THE DEGREE OF RANDOMNESS IN A STATIONARY TIME SERIES[1]

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1. Introduction. Let $\{X_n\}_{-\infty}^{\infty}$ be a wide sense stationary discrete time series, [2]. Then by definition the random variables $\{X_n\}$ are square integrable and the inner products $(X_n, X_{n+k}) = a(k)$ are independent of n. (We shall assume, without essential loss of generality, that $a(0) = 1$.) The covariance function $a(\cdot)$ is a positive definite function on the integers, and is therefore the Fourier-Stieltjes transform of a probability measure μ (the spectral measure of the process) on the unit circle (or $[0, 2\pi)$)

$$(*) \qquad a(k) = \int_0^{2\pi} \exp(-ik\theta)\, d\mu(\theta).$$

The process $\{X_n\}$ is called white noise if $a(k) = 0$ if $k \neq 0$, and this condition is of course equivalent to $d\mu(\theta) = d\theta/2\pi$. If the $\{X_n\}$ have zero expectations then this means that the process is uncorrelated. We shall consider here the question of measuring numerically the extent of deviation of a general stationary time series from white noise. More precisely, we shall examine the relation between an invariant defined directly from the process and ones defined in terms of the spectral measure μ. The methods used are based on the theory of linear operators in Hilbert space. This may seem rather indirect, but it seems to us to be the simplest way of approaching the problem. Our notation in this connection conforms to that in [1].

2. The main result. We first define a numerical invariant directly from our fixed process $\{X_n\}$. If F and G are two disjoint sets of integers, we let

$$\rho_{F,G} = \sup \rho(X_F, X_G) = \sup |(X_F, X_G)|/|X_F|\,|X_G|$$

where the supremum runs over all finite linear combinations X_F of the X_i $(i \in F)$ and finite linear combinations X_G of the X_i $(i \in G)$. (Although $|f|$ will denote the L_2 norm of a random variable f, $|\alpha|$ the modulus of a complex number α, and $|T|$ the norm of a linear transformation T, no confusion should result.) Then we define $\rho = \sup_{F \cap G = \phi} \rho_{F,G}$. The case $\rho = 0$ is white noise, whereas $\rho = 1$ is the opposite extreme. We can define other invariants in terms of the spectral measure μ as follows. We write $d\mu = d\mu_s + w\, d\theta/2\pi$ where μ_s is singular with respect to Lebesgue measure and $w \geqq 0$ is integrable. If $\mu_s = 0$, let $\alpha_1 = \text{ess min } w$, $\alpha_2 = \text{ess max } w$ and $\alpha = \alpha_1/\alpha_2$. If $\mu_s \neq 0$, we extend the definitions by setting $\alpha_1 = 0$, $\alpha_2 = \infty$, $\alpha = 0$. Thus $0 \leqq \alpha \leqq 1$ and $\alpha = 1$ corresponds to white noise. Note

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that $\alpha > 0$ if and only if μ is absolutely continuous and the spectral density w is essentially bounded from 0 and $+\infty$. We now come to our main result.

THEOREM 1. *If $\{X_n\}$ is a discrete stationary time series, then the constants defined above satisfy*

$$\alpha \leqq (1-\rho)/(1+\rho), \qquad \alpha_1 \geqq (1-\rho)/(1+\rho); \qquad 1/\alpha_2 \geqq (1-\rho)/(1+\rho)$$

and hence $\alpha \geqq ((1-\rho)/(1+\rho))^2$.

As a special case of the theorem we see that the opposite extreme of white noise can be characterized in terms of either ρ or α, namely $\rho = 1$ holds if and only if $\alpha = 0$. We shall show further at the end of this note that the first three inequalities above are best possible. In particular Theorem 2 gives a sufficient condition for the first inequality to be an equality.

3. Proof of the theorem. For our purposes we may realize the process in terms of its spectral measure μ. In L_μ^2 (measurable functions on $[0, 2\pi)$ square integrable with respect to μ.), let $X_n(\theta) = \exp(in\,\theta)$. Then we have a process with the given covariance function $a(\cdot)$.

We first reduce the theorem to the case $\mu_s = 0$. If $\mu_s \neq 0$, then by definition of the α's it suffices to show that $\rho = 1$. If we take $G_n = \{-n, -n+1, \cdots, -1\}$ and $F_n = \{0, 1, 2, \cdots, n\}$, it follows from a recent result of Helson and Szegö, ([3] p. 123) that $\sup_n \rho_{F_n, G_n} = 1$. Thus surely $\rho = 1$ and we are done. We may also assume that the process is non-deterministic, for if not, then $\rho = 1$ trivially, and one must only verify that $\alpha = 0$. But this is clear since $\int_0^{2\pi} \log w \, d\theta = -\infty$.

It is convenient now to choose another realization of the process; namely, in $L_2(0, 2\pi)$ let $X_n(\theta) = w(\theta)^{\frac{1}{2}} \exp(in\,\theta)$. Let D be the set of all finite linear combinations of the X_n. Then D is a dense linear manifold in the Hilbert space $H = L_2(0, 2\pi)$. If L is an arbitrary set of integers we define a linear transformation U_L of D onto itself by the formulas

$$U_L X_n = X_n \quad \text{if } n \in L, \qquad U_L X_n = -X_n \quad \text{if } n \notin L.$$

Next, let β be the supremum of the norms of the linear transformations U_L (β will be $+\infty$ if for example some U_L is unbounded). To be precise

$$\beta = \sup_{L, X \in D, |X|=1} |U_L X|.$$

LEMMA 1. *We have $\beta^{-2} = (1-\rho)/(1+\rho)$. (The left side is read as 0 if $\beta = +\infty$.)*

PROOF. Let X be a unit vector in D and L a set of integers. We write $X = X' + X''$ where X' is a linear combination of the X_n for $n \in L$ and X'' is a linear combination of the X_n for $n \notin L$. Then $U_L X = X' - X''$ and the subspace of D spanned by X' and X'' is invariant under U_L. By assumption X' and X'' lie at an angle θ with $\cos\theta = \rho(X', X'') \leqq \rho$.

We thus come to the following question: Given a two dimensional Hilbert space and vectors e_1 and e_2 with $\cos\theta = \cos(e_1, e_2) = k < 1$, what is the norm of the linear transformation T defined by $Te_1 = e_1$, $Te_2 = -e_2$. It is easy to see that $|Tx|/|x|$ is maximal when cos (x, e_1) or cos (x, e_2) is cos $(\theta/2)$, and we find by simple trigonometry that $|T|^2 = (1 + k)/(1 - k)$.

Returning to the situation of the lemma, we immediately deduce that $|U_L|^2 \leqq (1+\rho)/(1-\rho)$ for any L. On the other hand, by suitable choice of L and a vector $X = X' + X''$ we can force $\rho(X', X'')$ arbitrarily close to ρ. It is then immediate that $\beta^2 = \sup_L|U_L|^2 = (1 + \rho)/(1 - \rho)$ and the lemma is proved.

The first inequality of Theorem 1 will follow now from the following lemma.

Lemma 2. *We have* $\alpha \leqq \beta^{-2}$.

Proof. We may assume that $\alpha > 0$, for otherwise there is nothing to prove. Then w and hence $w^{\frac{1}{2}}$ are bounded from 0 and $+\infty$. Let M be the linear transformation on $L_2(0, 2\pi)$ defined by $Mf = w^{\frac{1}{2}}f$. Then M is bounded with a bounded inverse given by $M^{-1}f = w^{-\frac{1}{2}}f$. One knows the norms of these operators (cf. Lemma 4); $|M| = \text{ess sup } (w^{\frac{1}{2}}) = \alpha_2^{\frac{1}{2}}$ and $|M^{-1}| = \alpha_1^{-\frac{1}{2}}$.

If L is a set of integers, we define a transformation V_L analogous to U_L by

$$V_L(\exp (in\ \theta)) = \exp (in\ \theta) \quad \text{if } n \in L$$

$$V_L \exp (in\ \theta) = -\exp (in\ \theta) \quad \text{if } n \notin L.$$

Then V_L is a unitary transformation and $|V_L| = 1$. We see by an easy calculation that $MV_LM^{-1}X_n = U_LX_n$. By linearity $MV_LM^{-1} = U_L$ on all of D (finite linear combinations of the X_n). The left hand side of the above is a bounded operator on all of $L_2(0, 2\pi)$ and so U_L may be extended (uniquely) to a bounded operator which we also call U_L. We derive at once a norm inequality

$$|U_L| = |MV_LM^{-1}| \leqq |M|\,|V_L|\,|M^{-1}| = |M|\,|M^{-1}| = \alpha^{-\frac{1}{2}}.$$

Thus we find that $\beta \leqq \alpha^{-\frac{1}{2}}$ and our lemma is proved.

The reverse inequalities for α_1 and α_2 lie somewhat deeper. The procedure will be essentially to reverse the argument given in Lemma 2. Theorem 1 will then be proved once we have established the following lemma.

Lemma 3. $\alpha_1^{-1} \leqq \beta^2$ *and* $\alpha_2 \leqq \beta^2$.

Proof. We may clearly assume that β is finite and in that case the linear transformations U_L defined on D may be uniquely extended to bounded operators on $L_2(0, 2\pi)$. The set of U_L form an abelian group under multiplication. In fact $U_L^{-1} = U_L$ and $U_LU_M = U_K$ where $K = (L \cap M) \cup (L' \cap M')$ with S' denoting the complement of a set of integers. Since the group is uniformly bounded in norm (by β) one can conclude by a standard theorem in operator theory, [4], that it is similar to a unitary group. This means that there is a bounded invertible operator Q such that $QV_L\,Q^{-1}$ is a unitary operator for each L. Since this theorem may not be familiar, and since we shall need some more detailed information about Q, we include a proof.

Let G be a bounded abelian group of operators on a Hilbert space H. We denote by K the closure in the weak operator topology ([1] p. 476) of the set of all convex combinations of the V^*V for $V \in G$. (V^* denotes the adjoint of V.) If β is a bound for $|V|$, then K is contained in the ball of operators of norm less than or equal to β^2. Since this ball is compact, K is a compact convex set. Moreover G operates as a group of continuous linear transformations on K as follows: $\pi_U(S) = U^*SU$ for $S \varepsilon K$, $U \varepsilon G$. If $S = V^*V$ for some $V \varepsilon G$, $U^*V^*VU = (VU)^* VU$ and it follows at once that $\pi_U(K) \subset K$. One now applies the Markoff-Kakutani fixed point theorem, ([1] p. 456), to extract a fixed point P in K; that is, $U^*PU = P$ for all U in G.

The set K consists of non-negative self adjoint operators and if $x \varepsilon H$ and $V \varepsilon G$,

$$(V^*V x,x) = (Vx, Vx) \geqq \beta^{-2} (x, x)$$

since $|V^{-1}| \leqq \beta$. By convexity $(Sx, x) \geqq \beta^{-2}(x, x)$ holds for all $S \varepsilon K$. It follows that S is invertible with $|S^{-1}| \leqq \beta^2$. This holds in particular for P, and now we may extract a unique positive invertible square root Q of P. We have $|Q| \leqq \beta$, $|Q^{-1}| \leqq \beta$. An easy calculation shows that QVQ^{-1} is unitary for $V \varepsilon G$, and the proof is complete.

Let us return to the proof the lemma; G is now the set of U_L. Since $U_L X_n = \pm X_n$, $(U_L^* U_L X_n X_n) = (U_L X_n, U_L X_n) = (X_n, X_n) = 1$ for all n. By convexity, $(S X_n, X_n) = 1$ for all $S \varepsilon K$. In particular it holds for the fixed point $P = Q^2$, and we find that $|QX_n| = 1$ for all n. Since QU_LQ^{-1} is unitary it follows easily that the vectors QX_n are orthogonal to each other. Further, the X_n span a dense submanifold and since Q is bounded and invertible, the QX_n must be a complete orthonormal system in $L_2(0, 2\pi) = H$. We can therefore find a (unique) unitary transformation T of H onto itself such that $TQX_n = \exp(in\ \cdot)$. Let $M = Q^{-1}T^{-1}$ $(M^{-1} = TQ)$, and we have $|M| \leqq \beta$ and $|M^{-1}| \leqq \beta$.

We know that $M(\exp(in\ \cdot)) = X_n = w^{\frac{1}{2}} \exp(in\ \cdot)$ and since M is linear, $M(q) = w^{\frac{1}{2}} q$ for any trigonometric polynomial. If $f \varepsilon L_2(0, 2\pi)$, we may find a sequence q_n of trigonometric polynomials such that q_n converges to f in L_2. Then $M(q_n)$ converges to $M(f)$ in L_2 since M is bounded. We may, by passing to a subsequence, assume that q_n and $M(q_n)$ converge almost everywhere to f and $M(f)$ respectively. But then $M(q_n) = w^{\frac{1}{2}}q_n$ converges almost everywhere to $w^{\frac{1}{2}}f$, and it follows that $M(f) = w^{\frac{1}{2}}f$ for any f in L_2. Thus $M^{-1}(w^{\frac{1}{2}}f) = f$ for any f in L_2. Since M maps L_2 onto itself, every g in L_2 is of the form $g = w^{\frac{1}{2}}f$ with f in L_2, and it follws that $M^{-1}(g) = w^{-\frac{1}{2}}g$ for any g in L_2. To conclude the argument we make use of the following fact whose proof we omit.

Lemma 4. *If l is a measurable function and $Lf = l \cdot f$ defines a bounded linear transformation of $L_2(0, 2\pi)$ into itself of norm $|L|$, then* ess sup $|l| = |L|$.

We apply this to M and M^{-1} to find that

$$\alpha_2^{\frac{1}{2}} = \text{ess sup } w^{\frac{1}{2}} = |M| \leqq \beta, \quad \text{and} \quad \alpha_1^{-\frac{1}{2}} = \text{ess sup } w^{-\frac{1}{2}} = |M^{-1}| \leqq \beta.$$

Lemma 3 and Theorem 1 are now completely proved.

4. Some examples. We shall consider now a special case and some examples of the previous theorem. In particular, we give a sufficient condition for the first inequality of Theorem 1 to be an equality. Unfortunately, we do not know of any example where equality does not hold.

THEOREM 2. *Let the spectral measure of the process* $\{X_n\}$ *be absolutely continuous with density* w. *Assume that* w *is continuous,* $w(0) = w(2\pi)$ *and never vanishes. Further suppose that there exists* θ_1 *and* θ_2 *such that* $w(\theta_1) = \alpha_1$, $w(\theta_2) = \alpha_2$ *with* $\theta_1 - \theta_2 = \pm\pi$. *Then* $\alpha = (1 - \rho)/(1 + \rho)$.

PROOF. The final condition on w asserts that exp $(i\ \theta_1)$ and exp $(i\ \theta_2)$ are antipodal points on the unit circle in the complex plane. Consider the process $X_n^* = X_n \cdot \exp(-in\ \theta_1)$. Its spectral density w^* is given by $w^*(\theta) = w(\theta - \theta_1)$, and so the constants α, α_1 and α_2 are the same for the two processes. (We are viewing w and w^* as periodic functions on the line.) A moment's consideration shows one that $\{X_n\}$ and $\{X_n^*\}$ have the same constant ρ. Now w^* has a minimum at 0 and a maximum at π. We may then assume without loss of generality that this is true of w.

In the proof of Lemma 2, we have an inequality $|U_L| = |MV_L M^{-1}| \leqq |M| |V_L| |M^{-1}| = \alpha^{-\frac{1}{2}}$. In virtue of Lemma 1, it is enough to show sup $|U_L| = \alpha^{-\frac{1}{2}}$. Under the conditions of Lemma 2, we shall prove that $U_L = \alpha^{-\frac{1}{2}}$ when L is the set of all odd integers. Let $y_n(\theta) = w^{\frac{1}{2}}(\theta)\ D_n(\theta)$ where $D_n(\theta) = 1/(n+1) \sum_0^n$ exp $(in\ \theta)$. We observe that $|D_n(\theta)|^2 = F_n(\theta)$ is the Fejér kernel. Then $|y_n|^2 = \int_0^{2\pi} |y_n(\theta)|^2\ d\theta/2\pi = \int_0^{2\pi} w(\theta)\ F_n(\theta)\ d\theta/2\pi$ and $\lim |y_n|^2 = w(0) = \alpha_1$. Now $M^{-1}y_n(\theta) = F_n(\theta)$ and $V_L F_n(\theta) = F_n(\theta + \pi)$. Finally we find that

$$U_L y_n(\theta) = MV_L M^{-1} y_n(\theta) = w^{\frac{1}{2}}(\theta)\ F_n(\theta + \pi).$$

It follows just as before that $|U_L y_n|^2 = \int_0^{2\pi} w(\theta)\ F_n(\theta + \pi)\ d\theta/2\pi$ and hence $\lim |U_L y_n|^2 = w(\pi) = \alpha_2$. Thus

$$|U_L|^2 \geqq \lim |U_L y_n|^2/|y_n|^2 = \alpha_2/\alpha_1 = \alpha^{-1},$$

and our theorem is proved.

REMARKS.

(1) With a suitable reformulation one can drop the hypothesis that w be continuous. One simply has to make use of the fact that

$$\int_0^{2\pi} w(\theta) F_n(\varphi - \theta)\ d\theta/2\pi \to w(\varphi)$$

for almost all ϕ if w is integrable ([5] **2** p. 90). We leave this to the interested reader.

(2) For processes satisfying the conditions of Theorem 2, the proof shows how to construct the linear combinations of the random variables so as to maximize the correlation coefficients. To be precise, let $\{X_n\}$ be as in Theorem 2 with $w(0) = \alpha_1$, $w(\pi) = \alpha_2$, then one can easily establish the following fact.

COROLLARY. *Let* $F_n = \{0, 2, \cdots, 2n\}$, $G_n = \{1, 3, \cdots, 2n+1\}$ $f_n = \sum X_k$ $(k \in F_n)$ *and* $g_n = \sum X_k$ $(k \in G_n)$. *Then* $\rho = \sup \rho(f_n, g_n)$.

(3) We have established that the first inequality of Theorem 1 is best possible. However if $\alpha > 0$, we may clearly construct processes $\{X_n\}$ satisfying the conditions of Theorem 2 with the given constant α so that α_1 is arbitrarily close to α or so that α_2^{-1} is arbitrarily close to α. This may be done just by writing

down suitable spectral densities w. It then follows that the second and third inequalities of Theorem 1 are also best possible.

(4) Finally let us consider an autoregressive process of order one ([2], p. 36). The covariance function a is given by $a(0) = 1$, $a(1) = b/2$, and $a(k) = 0$ for $|k| > 1$ where $|b| \leqq 1$. The normalization procedure of Theorem 2 simply rotates b in the complex so that it becomes negative. Then $w(\theta) = 1 + b \cos \theta$, which has a minimum value of $1 + b$ at 0 and maximum value of $1 - b$ at π. Thus $\alpha = 1 + b/1 - b$, and one can see either by the corollary or by direct verification in this simple case that $\rho = -b$.

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