ON A COMPLETE CLASS OF LINEAR UNBIASED ESTIMATORS FOR RANDOMIZED FACTORIAL EXPERIMENTS[1]

By S. Zacks[2]

Technion, Israel Institute of Technology, and Columbia University

1. Introduction and summary. Consider a factorial system of order $N = p^m$, which consists of m factors each at p levels. The factorial model relates the expected yield to the various treatment combinations in terms of a linear function of $N = p^m$ parameters $(\beta_0, \beta_1, \cdots, \beta_{N-1})$, which represent the main effects and interactions. A subset of $S = p^s (s < m)$ preassigned parameters is specified for estimation and testing of hypotheses. The other $N - S$ parameters are considered as nuisance ones. Unbiased estimates of the $S = p^s$ preassigned parameters may be obtained by different procedures of balanced random allocation designs (see Dempster [2], [3]). In the present paper we consider unbiased estimators with two types of randomized fractional replication, studied previously by Ehrenfeld and Zacks [4]. These procedures, designated by R.P.I. and R.P.II, are based on orthogonal fractional replication designs. As will be shown in a subsequent paper, both procedures have some optimal properties in cases where the nuisance parameters may assume arbitrary values. R.P.I. consists of choosing at random, with or without replacement, n blocks of treatment combinations from the set of $M = p^{m-s}$ blocks, constructed by confounding the nuisance parameters. R.P.II. consists of choosing at random n treatment combinations independently from each one of the p^s blocks, constructed by confounding the pre-assigned parameters. The estimator of the pre-assigned parameters studied in the previous paper is the "least squares estimator", commonly applied in fractional replication procedures (see Kempthorne [5], Cochran and Cox [1]). When all the nuisance parameters are zero then this estimator is the least-squares estimator, and thus the best unbiased linear estimator. However, if the nuisance parameters are not zero there is no uniformly best unbiased estimator. The first question raised is whether unbiased estimates of the pre-assigned parameters can be attained by choosing a block of treatment combinations, in a similar manner to R.P.I., but with unequal probabilities, (unbalanced designs). As proven in the present paper, unbiased estimates *cannot* be attained with a procedure that assigns unequal probabilities to different blocks. Thus, the class of linear unbiased estimators is studied with respect to R.P.I. The same estimators can be applied to R.P.II., or to any balanced allocation design that yields a factorial model with similar properties to those of R.P.I.

The statistical model adopted in the present paper is the orthogonal factorial

Received March 7, 1962; revised March 7, 1963.

[1] Part of a Ph.D. dissertation submitted in the Department of Industrial and Management Engineering, Columbia University, under the sponsorship of the U. S. Government, Department of State, Agency for International Development.

[2] Now at Stanford University.

model. Accordingly, the main effects and interactions are represented by orthogonal contrasts of the expected values of the observed random variable. This is a common practice for measuring main-effects and interactions (see Kempthorne [5]). The resulting algebra and numerical procedures are relatively simple and free of the difficult problems associated with the approach of Dempster [2], [3]. Basic notions as well as the required algebra for the factorial model are represented in Section 2. Theorems concerning the specification of the class of linear unbiased estimators, and the characterization of the subclass of conditional least squares estimators are given in Section 3. In Section 4 we prove that the subclass of conditional least squares estimators is complete. For this purpose we first derive an explicit formula for the variance-covariance matrix of any linear unbiased estimator with R.P.I.; and then we show that to any linear unbiased estimator not in the subclass of the conditional least-squares estimators one can find a conditional least-squares estimator so that the difference between the corresponding variance-covariance matrices is positive definite. For simplifying the arguments and notation, the definitions and theorems in Sections 3 and 4 are given relative to a particular choice of the pre-assigned parameters for special defining parameters, by which the classification of the treatment combinations into blocks in R.P.I. is carried out. In Section 5 the results are generalized for arbitrary sets of pre-assigned and defining parameters. In Section 6 the results are extended for R.P.II.

2. Basic notions and the statistical model. Consider a factorial system of order $N = p^m$, where p is a prime integer, $p > 1$. The set X of all N treatment combinations is represented by

$$X = ((i_0, \cdots, i_{m-1}): \quad i_j = 0, \cdots, p-1 \quad \text{for all} \quad j = 0, \cdots, m-1). \tag{2.1}$$

The jth coordinate of a point in X represents the i_jth level of factor $j(j = 0, \cdots, m-1)$. A *standard order* of the points x in X is given by the relationship between the coordinates of a point $x_v \equiv (i_0, \cdots, i_{m-1})$ and the order subscript $v = \sum_{j=0}^{m-1} i_j p^j$. Let $Y(x_v)$ be a random variable associated with x_v. The relationship between the expected value of $Y(x_v)$ and the treatment combination x_v is given, according to the statistical model, by a linear function of $N = p^m$ parameters $\beta_0, \beta_1, \cdots, \beta_{N-1}$ as follows:

$$EY(x_v) = \sum_{u=0}^{N-1} c_u^{(N)}(x_v)\beta_u \qquad \text{for all} \quad v = 0, \cdots, N-1 \tag{2.2}$$

and

$$Y(x_v) = EY(x_v) + \epsilon_v \qquad \text{for all} \quad v = 0, \cdots, N-1 \tag{2.3}$$

where $\epsilon_v(v = 0, \cdots, N-1)$ are identically distributed independent random variables, with $E\epsilon_v = 0$ and $E\epsilon_v^2 = \sigma^2$. These variables represent the experimental errors. The parameters $\beta_u(u = 0, \cdots, N-1)$ have the usual interpretation of main effects and interactions measured by linear orthogonal contrasts of $EY(x_v)$. Accordingly, the coefficients $c_u^{(N)}(x_v)$ which depend on x_v are related to the orthogonal polynomials of order p (see Ehrenfeld and Zacks [4]). For simplicity, let $c_{vu}^{(N)} \equiv c_u^{(N)}(x_v)$. Furthermore, let $(C^{(p)})$ be the matrix whose column vectors are the coefficients of orthogonal polynomials of order p. The matrix $(C^{(N)})$ is given by the recursive relationship

$$(C^{(N)}) = (C^{(p)}) \otimes (C^{(N/p)}) \tag{2.4}$$

for all $N = p^m$, $m = 1, 2, \cdots$, where $(C^{(1)}) \equiv 1$ (scalar); and where $A \times B$ is the Kronecker's direct multiplication of A by B. From (2.4) and the properties of the coefficients of orthogonal polynomials, the following properties of $(C^{(N)})$ can be proved:

(i) $(C^{(N)})$ is non-singular, and $(C^{(N)})'(C^{(N)}) = (\Delta^{(N)})$; where $(\Delta^{(N)})$ is a diagonal matrix with

$$d_u^{(N)} = \sum_{v=0}^{N-1} (C_{vu}^{(N)})^2 \quad \text{for every} \quad u = 0, \cdots, N-1. \tag{2.5}$$

In particular, for $p = 2$ we have: $(C^{(N)})'(C^{(N)}) = (C^{(N)})(C^{(N)})' = NI^{(N)}; N = 2^m$ and $I^{(N)}$ being the identity matrix of order N.

(ii) The sum of the elements in any column vector of $(C^{(N)})$ excluding the first column, is zero. All the elements in the first column, and in the last row, are unit elements.

(iii) The elements of $(C^{(N)})$ are related to those of $(C^{(S)})$ and $(C^{(M)})$, $N = 2^m$, $S = 2^s$, $M = 2^{m-s}$, $m > s$, according to the relationship: $(C^{(N)}) = (C^{(M)}) \otimes (C^{(S)})$, i.e.,

$$c_{i+jS,t}^{(N)} = c_{ir_t}^{(S)} \cdot c_{jq_t}^{(M)} \quad \text{for all} \quad i = 0, \cdots, S-1; \tag{2.6}$$

where $t = r_t + q_t S(r_t = 0, \cdots, S-1; q_t = 0, \cdots, M-1)$.

(iv) In a 2^m factorial system, the elements of $(C^{(N)})$ in any row, and in columns corresponding to β_{u_1}, β_{u_2} and β_k are related by the rule:

$$c_{vk}^{(N)} = c_{vu_1}^{(N)} \cdot c_{vu_2}^{(N)} \quad \text{for all} \quad v = 0, \cdots, N-1 \tag{2.7}$$

where u_1, u_2 and k are related in the following way: If $u_1 = \sum_{j=0}^{m-1} \lambda_j z^j (\lambda_j = 0, 1$ for all $j = 0, \cdots, m-1)$ and $u_2 = \sum_{j=0}^{m-1} \lambda_j' 2^j (\lambda_j' = 0, 1$ for all $j = 0, \cdots, m-1)$ then $k = \sum_{j=0}^{m-1} \lambda_j'' 2^j$ where $\lambda_j'' \equiv \lambda_j + \lambda_j'$ (mod. 2) for all $j = 0, \cdots, m-1$.

The structural relationships between the $N = p^m$ parameters β_u are considered in terms of a set $B = ((\lambda_0, \cdots, \lambda_{m-1}) : \lambda_j = 0, \cdots, p-1$ for all $j = 0, \cdots, m-1)$. Every parameter β_u is represented by an m-tuple $(\lambda_0, \cdots, \lambda_{m-1})$ such that $u = \sum_{j=0}^{m-1} \lambda_j p^j$. An operator $\otimes$ of direct multiplication is defined over the set B as follows: if $\beta \equiv (\lambda_0, \cdots, \lambda_{m-1})$ and $\beta' \equiv (\lambda_0', \cdots, \lambda_{m-1}')$ then $\beta \otimes \beta' \equiv (\lambda_0'', \cdots, \lambda_{m-1}'')$ where $\lambda_j'' \equiv \lambda_j + \lambda_j'$ (mod. p) for every $j = 0, \cdots, m-1$. It follows immediately that B is a group with respect to the operator $\otimes$, with unit element $\beta_0 \equiv (0, 0, \cdots, 0)$; and the inverse of $\beta \equiv (\lambda_0, \cdots, \lambda_{m-1})$ is

$\beta^{-1} = (\lambda_0^*, \cdots, \lambda_{m-1}^*)$ where $\lambda_j^* \equiv p - \lambda_j \pmod{p}$ for every $j = 0, \cdots, m-1$. Let $[\beta]^a (a = 0, \cdots, p-1)$ denote the direct multiplication of β itself a-times, where $[\beta]^0 \equiv \beta_0$. A parameter β_u is said to be *independent* of a set of n parameters $\{\beta_{u_1}, \cdots, \beta_{u_n}\}$ if there are no n numbers $a_1, \cdots, a_n$ such that $\sum_{k=1}^n a_k > 0$, and $\beta_u \equiv [\beta_{u_1}]^{a_1} \otimes [\beta_{u_2}]^{a_2} \otimes \cdots \otimes [\beta_{u_n}]^{a_n}$. Every set of $s (s \leqq m)$ independent parameters in B generates a subgroup of order $S = p^s$. Let $\alpha' = (\beta_0, \beta_{t_1}, \cdots, \beta_{t_{S-1}})$ be a vector of $S = p^s (s < m)$ pre-assigned parameters, subject for estimation and testing of hypotheses, where $t_k < t_{k+1}$ and $t_0 \equiv 0$ for all $k = 0, \cdots, S-1$. Let $\{\beta_{d_0}, \cdots, \beta_{d_{m-s-1}}\}$ be a set of $m - s$ independent parameters, disjoint of the set of $S = p^s$ pre-assigned parameters. The $N = p^{m-s}$ parameters in the subgroup generated by $\{\beta_{d_0}, \cdots, \beta_{d_{m-s-1}}\}$ are called *defining parameters*. The $N = p^m$ treatment combinations x in X are classified into $M = p^{m-s}$ disjoint subsets $X_v (v = 0, \cdots, M-1)$ called blocks, relative to a given set of $m - s$ independent defining parameters, according to the system of linear equations:

$$(2.8) \qquad \sum_{j=0}^{m-1} i_j \lambda_{j,d_k} \equiv a_k \pmod{p} \qquad \text{for all } k = 0, \cdots, m-s-1$$

where $\beta_{d_k} \equiv (\lambda_{0,d_k}, \lambda_{1,d_k}, \cdots, \lambda_{m-1,d_k})$. Every $x \equiv (i_0, \cdots, i_{m-1})$ which satisfies (2.8) for a given $(m-s)$-tuple $(a_0, \cdots, a_{m-s-1})$, where $a_k = 0, \cdots, p-1$, is contained in X_v with $v = \sum_{k=0}^{m-s-1} a_k p^k$. For the objectives of the present study we can assume, without loss of generality, that the vector of pre-assigned parameters is $\alpha' = (\beta_0, \beta_1, \cdots, \beta_{S-1})$, and that the subgroup of defining parameters is generated by the parameters $\{\beta_S, \beta_{2S}, \cdots, \beta_{N-S}\}$, representing the $(m - s)$ "main effects" which are not in α. In Section 5 we show how to treat the general case. Relative to this set of independent defining parameters, the blocks of treatment combinations are:

$$(2.9) \quad X_v = \left\{ x: \text{if } x \equiv (i_0, \cdots, i_{m-1}) \text{ and } t = \sum_{j=0}^{m-1} i_j p^j \text{ then } t \equiv v (\text{mod.}\, p^{m-s}) \right\}.$$

Let $Y(X_v)$, $v = 0, \cdots, M-1$, be a vector of random variables associated with a block X_v; i.e., $Y(X_v)' = (Y(x_{vS}), Y(x_{1+vS}), \cdots, Y(x_{(v+1)S-1}))$. The statistical model is

$$(2.10) \qquad Y(X_v) = (C^{(S)})\alpha + \sum_{u=1}^{M-1} c_{vu}^{(M)} (C^{(S)}) \beta_{(u)} + \epsilon_v$$

for all $v = 0, \cdots, M-1$; where:

(i) $\beta_{(u)} (u = 1, \cdots, M-1)$ designates the vector of parameters alias to the components of α with respect to the defining parameter β_{uS}, i.e., $\beta'_{(u)} = (\beta_{uS}, \beta_{1+uS}, \cdots, \beta_{(u+1)S-1})$.

(ii) ϵ_v is a random vector, independent of X_v, with $E\epsilon_v = 0$ and $\mathfrak{D}(\epsilon_v) = \sigma^2 I^{(S)}$ for all $v = 0, \cdots, M-1$.

Denote by β the vector of all nuisance parameters, i.e.,

$$\beta' = (\beta'_{(1)}, \beta'_{(2)}, \cdots, \beta'_{(M-1)}).$$

Then, the statistical model (2.10) can be written in the form

$$Y(X_v) = (C^{(S)})\alpha + (H_v)\beta + \epsilon_v \quad \text{for all } v = 0, \cdots, M-1 \tag{2.11}$$

where (H_v) is a random rectangular matrix of order $S \times (N - S)$, given by:

$$(H_v) = (c_{v1}^{(M)}, c_{v2}^{(M)}, \cdots, c_{v(M-1)}^{(M)}) \otimes (C^{(S)}). \tag{2.12}$$

3. The class of unbiased linear estimators. In the sequel we confine the discussion to factorial systems of order 2^m. As before, let $N = 2^m$, $M = 2^{m-s} (s < m)$ and $S = 2^s$. We prove first that unbiased estimates of α can be obtained only under a randomization procedure with equal probabilities, *i.e.*, $\xi = (1/M)1^{(M)}$.

THEOREM 3.1. *Let $Y(X_i)$ be a random vector associated with a block of treatment combinations $X_v(v = 0, \cdots, M-1)$ chosen with a probability vector $\xi' = (\xi_0, \cdots, \xi_{M-1})$. Then $E_\xi Y(X_v)$ is independent of the vector of nuisance parameters β if, and only if, $\xi = (1/M)1^{(M)}$.*

PROOF.

(i) Let $\xi = (1/M)1^{(M)}$ then according to (2.11) $E_\xi Y(X_v) = (C^{(S)})\alpha + (1/M)\sum_{v=0}^{M-1} (H_v)\beta$ since ϵ_v is independent of (H_v) and $E\epsilon_v = 0$. Moreover, from (2.12) it follows that

$$\sum_{v=0}^{M-1} (H_v)\beta = \left(\sum_{v=0}^{M-1} c_{v1}^{(M)}, \sum_{v=0}^{M-1} c_{v2}^{(M)}, \cdots, \sum_{v=0}^{M-1} c_{v(M-1)}^{(M)}\right) \otimes (C^{(S)})\beta = 0,$$

since $1^{(M)\prime}(C^{(M)}) = (M, 0, 0, \cdots, 0)$.

(ii) If $E_\xi(H_v)\beta = 0$ then it is necessary that $\sum_{v=0}^{M-1} \xi_v c_{vu}^{(M)} = 0$ for all $u = 1, \cdots, M-1$; since β is arbitrary and $(C^{(S)})$ non-singular. This necessary condition can be written in the form $\xi'(C^{(M)}) = (1, 0, 0, \cdots, 0)$ for all probability vectors ξ. Multiplying both sides, from the right, by $(C^{(M)})'$ we arrive at the necessary condition: $M\xi' = (1, 0, \cdots, 0)(C^{(M)})' = (1, 1, \cdots, 1)$. Or, equivalently, $\xi = (1/M)1^{(M)}$.

The randomization procedure in which each block X_v is assigned probability $1/M$ will be denoted henceforth, by R.P.I.; and the subscript ξ will be omitted from expectation and variances operators. The following theorem characterizes the general structure of all linear unbiased estimators with R.P.I.

THEOREM 3.2. *Every linear unbiased estimator of α with* R.P.I. *is of the form:*

$$\hat{\alpha}(\gamma, F_v) = (1/S)(C^{(S)})'[Y(X_v) - (H_v)\gamma] + (F_v)Y(X_v) \tag{3.1}$$

for all $v = 0, \cdots, M-1$. Where γ is any fixed vector of order $N - S$; (F_v) is an $S \times S$ matrix, independent of ϵ_v with the properties:

(i) $E(F_v) = (0)$

(ii) $E(F_v)(H_v) = (0)$.

PROOF. To prove that conditions (i) and (ii) are sufficient for the representation (3.1), we substitute in (3.1) expression (2.11) for $Y(X_v)$. We arrive at the form

$$\begin{aligned}\hat{\alpha}(\gamma, F_v) = \alpha &+ (1/S)(C^{(S)})'(H_v)(\beta - \gamma) + (F_v)(C^{(S)})\alpha \\ &+ (F_v)(H_v)\beta + [(1/S)(C^{(S)})' + (F_v)]\epsilon.\end{aligned} \tag{3.2}$$

Hence

$$E\{\hat{\alpha}(\gamma, F_\iota)\} = \alpha + (1/S)(C^{(S)})'E\{(H_v)\}(\beta - \gamma) + E\{(F_v)\}(C^{(S)})\alpha + E\{(F_v)(H_v)\}\beta + E\{[(1/S)(C^{(S)})' + (F_v)]\}E\{\epsilon\}. \tag{3.3}$$

To prove that Conditions (i) and (ii) are necessary for the representation (3.1) let $\hat{\alpha}$ be any linear unbiased estimator of α, and write:

$$\hat{\alpha} = f_v + (A_v)Y(X_v) \qquad \text{for all } v = 0, \cdots, M-1 \tag{3.4}$$

where f_v is a random vector of order $S \times 1$, (A_v) a square random matrix of order $S \times S$. Both (A_v) and f_v may depend on $(C^{(S)})$ and on (H_v) but not on $Y(X_v)$. Since $(C^{(S)})$ is non-singular we can write:

$$(A_v) = (C^{(S)})^{-1} + (F_v) \qquad \text{for all } v = 0, \cdots, M-1. \tag{3.5}$$

By substituting (3.5) in (3.4) we get:

$$\hat{\alpha} = f_v + [(C^{(S)})^{-1} + (F_v)]Y(X_v) = (1/S)(C^{(S)})'Y(X_\tau) + f_v + (F_v)Y(X_v).$$

From the non-singularity of the matrices $(C^{(M)})$ and $(C^{(S)})$ it follows that to every set of M vectors $\{f_0, \cdots, f_{M-1}\}$ such that $\sum_{v=0}^{M-1} f_v = 0$ corresponds a unique vector γ in $E^{(N-S)}$ which satisfies the system of equations: $-(C^{(S)})^{-1}(H_v)\gamma = f_v$ for every $v = 0, \cdots, M-1$. The condition that $\sum_{v=0}^{M-1} f_v = 0$ is necessary for the unbiasedness of $\hat{\alpha}$, since

$$E(1/S)(C^{(S)})'Y(X_v) = \alpha.$$

Otherwise Ef_v will be a linear function of α and β. Hence the form of an unbiased estimator is reduced to (3.1). Furthermore, since

$$E(1/S)(C^{(S)})'[Y(X_v) - (H_v)\gamma] = \alpha,$$

a necessary condition for the unbiasedness of (3.1) is that $E(F_v)Y(X_v) = 0$. Substituting (2.11) for $Y(X_v)$ we get the necessary condition:

$$E[(F_v)(C^{(S)})\alpha + (F_v)(H_v)\beta] = 0. \tag{3.6}$$

According to (2.12) we can write Condition (3.6) in the form:

$$[E\{(F_v)(1, c_{v1}^{(M)}, \cdots, c_{v(M-1)}^{(M)}) \otimes (C^{(S)})\}]\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0. \tag{3.7}$$

Since (α, β) is an arbitrary vector, a necessary condition for (3.7) is:

$$E[(F_v)(1, c_{v1}^{(M)}, \cdots, c_{v(M-1)}^{(M)}) \otimes (C^{(S)})] = [E(F_v)(C^{(S)}), E(F_v)(H_v)] = (0) \tag{3.8}$$

which is equivalent to Conditions (i) and (ii).

A subclass of linear unbiased estimators of α with R.P.I. is the set of all $\hat{\alpha}(\gamma, F_v)$ with $(F_v) \equiv (0)$ for all $v = 0, \cdots, M-1$. These estimators are given by:

$$\hat{\alpha}(\gamma) = (1/S)(C^{(S)})'[Y(X_v) - (H_v)\gamma], \qquad \gamma \in E^{(N-S)}. \tag{3.9}$$

We notice that the class of all $\hat{\alpha}(\gamma)$, $\gamma \varepsilon E^{(N-S)}$, constitutes the set of all parametric solutions to the normal equations:

$$\begin{aligned}&[(1, c_{v1}^{(M)}, \cdots, c_{v(M-1)}^{(M)}) \otimes (C^{(S)})]'[(1, c_{v1}^{(M)}, \cdots, c_{v(M-1)}^{(M)})\\ &\otimes (C^{(S)})]\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = [(1, c_{v1}^{(M)}, \cdots, c_{v(M-1)}^{(M)}) \otimes (C^{(S)})]'Y(X_v)\end{aligned} \tag{3.10}$$

when substituting γ for the unknown β. The estimators $\hat{\alpha}(\gamma)$ are called, therefore, *conditional least-squares estimators* (c.l.s.e.). In the next section it is proven that the class of all c.l.s.e. estimators, $\hat{\alpha}(\gamma)$, is *complete*. That is, if $\hat{\alpha}(\gamma, F_v)$ is not a c.l.s.e. ($(F_v) \not\equiv (0)$) and if $\updownarrow(\hat{\alpha})$ denotes the variance covariance matrix of $\hat{\alpha}$, then $\updownarrow(\hat{\alpha}(\gamma, F_v)) - \updownarrow(\hat{\alpha}(\gamma))$ is positive definite.

4. The completeness of the class of conditional least-squares estimators. The following lemma is required for the derivation of the variance-covariance matrix of a linear unbiased estimator $\hat{\alpha}(\gamma, F_v)$ under R.P.I.

LEMMA 4.1. *Let $\hat{\alpha}(\gamma, F_v) = \hat{\alpha}(\gamma) + (F_v)Y(X_v)$ be a linear unbiased estimator of α under* R.P.I. *Then,*

$$E(C^{(S)})^{-1}(H_v)(\beta - \gamma)[(C^{(S)})\alpha + (H_v)\beta]'(F_v)' = (0) \tag{4.1}$$

for all α, β and γ.

PROOF.

$$\begin{aligned}&E(C^{(S)})^{-1}(H_v)(\beta - \gamma)[(C^{(S)})\alpha + (H_v)\beta]'(F_v)'\\ &\quad = E(C^{(S)})^{-1}(H_v)(\beta - \gamma)\alpha'(C^{(S)})'(F_v)'\\ &\qquad + E(C^{(S)})^{-1}(H_v)(\beta - \gamma)\beta'(H_v)'(F_v)'.\end{aligned} \tag{4.2}$$

According to Condition (ii) of Theorem 3.2, $E\{(H_v) \mid (F_v)\} = (0)$. Hence

$$\begin{aligned}&E\{(C^{(S)})^{-1}(H_v)(\beta - \gamma)\alpha'(C^{(S)})'(F_v)'\}\\ &\quad = E\{(C^{(S)})^{-1}E\{(H_v) \mid (F_v)\}(\beta - \gamma)\alpha'(C^{(S)})'(F_v)'\} = (0)\end{aligned}$$

for all α, β and γ. It remains to show that the second term in the right hand side of (4.2) is (0). Let

$$\begin{aligned}(K) &= (F_v)(H_v)\beta(\beta - \gamma)'(H_v)'(C^{(S)'})^{-1}\\ &= \sum_{u_1=1}^{M-1}\sum_{u_2=1}^{M-1} c_{vu_1}^{(M)} c_{vu_2}^{(M)}(F_v)(C^{(S)})\beta_{(u_1)}(\beta - \gamma)'_{(u_2)}\end{aligned} \tag{4.3}$$

where $\beta' = (\beta'_{(1)}, \beta'_{(2)}, \cdots, \beta'_{(M-1)})$. By virtue of (2.7), if $u_1 = \sum_{j=0}^{m-s-1} i_j 2^j$ and $u_2 = \sum_{j=0}^{m-s-1} i'_j 2^j (i_j, i'_j = 0, 1)$ then $c_{vu_1}^{(M)} c_{vu_2}^{(M)} = c_{vu_3}^{(M)}$, for every $v = 0, \cdots, M - 1$, where $u_3 = \sum_{j=0}^{m-s-1} i''_j 2^j$ and $i''_j \equiv i_j + i'_j \pmod{2}$ for every $j = 0, \cdots, m - s - 1$. Denote $u_3 \equiv u_1 \oplus u_2$. Thus,

$$(K) = \sum_{u_1=1}^{M-1}\sum_{u_2=1}^{M-1} c_{v,u_1\oplus u_2}^{(M)}(F_v)(C^{(S)})\beta_{(u_1)}(\beta - \gamma)'_{(u_2)}. \tag{4.4}$$

If $u_1 \oplus u_2 = 0$ then $c_{v,u_1\oplus u_2}^{(M)} = 1$ for all $v = 0, \cdots, M-1$ and $E\{c_{v,u_1\oplus u_2}^{(M)}(F_v)\} = E\{(F_v)\} = (0)$ by Condition (i) of Theorem 3.2. Furthermore, if $u_1 \oplus u_2 \neq 0$ then $E\{c_{v,u_1\oplus u_2}^{(M)}(F_v)\} = (0)$; since by Condition (ii) of Theorem 3.2,

$$E\{(F_v)(H_v)\} = E\{(F_v)[(c_{v1}^{(M)}, \cdots, c_{v(M-1)}^{(M)}) \otimes (C^{(S)})]\} = (0).$$

Hence, $E\{(K)\} = (0)$.

We turn now to the derivation of the variance-covariance matrix of a linear unbiased estimator under R.P.I.

THEOREM 4.2. *Let* $\alpha(\gamma, F_v) = (1/S)(C^{(S)})'[Y(X_v) - (H_v)\gamma] + (F_v)Y(X_v)$ *be any linear unbiased estimator of* α, *under* R.P.I. *Then, its variance-covariance matrix* $\ddagger(\hat{\alpha}(\gamma, F_v))$ *is given by*:

$$\begin{aligned} \ddagger(\hat{\alpha}(\gamma, F_v)) &= \sigma^2[(1/S)I^{(S)} + E(F_v)(F_v)'] \\ &\quad + (1/S^2)E\{(C^{(S)})'(H_v)(\beta - \gamma)(\beta - \gamma)'(H_v)'(C^{(S)})\} \\ &\quad + E\{(F_v)[(C^{(S)})\alpha + (H_v)\beta][(C^{(S)})\alpha + (H_v)\beta]'(F_v)'\}. \end{aligned} \tag{4.5}$$

PROOF. Denote by $\ddagger(\hat{\alpha}(\gamma, F_v) \mid (H_v), (F_v))$ the conditional variance-covariance matrix of $\hat{\alpha}(\gamma, F_v)$, given (H_v) and (F_v). According to the relationship of the total variance to the conditional variance, we write:

$$\begin{aligned} \ddagger(\hat{\alpha}(\gamma, F_v)) &= E\{\ddagger(\hat{\alpha}(\gamma, F_v) \mid (H_v), (F_v))\} \\ &\quad + \ddagger(E\{\hat{\alpha}(\gamma, F_v) \mid (H_v), (F_v)\}). \end{aligned} \tag{4.6}$$

The conditional variance-covariance matrix of $\hat{\alpha}(\gamma, F_v)$ is given by:

$$\begin{aligned} \ddagger(\hat{\alpha}(\gamma, F_v) \mid (H_v), (F_v)) &= E\{[(F_v) + (C^{(S)})^{-1}] \\ &\quad \cdot \epsilon\epsilon'[(F_v) + (C^{(S)})^{-1}]' \mid (F_v)\} \\ &= \sigma^2[(1/S)I^{(S)} + (F_v)(F_v)' + (C^{(S)})^{-1}(F_v)' \\ &\quad + (F_v)(C^{(S)\prime})^{-1}]. \end{aligned} \tag{4.7}$$

Thus, since $E\{(F_v)\} = (0)$ we have,

$$E\{\ddagger(\hat{\alpha}(\gamma, F_v) \mid (H_v), (F_v))\} = \sigma^2[(1/S)I^{(S)} + E\{(F_v)(F_v)'\}]. \tag{4.8}$$

The conditional expectation of $\hat{\alpha}(\gamma, F_v)$, given (H_v) and (F_v), is

$$\begin{aligned} E\{\hat{\alpha}(\gamma, F_v) \mid (H_v), (F_v)\} &= \alpha + (C^{(S)})^{-1}(H_v)(\beta - \gamma) \\ &\quad + (F_v)[(C^{(S)})\alpha + (H_v)\beta]. \end{aligned} \tag{4.9}$$

Since $\hat{\alpha}(\gamma, F_v)$ is unbiased, we have:

$$\begin{aligned} &\ddagger(E\{\hat{\alpha}(\gamma, F_v) \mid (H_v), (F_v)\}) \\ &\quad = E\{[(C^{(S)})^{-1}(H_v)(\beta - \gamma) + (F_v)[(C^{(S)})\alpha + (H_v)\beta]] \\ &\quad\quad \cdot [(C^{(S)})^{-1}(H_v)(\beta - \gamma) + (F_v)[(C^{(S)})\alpha + (H_v)\beta]]'\} \\ &\quad = (1/S^2)E\{(C^{(S)})'(H_v)(\beta - \gamma)(\beta - \gamma)'(H_v)'(C^{(S)})\} \\ &\quad\quad + E\{(F_v)[(C^{(S)})\alpha + (H_v)\beta][(C^{(S)})\alpha \\ &\quad\quad + (H_v)\beta]'(F_v)'\} + E\{(A)\} + E\{(A)'\} \end{aligned} \tag{4.10}$$

where $(A) = (C^{(S)})^{-1}(H_v)(\beta - \gamma)[(C^{(S)})\alpha + (H_v)\beta]'(F_v)'$. Finally, according to Lemma 4.1, $E\{(A)\} = E\{(A)'\} = (0)$.

COROLLARY 4.2.1. *The variance-covariance matrix of a* c.l.s.e., $\hat{\alpha}(\gamma)$, *is*:

$$\ddagger(\hat{\alpha}(\gamma)) = (\sigma^2/S)I^{(S)} + (1/S^2)E\{(C^{(S)})'(H_v)(\beta-\gamma)(\beta-\gamma)'(H_v)'(C^{(S)})\} \tag{4.11}$$
$$= (\sigma^2/S)I^{(S)} + \sum_{u=1}^{M-1}(\beta-\gamma)_{(u)}(\beta-\gamma)'_{(u)}$$

i.e., if $\hat{\alpha}_i(\gamma)(i = 0, \cdots, S-1)$ denote the components of $\hat{\alpha}(\gamma)$, then

$$\text{cov}(\hat{\alpha}_i(\gamma), \hat{\alpha}_j(\gamma)) = \sigma^2/S + \sum_{u=1}^{M-1}(\beta_{i+uS} - \gamma_{i+uS})^2, \qquad \text{if } i = j \tag{4.12}$$
$$= \sum_{u=1}^{M-1}(\beta_{i+uS} - \gamma_{i+uS})(\beta_{j+uS} - \gamma_{j+uS}), \qquad \text{if } i \neq j.$$

COROLLARY 4.2.2. *If* $\hat{\alpha}(\gamma, F_v)$ *is any linear unbiased estimator of* α, *under* R.P.I., *with* $(F_v) \not\equiv (0)$ *then* $\ddagger(\hat{\alpha}(\gamma, F_v)) - \ddagger(\hat{\alpha}(\gamma))$ *is positive definite.*

Indeed, according to (4.5) and (4.11), the difference of the variance-covariance matrices of $\hat{\alpha}(\gamma, F_v)$ and $\hat{\alpha}(\gamma)$, with the same γ, is

$$\ddagger(\hat{\alpha}(\gamma, F_v)) - \ddagger\hat{\alpha}(\gamma) = \sigma^2E\{(F_v)(F_v)'\} + E\{(F_v)[(C^{(S)})\alpha + (H_v)\beta][(C^{(S)})\alpha + (H_v)\beta]'(F_v)\}. \tag{4.13}$$

If $(F_v) \not\equiv (0)$ then $x'\sigma^2(F_v)(F_v)'x = \sigma^2[(F_v)'x]'[(F_v)x] > 0$ for all $x \not\equiv 0$. Hence $\sigma^2E\{(F_v)(F_v)'\}$ is positive definite. Moreover,

$$x'E\{(F_v)[(C^{(S)})A + (H_v)\beta][(C^{(S)})\alpha + (H_v)\beta]'(F_v)'\}x \geqq 0$$

for every $x \neq 0$ and every α and β. Thus (4.13) is positive definite. This establishes the completeness of the class of c.l.s.e.

5. The general least-squares estimators for R.P.I. Let

$$\beta_{(0)} = (\beta_0, \beta_{t_1}, \cdots, \beta_{t_{S-1}})', \qquad t_k < t_{k+1}(k = 1, \cdots, 2^s - 1)$$

be a vector of any $S = 2^s$ pre-assigned parameters, and let $\{\beta_{d_0}, \cdots, \beta_{d_{m-s-1}}\}$ be any set of $(m - s)$ defining parameters independent of the pre-assigned ones. Let $\{X_v : v = 0, \cdots, M-1\}$ be the corresponding $M = 2^{m-s}$ blocks of treatment combinations, constructed according to (2.8). Define by $\beta_{(u)}(u = 1, \cdots, M-1)$ the vector of S parameters obtained by multiplying each of the components of $\beta_{(0)}$ by β_u^*, i.e., $\beta_{(u)} = (\beta_0 \otimes \beta_u^*, \beta_{t_1} \otimes \beta_u^*, \cdots, \beta_{t_{S-1}} \otimes \beta_u^*)'$, where

$$\{\beta_u^* : u = 0, \cdots, M-1\}$$

is the subgroup generated by the $(m - s)$ defining parameters. If $\beta_{d_k} \equiv (\lambda_{0,d_k}, \cdots, \lambda_{m-1,d_k})$ for $k = 0, \cdots, m-s-1$, define

$$b_{vu} = (-1)^{\sum_{j=0}^{m-s-1} i_j'(i_j - L(d_j))}; \qquad v, u = 0, \cdots, M-1 \tag{5.1}$$

where $v = \sum_{j=0}^{m-s-1} i_j 2^j$, $u = \sum_{j=0}^{m-s-1} i_j' 2^j$ and $L(d_j) \equiv \sum_{k=0}^{m-1} \lambda_{k,d_j}$. Then, as shown previously by Ehrenfeld and Zacks [4] the statistical model can be written in the form:

$$Y(X_v) = \sum_{u=0}^{M-1} (P_{vu}^{(S)})\beta_{(u)} + \epsilon_v = (P_{v0}^{(S)})\alpha + (H_v)\beta + \epsilon_v \tag{5.2}$$
$$\text{for all } v = 0, \cdots, M-1$$

where $\alpha = \beta_{(0)}$, $\beta' = (\beta_{(1)}, \cdots, \beta_{(M-1)})$; $(P_{v0}^{(S)})$ is a $S \times S$ matrix of the coefficients in $(C^{(N)})$, $N = 2^m$, corresponding to the x's in X_v and to the β's in α, arranged in the standard order; and where,

$$(H_v) = (b_{v1}^{(M)}, \cdots, b_{v(M-1)}^{(M)}) \otimes (P_{v0}^{(S)}) \tag{5.3}$$

for all $v = 0, \cdots, M - 1$. Substituting $(P_{v0}^{(S)})$ for $(C^{(S)})$ in (3.1), and (H_v) which is given by (5.3), we get the general representation of a linear unbiased estimator. Furthermore, the general representation of c.l.s.e. is given by:

$$\hat{\alpha}(\gamma) = (1/S)(P_{v0}^{(S)})'[Y(X_v) - (H_v)\gamma], \qquad \gamma \in E^{(N-S)}. \tag{5.4}$$

It is readily verified that all the theorems given in Sections 3 and 4 hold.

6. Conditional least-squares estimators with R.P.II. In R.P.II. we have to require that the $S = 2^s$ pre-assigned parameters will constitute a *sub*-group since in this procedure the pre-assigned parameters have the role of the defining ones. Accordingly, we will assume that the pre-assigned parameters are the first 2^s ones. Otherwise, relabel the factors and apply a linear transformations on their levels (reflections) to obtain it. The $S = 2^s$ blocks are, accordingly, the following sets:

$$X_i = \{x: c_j^{(N)}(x_{i+uS}) = c_j^{(S)}(x_i) \text{ for all } i = 0, \cdots, S-1, \tag{6.1}$$
$$j = 0, \cdots, S-1 \text{ and } u = 0, \cdots, M-1\}.$$

Let $Y_{ij_k} \equiv Y(x_{i+j_kS})$, $(k = 1, \cdots, n)$ be the random variable associated with the kth treatment combination chosen at random from the ith block. Without loss of generality, let $n = 1$. Accordingly, the statistical model for R.P.II. can be written as

$$Y_{ij_i} = \sum_{u=0}^{S-1} c_{iu}^{(S)}\alpha_u + \sum_{t=0}^{N-S-1} c_{i+j_iS,t+S}^{(N)}\beta_{t+S} + \epsilon_{ij_i} \tag{6.2}$$

where ϵ_{ij_i} is a random variable independent of x_{ij_i}, with $E\epsilon = 0$, $E\epsilon^2 = \sigma^2$. By virtue of (2.6) we have

$$c_{i+j_iS,t+S}^{(N)} = c_{ir_t}^{(S)} \cdot c_{j_iq_t}^{(M)} \tag{6.3}$$

where $M = 2^{m-s}$, $r_t \equiv t \pmod{S}$ and $q_t = 1 + [t/S]$. This statistical model can thus be written, in a matrix form, as

$$Y = (C^{(S)})\alpha + (H)\beta + \epsilon \tag{6.4}$$

where Y is the vector of observation of order S; ϵ is a random vector of order S independent of the x's with $E\epsilon = 0$, $E\epsilon\epsilon' = \sigma^2 I^{(S)}$, and where

$$(H) = ((G_1)(C^{(S)}), (G_2)(C^{(S)}), \cdots, (G_{M-1})(C^{(S)})) \tag{6.5}$$

each (G_u), $u = 1, \cdots, M - 1$, is a diagonal matrix of order $S \times S$, whose diagonal elements are $g_{ii}^{(u)} = c_{jiu}^{(M)} (i = 0, \cdots, S - 1)$.

Replacing (H_v) in (3.1) and (3.9) by (H) of R.P.II. (6.5) we get the corresponding class of linear unbiased estimators, and the subclass of c.l.s.e. for R.P.II. It is easy to check that all the theorems of Section 2 and 3 hold.

7. Discussion. A *complete* class of linear unbiased estimators for randomized fractional replication designs has been presented. Each estimator belonging to this class is a conditional least-squares estimator, which can be interpreted as the least-squares estimator for a 2^s fractional system *adjusted* to balance for the effect of the nuisance parameters, according to the block chosen and the information available concerning the nuisance parameters. Indeed, the estimator $\hat{\alpha}_0 = (1/S)(C^{(S)})'Y$ is the least-squares estimator of α in a $2^s = S$ factorial system. $\hat{\alpha}(\gamma) = \hat{\alpha}_0 - (1/S)(C^{(S)})'(H_v)\gamma$ is an adjustment of $\hat{\alpha}_0$ by $-(1/S)(C^{(S)})'(H_v)\gamma$, which depends on the block chosen, (H_v), and the point γ chosen. A-priori information concerning the vector of nuisance parameters β may often be available. Let $\{\pi_t(\beta) : t = 0, \cdots, N - S - 1\}$ be a set of all the marginal a-priori distributions of the components of β. Then $\hat{\alpha}(\gamma^0)$, with $\gamma_t^0 = E_{\pi_t}\{\beta\}$ for all $t = 0, \cdots, N - S - 1$, is a *Bayes* conditional least-squares estimator, relative to the loss function, given by the trace of the variance-covariance matrix of $\hat{\alpha}(\gamma)$. In the following paper the problem of choosing a conditional least-squares estimator will be studied in a more general framework.

Acknowledgment. The author wishes to acknowledge Professor S. Ehrenfeld for his helpful comments and encouragement.

REFERENCES

[1] COCHRAN, W. G. and G. M. COX (1957). *Experimental Designs*, 2nd ed. Wiley, New York.

[2] DEMPSTER, A. P. (1960). Random allocation designs I: On general classes of estimation methods. *Ann. Math. Statist.* **31** 885–905.

[3] DEMPSTER, A. P. (1961). Random allocation designs II: Approximative theory for simple random allocations. *Ann. Math. Statist.* **32** 387–405.

[4] EHRENFELD, S. and S. ZACKS (1951). Randomization and factorial experiments. *Ann. Math. Statist.* **32** 270–297.

[5] KEMPTHORNE, O. (1952). *The Design and Analysis of Experiments*. Wiley, New York.