for publication the authors' attention was drawn to the work of Box and Behnken (1960 b). They have obtained some second order designs by following a procedure which uses balanced incomplete block designs in the same manner as described below. They did not, however, extend the method to include other complementary sets of points which, as will be shown, allow one to obtain rotatable second and third order designs based on *any* balanced incomplete block design.

In the present paper a method has been given by using the properties of balanced incomplete block designs through which second order rotatable designs with any number of factors, with a reasonably small number of points, can be obtained. By extending the method, third order rotatable designs, both sequential and nonsequential, up to 15 factors have been obtained with the help of doubly balanced incomplete block designs and complementary B.I.B. designs.

2. Rotatable designs. Let there be v variates, each at s levels. If a design be formed with N of the s^v treatment combinations, it can be written as the following $N \times v$ matrix, which we shall call the design matrix:

$$\begin{bmatrix} x_{11} & x_{21} & x_{31} & \cdots & x_{v1} \\ x_{12} & x_{22} & x_{32} & \cdots & x_{v2} \\ \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & & \cdots \\ x_{1N} & x_{2N} & x_{3N} & \cdots & x_{vN} \end{bmatrix}.$$

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For convenience, a variate x_i has been associated with the ith factor to denote its levels. The treatment combinations will hereafter be called points of the design. According to Box and Hunter (1957) a design of the above form will be a rotatable design of order d if a response polynomial surface

$$y = \beta_0 + \sum_i \beta_i x_i + \sum_{i \leqq j} \beta_{ij} x_i x_j + \sum_{i \leqq j \leqq k} \beta_{ijk} x_i x_j x_k + \cdots$$

of order d of the response y as obtained from the treatments, on the variates x_i, $i = 1, 2, \cdots, v$, with some suitable origin and scale, can be so fitted that the variance of the estimated response from any treatment is a function of the sum of squares of the levels of the factors in that treatment combination. In other words, the variance of the estimated response at a point is a function of the square of the distance of the point from a suitable origin, so that the variances of all estimated responses at points equidistant from the origin are the same. When the response surface is of the second degree (i.e., $d = 2$), such constancy of variance is possible if the design points are so selected as to satisfy the following relations:

(A) $\sum x_i = 0$, $\sum x_i x_j = 0$, $\sum x_i x_j^2 = 0$, $\sum x_i^3 = 0$, $\sum x_i x_j^3 = 0$, $\sum x_i x_j x_k^2 = 0$, $\sum x_i x_j x_k = 0$, $\sum x_i x_j x_k x_l = 0$ for all $i \neq j \neq k \neq l$.

(B) (i) $\sum x_i^2 = \text{constant} = N\lambda_2$
(ii) $\sum x_i^4 = \text{constant} = 3N\lambda_4$ for all i.

(C) $\sum x_i^2 x_j^2 = \text{constant}$ for all $i \neq j$.

(D) $\sum x_i^4 = 3 \sum x_i^2 x_j^2$ for all $i \neq j$.

(E) $\lambda_4/\lambda_2^2 > v/(v + 2)$.

In the above relations, the summation is over the design points.

In the case of third order designs the following further relations also should be satisfied:

(A_1) Each of the sums of powers or products of powers of x_i's, in which at least one power is odd, is zero.

(B_1) $\sum x_i^6 = \text{constant} = 15\, N\lambda_6$ for all i.

(C_1) (i) $\sum x_i^2 x_j^4 = \text{constant}.$
(ii) $\sum x_i^2 x_j^2 x_k^2 = \text{constant}$ for all $i \neq j \neq k$.

(D_1) (i) $\sum x_i^6 = 5 \sum x_i^2 x_j^4.$
(ii) $\sum x_i^2 x_j^4 = 3 \sum x_i^2 x_j^2 x_k^2$.

(E_1) $(\lambda_2 \lambda_6/\lambda_4^2 > (v + 2)/(v + 4)$, the summations being over the design points.

3. Method of construction of rotatable designs. Each point in a design is essentially a combination of the levels of different factors. We thus propose first to take some unknown levels, to be denoted by a, b, c etc., excepting that some of these may be zero also, and to get a factorial design in v factors, say, out of these unknown levels. Thus, if there are four factors, each at two levels denoted by a and b, the 16 combinations will be of the form

$$\begin{array}{cccc} a & a & a & a \\ a & b & b & b \\ b & a & a & b \\ a & a & b & a \\ & \cdots & & \end{array}$$

etc.

Next we shall have another design in v factors of the form 2^v where the two levels of each factor are $+1$ and -1. We can now get one more set of combinations when any combination of the first design is associated with any combination of the second design, 2^v by multiplying the corresponding entries, that is, the levels of the same factor in the two combinations and writing the products in the same order. This method of association of any two combinations of the two designs will hereafter be called multiplication.

For example, in the design with four factors if the combination $a \quad b \quad b \quad b$ be "multiplied" by each of the 2^4 combinations of the levels $+1$ and -1, we shall get the following 16 combinations:

$$\begin{array}{cccccccc} a & b & b & b & \text{when multiplied by} & 1 & 1 & 1 & 1 \\ a & b & b & -b & \text{when multiplied by} & 1 & 1 & 1 & -1 \\ a & b & -b & b & \text{when multiplied by} & 1 & 1 & -1 & 1 \\ a & b & -b & -b & \text{when multiplied by} & 1 & 1 & -1 & -1 \end{array}$$

etc.

If one of the unknown levels, say, a be zero, all the sixteen combinations will not be distinct, but only eight of them will be distinct, as by associating $+1$ and -1 with zero we get the same thing. We shall consider, in future, only those combinations which are distinct unless otherwise mentioned. Thus, by "multiplying" any combination of the first design with all the combinations of the second design, 2^v, we shall get 2^p distinct combinations where p denotes the number of non-zero unknown levels in the combination considered of the first design. As a matter of fact, if there be only p unknowns in a combination together with some zeros, we have to multiply only the non-zero levels in the combination with each of the 2^p combinations of $+1$ and -1.

We have by now come across three types of combinations, namely, (i) Factorial combinations of the unknown levels a, b, etc. together with 0. (ii) Factorial combinations of levels $+1$ and -1. (iii) Combinations when each 0, a, b etc. is associated with $+1$ and -1 through "multiplication".

The first type of factorial combinations will be called combinations of unknown levels and the second will be called associate combinations. The third combinations will actually constitute the design points and hence they will be referred to as the design points.

It will be seen easily that if a design be formed by including all the distinct points which are got by "multiplying" any combination of the unknown levels with all the associate combinations, these points will always satisfy relations A and A_1. When $v > 4$, or $p > 4$, the relations A and A_1 will also be satisfied when a suitable fraction of the 2^v or 2^p associate combinations, as the case may be, are so chosen for "multiplication" to obtain a second order rotatable design that no interaction with less than five factors is confounded in these associate combitions. In case of third order designs the fraction should be so chosen that no interaction with less than seven factors is confounded. For satisfying the other relations, B, C, D, E one or more combinations of the unknown levels will have to be chosen suitably. A method for the choice of such combinations through which second order rotatable designs can be obtained has been described below.

Let there be a balanced incomplete block design with the parameters[1] (v, b^*, r, k, λ). Let us write the design in the form of a $b^* \times v$ matrix, the elements of which are zero and a. If in any block a particular treatment occurs the element in that block corresponding to that treatment will be a, otherwise, zero. Each row of the matrix corresponding to a block of the B.I.B. design can be considered to give a combination of zero and the unknown level a. By "multiplying" each of these "b^*" combinations thus obtained through the B.I.B. design with 2^k, since $p = k$ here, or a suitable fraction of the associate combinations, we shall get a number of design points less than or equal to $b^* \times 2^k$. These points which we will denote as a-(v, b^*, r, k, λ) $\times$ 2^k (or a suitable fraction of 2^k) will satisfy all relations except D and E, as constancy of replication will satisfy relation B and that of replication of pair of treatments will satisfy relation C. These points can also be obtained through the method of Box and Behnken (1960 b) if the asterisks used by them in the B.I.B. designs only, be replaced by a and then the sign $+$ or $-$ of 1 presented as column, be associated with a in the same manner described therein.

4. Second order rotatable designs. Box and Behnken (1960 b) have shown that the points obtained through a B.I.B. design with $r = 3\lambda$, together with at least one central point, will always give a second order rotatable design in v factors. Their designs will not involve a, but the design obtained through the present method will involve a, which has to be obtained from the relation $\sum x_i^2 = N$, where N denotes the total number of design points.

There are four B.I.B. designs up to 16 variates satisfying the relation $r = 3\lambda$. Two of these designs, viz., for 4 and 7 variates, are presented in the above reference, while the other two designs for 10 and 16 variates, are presented in Appendix I for the sake of completion. These two designs have also been reported by Box, G. E. P. and Behnken, D. W. (1958). We have described below the method of obtaining second order rotatable designs through any B.I.B. design where $r \neq 3\lambda$.

[1] As the symbol b has been used in this paper to denote an unknown level of the factors together with a, c, etc., the symbol b^* has been used in this paper to denote the number of blocks in a balanced incomplete block design.

If the relation $r = 3\lambda$ does not hold in any B.I.B. design, we can always get a second order rotatable design through it by taking some more combinations involving one more unknown level, b, and then by "multiplying" them with the requisite number of associate combinations. The combinations to be taken are either the v combinations,

$$\begin{matrix} b & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ & \cdots & & \cdots & \cdots \\ & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b \end{matrix}$$

obtained from the combination $(b,\ 0\ 0 \cdots 0)$ by permuting over the different factors, or the combination $(b \quad b \cdots b)$ according as $r < 3\lambda$ or $r > 3\lambda$.

We have so far used two types of combinations, viz., one involving the unknown level a and the other involving b. The combinations obtained through B.I.B. design will hereafter be called a-combinations, while the v combinations obtained from $(b\ 0\ 0 \cdots 0)$ will be called combinations of the type $(b\ 0 \cdots 0)$. The design points obtained by the combinations of type $(b\ 0\ 0 \cdots 0)$ and combination $(b\ b \cdots b)$ after "multiplication" with the requisite associate combinations will hereafter be denoted respectively as $(b\ 0 \cdots 0) \times 2^1$ and $(b\ b \cdots b) \times$ suitable fraction of 2^v.

In the above designs $\sum x_i^4$ and $\sum x_i^2 x_j^2$ will be functions of a and b. From the relation $\sum x_i^4 = 3 \sum x_i^2 x_j^2$, we shall get an equation connecting a and b. This equation will always give a positive solution of a^2/b^2, provided that the extra combinations are suitably chosen, taking into account which of $r < 3\lambda$ or $r > 3\lambda$ holds. For determining the unknown levels a and b, we have one more equation, viz., $\sum x_i^2 = N$, where N is the total number of points including the central points.

For example, in the design $v = 8$, $k = 2$, $r = 7$, $b^* = 28$, $\lambda = 1$, $r > 3\lambda$, and hence the combination $(b \quad b \cdots b)$ has to be taken together with the 28 a-combinations given by the B.I.B. design in 8 factors. The design points will be (i) a-$(v = 8,\ k = 2,\ r = 7,\ b^* = 28,\ \lambda = 1) \times 2^2$, and (ii) $(b, b \cdots b) \times (\frac{1}{4}$ replicate of $2^8)$.

In this design we have $\sum x_i^4 = 28a^4 + 64b^4$, $\sum x_i^2 x_j^2 = 4a^4 + 64b^4$. Hence, relation D gives the equation

$$28a^4 + 64b^4 = 3(4a^4 + 64b^4),$$

whence

$$b^4/a^4 = \tfrac{1}{8}.$$

The above equation, together with $\sum x_i^2 = 28a^2 + 64b^2 = N$, will completely determine the two unknowns a and b. The number of points in this design will be $112 + 64 = 176$. No central points are necessary in this design, though they may be added if otherwise necessary.

By properly choosing the B.I.B. designs the number of design points can be reduced. A list of second order rotatable designs, together with the additional type of combinations, when necessary, to be taken for the construction of such designs up to 16 factors, is presented in Appendix I, together with relevant details. Designs for larger number of factors can, however, be obtained on the same lines. It will be seen that all the designs obtainable through B.I.B. designs have either 3 or 5 levels, according as the extra combinations with b are taken or not.

5. Third order rotatable designs. Third order rotatable designs can be both non-sequential and sequential. If the design points satisfy all the requirements mentioned in Section 2 and are tried in one occasion, they form a non-sequential third order rotatable design. Alternatively, Gardiner et al. (1959) have shown that a third order rotatable design can be performed sequentially by dividing all the design points into two groups, each forming the contents of a block. If the design points in the first block form a second order design and the inclusion of the additional points of the second block makes the whole a third order design, this design with the points in both the blocks is called a sequential third order rotatable design. The points in the second block will be tried when the fit as obtained from the first block happens to be inadequate.

For the estimation of the polynomial coefficients independently of block effects through such designs, there is a further condition to be satisfied, viz.,

$$(\textstyle\sum_1 x_i^2)/(\sum_2 x_i^2) = (n_1 + n_{10})/(n_2 + n_{20})$$

where n_1 and n_2 are the numbers of treatments excluding the central points in the two blocks respectively; n_{10} and n_{20} denote the central points which may have to be added to the two blocks and $\sum_1$ and $\sum_2$ denote the summation over the points in the first and second blocks respectively.

As $\sum_1 x_i^2$ and $\sum_2 x_i^2$ are functions of the levels of the factors, which can be evaluated from the other relations of the design, the above relation can always be satisfied by suitably choosing n_{10} and n_{20}.

The a-combinations chosen through B.I.B. designs for the construction of second order rotatable designs do not usually satisfy the relations C_1 (ii) together with D, D_1 (i) and D_1 (ii) and E as required for third order rotatable designs. If the B.I.B. design happens to be doubly balanced i.e. in addition to pairs of treatments occurring a constant number of times, λ, the triplets of treatments also occur a constant number of times, μ, in the blocks (Calvin, 1954), the relation C_1 (ii) is also satisfied. For satisfying the other relations not yet satisfied, viz. D, D_1 (i) and D_1 (ii) and E, we have to introduce combinations involving fresh unknowns which can be evaluated by solving the equations obtained through D, D_1 (i) and D_1 (ii). For example, each of the following designs are doubly balanced. For the sake of convenience only the numerical values of the parameters (v, k, r, b^*, λ and μ) of the doubly B.I.B. designs have been shown in brackets in the same order in which they are written above. In the future, the parameters of B.I.B. designs also will be presented similarly except for μ.

(i) (3, 2, 2, 3, 1, 0)
(ii) (4, 3, 3, 4, 2, 1)
(iii) (5, 3, 6, 10, 3, 1)
(iv) (6, 3, 10, 20, 4, 1)
(v) (8, 4, 7, 14, 3, 1)
(vi) (9, 3, 28, 84, 7, 1)
(vii) (10, 4, 12, 30, 4, 1)
(viii) (11, 5, 15, 33, 6, 2)
(ix) (12, 6, 11, 22, 5, 2).

With the help of each of these designs which will supply us the a-combinations as described earlier for second order rotatable designs, third order designs, both sequential and non-sequential, can be obtained by taking further one or more of the combinations of the type $(b\ 0\ 0 \cdots 0)$, $(c\ c\ 0\ 0 \cdots 0)$, $(d\ d \cdots d)$ involving fresh unknown levels b, c, d and multiplying them with the associate combinations as earlier. The combination $(c\ c\ 0\ 0 \cdots 0)$ will give $\binom{v}{2}$ combinations when permuted over all the v factors and these $v(v-1)/2$ combinations will hereafter be called combinations of type $(c\ c\ 0\ 0 \cdots 0)$. The design points obtained from the combinations of type $(c\ c\ 0\ 0 \cdots 0)$ after multiplying each one of them with the 2^2 associate combinations will be denoted as $(c\ c\ 0 \cdots 0) \times 2^2$. The other two types of combinations have been described earlier. Sometimes it becomes necessary to include in the same design more than one set of the same type for getting positive solutions for all the levels.

As an example, we can get a third order non-sequential rotatable design in 9 factors with the help of the following design points:

(i) 672 points from a-(9, 3, 28, 84, 7, 1) $\times\ 2^3$
(ii) 256 points from $(b\ b \cdots b) \times \frac{1}{2}$ repl. 2^9
(iii) 256 points from $(c\ c \cdots c) \times \frac{1}{2}$ repl. 2^9
(iv) 18 points from $(d\ 0 \cdots 0) \times 2$.

The equations for solving the unknowns come out as

From D: $(28 \times 8)a^4 + 256(b^4 + c^4) + 2d^4 = (21 \times 8)a^4 + 3 \times 256(b^4 + c^4)$

From D_1(i): $(28 \times 8)a^6 + 256(b^6 + c^6) + 2d^6 = (35 \times 8)a^6 + 5 \times 256(b^6 + d^6)$

From D_1(ii): $(7 \times 8)a^6 + 256(b^6 + c^6) = 3 \times 8a^6 + 3 \times 256(b^6 + c^6)$.

Solving these equations we get

$$b^2/a^2 = 0.392768$$

$$c^2/a^2 = 0.122376$$

$$d^2/a^2 = 3.914868.$$

The value of a can be obtained from $\sum x_i^2 = N$. This design contains 1202 points.

Sequential third order designs can be obtained with the help of the same types of combinations, viz., a-combinations through the doubly B.I.B. designs, together with one or more of the types of combinations $(b\ 0\ 0 \cdots 0)$, $(c\ c\ 0 \cdots 0)$ and $(d\ d \cdots d)$. For example, we can get a sequential third order rotatable design in 8 factors with the help of the following design points:

Block I (i) 128 points of $(d\ d \cdots d) \times (\frac{1}{2}$ replicate of $2^8)$
(ii) 16 points of $(e\ 0\ 0 \cdots 0) \times 2$
Block II (iii) 224 points of a-(8, 4, 7, 14, 3, 1) $\times 2^4$
(iv) 112 points of $(c\ c\ 0 \cdots 0) \times 2^2$

The design relations will lead to the following equations. From relations

$$(D): 112a^4 + 28c^4 + 128d^4 + 2e^4 = 144a^4 + (3 \times 128)d^4 + 12c^4$$
$$(D_1)\ (\text{i}): 112a^6 + 28c^6 + 128d^6 + 2e^6 = 240a^6 + (5 \times 128)d^6 + 20c^6$$
$$(D_1)\ (\text{ii}): 48a^6 + 128d^6 + 4c^6 = 48a^6 + (3 \times 128)d^6$$

There is one more relation to make each block a second order rotatable design. This relation gives $2e^4 + 128d^4 = (3 \times 128)\ d^4$. Putting $a^2/d^2 = s$, $c^2/d^2 = u$, $e^2/d^2 = t$, the equations become

$$8u^2 + t^2 = 16s^2 + 128$$
$$4u^3 + t^3 = 64s^3 + 256$$
$$4u^3 = 2 \times 128$$
$$t^2 = 128,$$

whence $u = 4$, $t = 128^{\frac{1}{2}}$, and $s = 8^{\frac{1}{2}}$. The value of d can be obtained from $\sum x_i^2 = N$. The number of central points to be added to the two blocks to ensure estimation of the polynomial coefficients independently of block effects will be determined from

$$(\textstyle\sum_1 x_i^2)/(\sum_2 x_i^2) = (144 + n_{10})/(336 + n_{20}),$$

where $\sum_1 x_i^2$ is summed over the points in the first block and $\sum_2 x_i^2$ is summed over the points in the second block.

As $\sum_1 x_i^2$ and $\sum_2 x_i^2$ are functions of the unknown levels, which have been obtained by solving the equations gotten from the different relations to be satisfied, n_{10} and n_{20}, the number of central points to be added to the first and second block respectively, can be obtained from the above relation. Actually, $\sum_1 x_i^2 = 128d^2 + 2e^2$ and $\sum_2 x_i^2 = 112a^2 + 28c^2$.

Substituting for s, u and t obtained earlier, n_{10} and n_{20} can be obtained from

$$(64 + t)/(56s + 14u) = (144 + n_{10})/(336 + n_{20}).$$

Thus, we get a sequential third order rotatable design for 8 factors in 480 non-central points.

6. Third order designs obtained through complementary B.I.B. Designs. A B.I.B. design, not necessarily doubly balanced, is taken together with its complementary B.I.B. design, repeated if necessary, for generating the a-combinations as before. We can now get points through these a-combinations which will satisfy C_1(ii), as μ will be a constant in the combined B.I.B. designs, together with all the other relations excepting D, D_1(i) and D_1(ii), E. For satisfying these relations we have to take one or more of the types of combinations $(b\ 0\ 0 \cdots 0)$, $(c\ c\ 0 \cdots 0)$ and $(d\ d \cdots d)$ involving fresh unknowns.

For example, a non-sequential third order rotatable design in 10 factors can be obtained with the following points:

(i) (18×32) points of a-(10, 5, 9, 18, 4) $\times 2^5$,

(ii) (18×32) points of a-(10, 5, 9, 18, 4) $\times 2^5$, the design in (ii) being the complementary B.I.B. design of the design in (i).

(iii) 20 points of $(b\ 0 \cdots 0) \times 2$,

(iv) 180 points of $(d\ d\ 0 \cdots 0) \times 2^2$,

(v) 20 points of $(c\ 0 \cdots 0) \times 2$.

Here $\mu = 3$ in the combined designs.

The relations D, D_1(i) and (ii) give the equations,

$$(18 \times 32)a^4 + 2b^4 + 2c^4 + 36d^4 = (24 \times 32)a^4 + 12d^4.$$

$$(18 \times 32)a^6 + 2b^6 + 2c^6 + 36d^6 = (40 \times 32)a^6 + 20d^6.$$

$$(8 \times 32)a^6 + 4d^6 = (9 \times 32)a^6.$$

Putting $b^2/a^2 = s$, $c^2/a^2 = t$, $d^2/a^2 = u$, we get $u = 2$, $s^2 + t^2 = 48$, and $s^3 + t^3 = 288$. Solving, we get $s = 6.494805$, $t = 2.411955$. Thus, we get a non-sequential third order rotatable design in 1372 points.

Sequential third order designs can also be constructed with the help of the complementary B.I.B. designs together with three other types of combinations involving fresh unknowns.

For example, with the following points we can get a sequential third order design for 7 factors:

Block I (i) 112 points of a-(7, 4, 4, 7, 2) $\times 2^4$

(ii) 14 points of $(b\ 0\ 0 \cdots 0) \times 2$

Block II (iii) 112 points from a-(7, 3, 3, 7, 1) $\times 2^3$,

the design in (iii) being the complementary design of the B.I.B. design in (i). Each of the 56 points in (iii) is to be repeated once more.

Here, $\mu = 1$ in the combined designs. The complementary B.I.B. design in (iii) has to be repeated once more as each a-combination from the first B.I.B. design gives 16 design points on "multiplication" with the associate combinations and each a-combination from the complementary B.I.B. design gives only 8 combinations on "multiplication" with the associate combinations. Hence unless all the points obtained from the a-combinations of the complementary design be repeated once, $\sum x_i^2 x_j^2 x_k^2$ will not be constant for all i, j, k, $i \neq j \neq k$.

In the case of the above design points condition D_1(ii) is satisfied, as $\lambda = 3\mu$ and the requirement that each block is a second order rotatable design is satisfied as $r = 3\lambda$ in block II.

Relations D and D_1(i) give the equations,

$$112a^4 + 2b^4 = (3 \times 48) \times a^4.$$

$$112a^6 + 2b^6 = (5 \times 48)a^6.$$

Putting $b^2/a^2 = t$, these equations become $t^2 = 16$ and $t^3 = 64$. Hence $t = 4$. a can be obtained from the equation $\sum x_i^2 = N$.

Numbers of central points to be added to the two blocks are given by the relation,

$$48a^2/(64a^2 + 2b^2) = (112 + n_{10})/(126 + n_{20}),$$

i.e.

$$48/(64 + 2t) = (112 + n_{10})/(126 + n_{20}).$$

Thus, we get a sequential third order rotatable design for 7 factors in 238 non-central points, with some central points to be added.

Appendices II and III present respectively non-sequential and sequential third order rotatable designs up to 15 factors obtained by utilizing doubly balanced incomplete block designs or by a B.I.B. design together with its complementary B.I.B. design. Some of the designs in the appendices, particularly for small number of factors, have been obtained by others through other methods.

APPENDIX I

List of Second Order Rotatable Designs

No. of Factors. (v)	Types of combinations with the associate design to be used for "multiplication."	Number of points from each of the combinations	Solutions in terms of a^2
Col. (1)	Col. (2)	Col. (3)	Col. (4)
3	a-(4, 2, 3, 6, 1) $\times 2^2$ $(b\ 0\ 0) \times 2$	12 6	$b^2/a^2 = \sqrt{2}$
4	a-(4, 3, 3, 4, 2) $\times 2^3$ $(b\ 0\ 0\ 0) \times 2$	32 8	$b^2/a^2 = 2\sqrt{3}$
5	a-(5, 2, 4, 10, 1) $\times 2^2$ $(b\ b\ b\ b\ b) \times \frac{1}{2}$ repl 2^5	40 16	$b^2/a^2 = 1/(2\sqrt{2})$
6(i)	a-(6, 2, 5, 15, 1) $\times 2^2$ $(b\ b \cdots b) \times \frac{1}{2}$ repl 2^6	60 32	$b^2/a^2 = 1/(2\sqrt{2})$
6(ii)	a-(6, 3, 5, 10, 2) $\times 2^3$ $(b\ 0\ 0 \cdots 0) \times 2$	80 12	$b^2/a^2 = 2$
8(i)	a-(8, 2, 7, 28, 1) $\times 2^2$ $(b\ b \cdots b) \times \frac{1}{4}$ repl 2^8	112 64	$b^2/a^2 = 1/(2\sqrt{2})$
8(ii)	a-(8, 4, 7, 14, 3) $\times 2^4$ $(b\ 0\ 0 \cdots 0) \times 2$	224 16	$b^2/a^2 = 4$
9	a-(9, 3, 4, 12, 1) $\times 2^3$ $(b\ b \cdots b) \times \frac{1}{4}$ repl 2^9	96 128	$b^2/a^2 = 1/(4\sqrt{2})$
10	a-(10, 4, 6, 15, 2) $\times 2^4$	240	
11	a-(11, 5, 5, 11, 2) $\times \frac{1}{2}$ repl 2^5 $(b\ 0\ 0 \cdots 0) \times 2$	176 22	$b^2/a^2 = 2\sqrt{2}$
12(i)	a-(12, 6, 11, 22, 5) $\times \frac{1}{2}$ repl 2^6 $(b\ 0 \cdots 0) \times 2$	704 24	$b^2/a^2 = 8$
12(ii)	a-(12, 2, 11, 66, 1) $\times 2^2$ $(b\ b \cdots b) \times \frac{1}{8}$ repl 2^{12}	264 512	$b^2/a^2 = 1/(4\sqrt{2})$
13(i)	a-(13, 4, 4, 13, 1) $\times 2^4$ $(b\ b \cdots b) \times \frac{1}{8}$ repl 2^{13}	208 1024	$b^2/a^2 = \frac{3}{16}$
13(ii)	a-(13, 3, 6, 26, 1) $\times 2^3$ $(b\ b \cdots b) \times \frac{1}{8}$ repl 2^{13}	208 1024	$b^2/a^2 = \frac{3}{16}$
14	a-(14, 2, 13, 91, 1) $\times 2^2$ $(b\ b \cdots b) \times \frac{1}{16}$ repl 2^{14}	364 1024	$b^2/a^2 = \frac{5}{16}$
15	a-(15, 7, 7, 15, 3) $\times \frac{1}{2}$ repl 2^7 $(b\ 0 \cdots 0) \times 2$	960 30	$b^2/a^2 = 8$
16	a-(16, 6, 6, 16, 2) $\times \frac{1}{2}$ repl 2^6	512	

APPENDIX II

List of Nonsequential Third Order Rotatable Designs
(Col. Nos. in Appendix II correspond to those in Appendix I)

Col. (1)	Col. (2)	Col. (3)	Col. (4)
3	a-(3, 2, 2, 3, 1, 0) $\times 2^2$	12	$b^2/a^2 = 2.109000$
	$(b\ 0\ 0) \times 2$	6	$c^2/a^2 = 0.852600$
	$(c\ 0\ 0) \times 2$	6	$d^2/a^2 = 0.629960$
	$(d\ d\ d) \times 2^3$	8	
4	a-(4, 3, 3, 4, 2, 1) $\times 2^3$	32	$a^2/d^2 = 0.793701$
	$(b\ 0\ 0\ 0) \times 2$	8	$b^2/d^2 = 2.577472$
	$(c\ 0\ 0\ 0) \times 2$	8	$c^2/d^2 = 0.957168$
	$(d\ d\ 0\ 0) \times 2^2$	24	
5(i)	a-(5, 3, 6, 10, 3, 1) $\times 2^3$	80	$b^2/a^2 = 3.247410$
	$(b\ 0\ 0\ 0\ 0) \times 2$	10	$c^2/a^2 = 1.205956$
	$(c\ 0\ 0\ 0\ 0) \times 2$	10	
5*(ii)	a-(5, 4, 4, 5, 3, 2) $\times 2^4$	80	$a^2/d^2 = 0.436790$
	$(b\ 0\ 0\ 0\ 0) \times 2$	10	$b^2/d^2 = 1.975158$
	$(c\ 0\ 0\ 0\ 0) \times 2$	10	$c^2/d^2 = 0.856008$
	$(d\ d\ 0\ 0\ 0) \times 2^2$	40	$e^2/d^2 = 1.000000$
	$(e\ 0\ 0\ 0\ 0) \times 2$	10	
6*(i)	a-(6, 3, 10, 20, 4, 1) $\times 2^3$	160	$a^2/d^2 = 2.519842$
	$(b\ 0 \cdots 0) \times 2$	12	$b^2/d^2 = 7.226732$
	$(c\ 0 \cdots 0) \times 2$	12	$c^2/d^2 = 3.683908$
	$(d\ d \cdots d) \times 2^6$	64	$e^2/d^2 = 7.000000$
	$(e\ 0 \cdots 0) \times 2$	12	
6*(ii)	a-(6, 4, 10, 15, 6, 3) $\times 2^4$	240	$a^2/d^2 = 0.436790$
	$(b\ 0 \cdots 0) \times 2$	12	$b^2/d^2 = 2.015918$
	$(c\ 0 \cdots 0) \times 2$	12	$c^2/d^2 = 1.465050$
	$(d\ d\ 0 \cdots 0) \times 2^2$	60	$e^2/d^2 = 1.000000$
	$(e\ 0\ 0 \cdots 0) \times 2$	12	$x^2/d^2 = 1.000000$
	$(x\ 0 \cdots 0) \times 2$	12	
6*(iii)	a-(6, 5, 5, 6, 4, 3) $\times 2^5$	192	$b^2/a^2 = 3.657940$
	$(b\ 0 \cdots 0) \times 2$	12	$c^2/a^2 = 1.272580$
	$(c\ 0 \cdots 0) \times 2$	12	$d^2/a^2 = 3.195920$
	$(d\ d\ 0 \cdots 0) \times 2^2$	60	$e^2/a^2 = 1.945794$
	$(e\ e\ 0 \cdots 0) \times 2^2$	60	$w^2/a^2 = 4.000000$
	$(w\ 0 \cdots 0) \times 2$	12	$x^2/a^2 = 5.000000$
	$(x\ 0 \cdots 0) \times 2$	12	

* Designs with asterisks have infinite number of solutions of which only one has been given.

APPENDIX II—*Continued*

Col. (1)	Col. (2)	Col. (3)	Col. (4)
7(i)	a-(7, 4, 4, 7, 2) $\times 2^4$	112	
	Complementary B. I. B. D.		
	a-(7, 3, 3, 7, 1) $\times 2^3$	112	$b^2/a^2 = 4$
	repeated once more		
	$(b\ 0 \cdots 0) \times 2$	14	
7(ii)	a-(7, 3, 15, 35, 5, 1) $\times 2^3$	280	$a^2/d^2 = 2.000000$
	$(b\ 0 \cdots 0) \times 2$	14	$b^2/d^2 = 7.542256$
	$(c\ 0 \cdots 0) \times 2$	14	$c^2/d^2 = 2.667280$
	$(d\ d \cdots d) \times \frac{1}{2}$ repl 2^7	64	
8*	a-(8, 4, 7, 14, 3, 1) $\times 2^4$	224	$a^2/d^2 = 1$
	$(b\ 0 \cdots 0) \times 2$	16	$b^2/d^2 = 4$
	$(c\ c\ 0 \cdots 0) \times 2^2$	112	$c^2/d^2 = 4$
	$(d\ d \cdots d) \times \frac{1}{2}$ repl 2^8	128	
9(i)	a-(9, 3, 28, 84, 7, 1) $\times 2^3$	672	$b^2/a^2 = 0.392768$
	$(b\ b \cdots b) \times \frac{1}{2}$ repl 2^9	256	$c^2/a^2 = 0.122376$
	$(c\ c \cdots c) \times \frac{1}{2}$ repl 2^9	256	$d^2/a^2 = 3.914868$
	$(d\ 0 \cdots 0) \times 2$	18	
9(ii)	a-(9, 5, 10, 18, 5) $\times 2^5$	576	
	Complementary B. I. B. D.		
	a-(9, 4, 8, 18, 3) $\times 2^4$	576	$b^2/a^2 = 5.944129$
	repeated once more.		
	$(b\ 0 \cdots 0) \times 2$	18	$c^2/a^2 = 4.546079$
	$(c\ 0 \cdots 0) \times 2$	18	$d^2/a^2 = 2.000000$
	$(d\ d\ 0 \cdots 0) \times 2^2$	144	
10(i)	a-(10, 4, 12, 30, 4, 1) $\times 2^4$	480	
	$(b\ b \cdots b) \times \frac{1}{2}$ repl 2^{10}	512	$b^2/a^2 = 0.248096$
	$(c\ c \cdots c) \times \frac{1}{2}$ repl 2^{10}	512	$c^2/a^2 = 0.064468$
	$(d\ 0 \cdots 0) \times 2$	20	$d^2/a^2 = 2.371260$
	$(d\ 0 \cdots 0) \times 2$	20	
	$(d\ 0 \cdots 0) \times 2$	20	
	$(d\ 0 \cdots 0) \times 2$	20	
	$(d\ 0 \cdots 0) \times 2$	20	
	$(d\ 0 \cdots 0) \times 2$	20	
10(ii)	a-(10, 5, 9, 18, 4) $\times 2^5$	576	
	Complementary B. I. B. D.		
	a-(10, 5, 9, 18, 4) $\times 2^5$	576	$b^2/a^2 = 6.494805$
	$(b\ 0 \cdots 0) \times 2$	20	$c^2/a^2 = 2.411955$
	$(c\ 0 \cdots 0) \times 2$	20	$d^2/a^2 = 2.000000$
	$(d\ d\ 0 \cdots 0) \times 2^2$	180	
11(i)	a-(11, 5, 15, 33, 6, 2) $\times 2^5$	1056	$b^2/a^2 = 5.443720$
	$(b\ 0 \cdots 0) \times 2$	22	$c^2/a^2 = 4.285562$
	$(c\ 0 \cdots 0) \times 2$	22	

APPENDIX II—*Continued*

Col. (1)	Col. (2)	Col. (3)	Col. (4)
11*(ii)	a-(11, 6, 6, 11, 3) $\times 2^6$	704	
	Complementary B. I. B. D.		
	a-(11, 5, 5, 11, 2) $\times 2^5$	704	$b^2/a^2 = 0.572357$
	repeated once more		
	$(b\ b \cdots b) \times \frac{1}{4}$ repl 2^{11}	512	$c^2/a^2 = 3.647317$
	$(c\ c\ 0 \cdots 0) \times 2^2$	220	$d^2/a^2 = 1.954600$
	$(d\ d\ 0 \cdots 0) \times 2^2$	220	$e^2/a^2 = 2.000000$
	$(e\ e\ 0 \cdots 0) \times 2^2$	220	
11*(iii)	a-(11, 3, 45, 165, 9, 1) $\times 2^3$	1320	$b^2/a^2 = 0.333192$
	$(b\ b \cdots b) \times \frac{1}{4}$ repl 2^{11}	512	$c^2/a^2 = 0.199449$
	$(c\ c \cdots c) \times \frac{1}{4}$ repl 2^{11}	512	$d^2/a^2 = 0.125000$
	$(d\ d \cdots d) \times \frac{1}{4}$ repl 2^{11}	512	$e^2/a^2 = 3.634241$
	$(e\ 0 \cdots 0) \times 2$	22	
12*(i)	a-(12, 6, 11, 22, 5, 2) $\times 2^6$	1408	$b^2/a^2 = 3.161774$
	$(b\ b\ 0 \cdots 0) \times 2^2$	264	$c^2/a^2 = 2.000797$
	$(c\ c\ 0 \cdots 0) \times 2^2$	264	$d^2/a^2 = 0.396850$
	$(d\ d \cdots d) \times \frac{1}{4}$ repl 2^{12}	1024	$e^2/a^2 = 2.000000$
	$(e\ e\ 0 \cdots 0) \times 2^2$	264	
12(ii)	a-(12, 6, 11, 22, 5) $\times 2^6$	1408	
	Complementary B. I. B. D.		
	a-(12, 6, 11, 22, 5) $\times 2^6$	1408	$b^2/a^2 = 8.000000$
	$(b\ 0 \cdots 0) \times 2$	24	$c^2/a^2 = 2.983100$
	$(c\ c\ 0 \cdots 0) \times 2^2$	264	$d^2/a^2 = 1.761050$
	$(d\ d\ 0 \cdots 0) \times 2^2$	264	
12(iii)	a-(12, 3, 55, 220, 10, 1) $\times 2^3$	1760	$b^2/a^2 = 0.282368$
	$(b\ b \cdots b) \times \frac{1}{4}$ repl 2^{12}	1024	$c^2/a^2 = 0.169032$
	$(c\ c \cdots c) \times \frac{1}{4}$ repl 2^{12}	1024	$d^2/a^2 = 3.301927$
	$(d\ 0 \cdots 0) \times 2$	24	
13*(i)	a-(13, 6, 12, 26, 5) $\times 2^6$	1664	
	Complementary B. I. B. D.		
	a-(13, 7, 14, 26, 7) $\times \frac{1}{2}$ repl 2^7	1664	$b^2/a^2 = 6.802642$
	$(b\ 0 \cdots 0) \times 2$	26	$c^2/a^2 = 4.660909$
	$(c\ 0 \cdots 0) \times 2$	26	$d^2/a^2 = 3.610148$
	$(d\ d\ 0 \cdots 0) \times 2^2$	312	$e^2/a^2 = 0.983286$
	$(e\ e\ 0 \cdots 0) \times 2^2$	312	
13(ii)	a-(13, 4, 44, 143, 11, 2) $\times 2^4$	2288	$a^2/d^2 = 3.030288$
	$(b\ b\ 0 \cdots 0) \times 2^2$	312	$b^2/d^2 = 7.578788$
	$(c\ c\ 0 \cdots 0) \times 2^2$	312	$c^2/d^2 = 3.180174$
	$(d\ d\ d \cdots d) \times \frac{1}{4}$ repl 2^{13}	2048	
14*	a-(14, 7, 13, 26, 6) $\times \frac{1}{2}$ repl 2^7	1664	
	Complementary B. I. B. D.		

APPENDIX II—*Concluded*

Col. (1)	Col. (2)	Col. (3)	Col. (4)
14*	a-(14, 7, 13, 26, 6) $\times$ $\frac{1}{2}$ repl 2^7	1664	
	(b 0 $\cdots$ 0) $\times$ 2	28	$b^2/a^2 = 6.782826$
	(c 0 $\cdots$ 0) $\times$ 2	28	$c^2/a^2 = 1.998312$
	(d d 0 $\cdots$ 0) $\times$ 2^2	364	$d^2/a^2 = 3.629538$
	(e e 0 $\cdots$ 0) $\times$ 2^2	364	$e^2/a^2 = 0.571362$
15	a-(15, 7, 7, 15, 3) $\times$ 2^7	1920	
	Complementary B. I. B. D.		
	a-(15, 8, 8, 15, 4) $\times$ $\frac{1}{2}$ repl 2^8	1920	
	(b 0 $\cdots$ 0) $\times$ 2	30	$b^2/a^2 = 4$
	(c 0 $\cdots$ 0) $\times$ 2	30	$c^2/a^2 = 4$
	(d d 0 $\cdots$ 0) $\times$ 2^2	420	$d^2/a^2 = 4$

APPENDIX III

List of Sequential Third Order Rotatable Designs

(Col. numbers in this Appendix correspond to those in the previous Appendices)

Col. (1)	Col. (2)	Col. (3)	Col. (4)
3	Block No. 1		
	a-(3, 2, 2, 3, 1, 0) $\times 2^2$	12	
	$(b\ 0\ 0) \times 2$	6	$b^2/a^2 = 1.41421$
	Block No. 2		
	$(c\ 0\ 0) \times 2$	6	$c^2/a^2 = 1.92849$
	$(d\ d\ d) \times 2^3$	8	$d^2/a^2 = 0.32390$
	$(w\ w\ w) \times 2^3$	8	$w^2/a^2 = 0.60000$
4	Block No. 1		
	$(d\ d\ 0\ 0) \times 2^2$	24	$d^2/a^2 = 1.259921$
	Block No. 2		
	a-(4, 3, 3, 4, 2, 1) $\times 2^3$	32	
	$(b\ 0\ 0\ 0) \times 2$	8	$b^2/a^2 = 3.247410$
	$(c\ 0\ 0\ 0) \times 2$	8	$c^2/a^2 = 1.205956$
5*	Block No. 1		
	$(c\ c\ 0\ 0\ 0) \times 2^2$	40	$c^2/d^2 = 4.000000$
	$(d\ d\ d\ d\ d) \times 2^5$	32	
	Block No. 2		
	a-(5, 4, 4, 5, 3, 2) $\times 2^4$	80	$a^2/d^2 = 1.587401$
	$(b\ 0\ 0\ 0\ 0) \times 2$	10	$b^2/d^2 = 6.415804$
	$(e\ 0\ 0\ 0\ 0) \times 2$	10	$e^2/d^2 = 3.103404$
	$(w\ 0\ 0\ 0\ 0) \times 2$	10	$w^2/d^2 = 5.000000$
	$(x\ 0\ 0\ 0\ 0) \times 2$	10	$x^2/d^2 = 5.000000$
6	Block No. 1		
	$(d\ d \cdots d) \times 2^6$	64	
	$(e\ 0 \cdots 0) \times 2$	12	$e^2/d^2 = 8.000000$
	Block No. 2		
	a-(6, 3, 10, 20, 4, 1) $\times 2^3$	160	$a^2/d^2 = 2.519842$
	$(b\ 0 \cdots 0) \times 2$	12	$b^2/d^2 = 5.039684$
	$(c\ 0 \cdots 0) \times 2$	12	$c^2/d^2 = 5.039684$
7	Block No. 1		
	a-(7, 3, 3, 7, 1) $\times 2^3$	112	
	repeated once more		
	Block No. 2		
	Complementary B. I. B. D.		
	a-(7, 4, 4, 7, 2) $\times 2^4$	112	
	$(b\ 0 \ldots 0) \times 2$	14	$b^2/a^2 = 4$
8	Block No. 1		
	$(d\ d \cdots d) \times \frac{1}{2}$ repl. 2^8	128	
	$(e\ 0 \cdots 0) \times 2$	16	$e^2/d^2 = 8\sqrt{2}$
	Block No. 2		
	a-(8, 4, 7, 14, 3, 1) $\times 2^4$	224	$a^2/d^2 = 2\sqrt{2}$
	$(c\ c\ 0 \cdots 0) \times 2^2$	112	$c^2/d^2 = 4$

Col. (1)	Col. (2)	Col. (3)	Col. (4)
9*(i)	Block No. 1		
	$(c\ c \cdots c) \times \frac{1}{2}$ repl 2^9	256	$c^2/b^2 = 0.899121$
	$(d\ 0 \cdots 0) \times 2$	18	$d^2/b^2 = 8.507872$
	$(e\ 0 \cdots 0) \times 2$	18	$e^2/b^2 = 2.563504$
	$(w\ 0 \cdots 0) \times 2$	18	$w^2/b^2 = 8.000000$
	$(x\ 0 \cdots 0) \times 2$	18	$x^2/b^2 = 8.000000$
	Block No. 2		
	a-(9, 3, 28, 84, 7, 1) $\times 2^3$	672	$a^2/b^2 = 3.023716$
	$(b\ b \cdots b) \times \frac{1}{2}$ repl 2^9	256	
9(ii)	Block No. 1		
	a-(9, 4, 8, 18, 3) $\times 2^4$	576	
	repeated once more		
	$(e\ 0 \cdots 0) \times 2$	18	$e^2/a^2 = 4.000000$
	Block No. 2		
	Complementary B. I. B. D. to above design		
	a-(9, 5, 10, 18, 5) $\times 2^5$	576	
	$(b\ 0 \cdots 0) \times 2$	18	$b^2/a^2 = 6.196762$
	$(c\ 0 \cdots 0) \times 2$	18	$c^2/a^2 = 1.264954$
	$(d\ d\ 0 \cdots 0) \times 2^2$	144	$d^2/a^2 = 2.000000$
10	Block No. 1		
	a-(10, 5 9, 18, 4) $\times 2^5$	576	
	$(b\ 0 \cdots 0) \times 2$	20	$b^2/a^2 = 6.494805$
	$(c\ 0 \cdots 0) \times 2$	20	$c^2/a^2 = 2.411955$
	Block No. 2		
	Complementary B. I. B. D. to above design		
	a-(10, 5, 9, 18, 4) $\times 2^5$	576	
	$(d\ d\ 0 \cdots 0) \times 2^2$	180	$d^2/a^2 = 2.000000$
11(i)	Block No. 1		
	$(b\ b\ 0 \cdots 0) \times 2^2$	220	$b^2/a^2 = 3.225490$
	$(c\ c\ 0 \cdots 0) \times 2^2$	220	$c^2/a^2 = 0.453387$
	$(d\ d \cdots d) \times \frac{1}{4}$ repl 2^{11}	512	$d^2/a^2 = 0.538609$
	Block No. 2		
	a-(11, 5, 15, 33, 6, 2) $\times 2^5$	1056	
	$(e\ e\ 0 \cdots 0) \times 2^2$	220	$e^2/a^2 = 1.851640$
11(ii)	Block No. 1		
	a-(11, 5, 5, 11, 2) $\times 2^5$	704	
	repeated once more		
	$(e\ e\ 0 \cdots 0) \times 2^2$	220	$e^2/a^2 = 1.511858$
	Block No. 2		
	Complementary B. I. B. D. to above design		
	a-(11, 6, 6, 11, 3) $\times 2^6$	704	
	$(b\ b \cdots b) \times \frac{1}{4}$ repl 2^{11}	512	$b^2/a^2 = 0.572357$
	$(c\ c\ 0 \cdots 0) \times 2^2$	220	$c^2/a^2 = 3.572516$
	$(d\ d\ 0 \cdots 0) \times 2^2$	220	$d^2/a^2 = 2.464714$

APPENDIX III—*Concluded*

Col. (1)	Col. (2)	Col. (3)	Col. (4)
12	Block No. 1		
	$(b\ b\ 0 \cdots 0) \times 2^2$	264	$b^2/a^2 = 2.806072$
	$(c\ c\ 0 \cdots 0) \times 2^2$	264	$c^2/a^2 = 1.485024$
	$(d\ d\ d \cdots d) \times \frac{1}{4}$ repl 2^{12}	1024	$d^2/a^2 = 0.396850$
	Block No. 2		
	a-(12, 6, 11, 22, 5, 2) $\times 2^6$	1408	
	$(e\ e\ 0 \cdots 0) \times 2^2$	264	$e^2/a^2 = 2.828427$
13*	Block No. 1		
	a-(13, 6, 12, 26, 5) $\times 2^6$	1664	
	$(e\ e\ 0 \cdots 0) \times 2^2$	312	$e^2/a^2 = 2.309401$
	Block No. 2		
	Complementary B. I. B. D. to above design		
	a-(13, 7, 14, 26, 7) $\times \frac{1}{2}$ repl 2^7	1664	
	$(c\ c\ 0 \cdots 0) \times 2^2$	312	$c^2/a^2 = 3.253305$
	$(d\ d\ 0 \cdots 0) \times 2^2$	312	$d^2/a^2 = 2.511237$
	$(b\ b\ b \cdots b) \times \frac{1}{4}$ repl 2^{13}	2048	$b^2/a^2 = 0.343768$
	$(w\ w\ 0 \cdots 0) \times 2^2$	312	$w^2/a^2 = 3.000000$
14*	Block No. 1		
	a-(14, 7, 13, 26, 6) $\times \frac{1}{2}$ repl 2^7	1664	
	$(e\ e\ 0 \cdots 0) \times 2^2$	364	$e^2/a^2 = 2.828427$
	Block No. 2		
	Complementary B. I. B. D. to above design		
	a-(14, 7, 13, 26, 6) $\times \frac{1}{2}$ repl 2^7	1664	
	$(b\ b \cdots b) \times \frac{1}{4}$ repl. 2^{14}	4096	$b^2/a^2 = 0.235260$
	$(c\ c\ 0 \cdots 0) \times 2^2$	364	$c^2/a^2 = 3.172259$
	$(d\ d\ 0 \cdots 0) \times 2^2$	364	$d^2/a^2 = 2.296065$
	$(w\ w\ 0 \cdots 0) \times 2^2$	364	$w^2/a^2 = 2.000000$
15	Block No. 1		
	a-(15, 7, 7, 15, 3) $\times 2^7$	1920	
	$(e\ e\ 0 \cdots 0) \times 2^2$	420	$e^2/a^2 = 2.412091$
	Block No. 2		
	Complementary B. I. B. D. to above design		
	a-(15, 8, 8, 15, 3) $\times \frac{1}{2}$ repl 2^8	1920	
	$(b\ b \cdots b) \times \frac{1}{8}$ repl 2^{15}	4096	$b^2/a^2 = 0.164658$
	$(c\ c\ 0 \cdots 0) \times 2^2$	420	$c^2/a^2 = 3.832317$
	$(d\ d\ 0 \cdots 0) \times 2^2$	420	$d^2/a^2 = 1.413329$

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