# SAMUEL STANLEY WILKS, 1906–1964

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Samuel Stanley Wilks died in his sleep at his home in Princeton, New Jersey, March 7, 1964. His unexpected death at the age of 57 years came as a shock to his many colleagues, students, and friends throughout the world, who now mourn his passing. The loss of this vigorous, talented, and devoted man is felt widely, especially among mathematical statisticians. It is particularly fitting to dedicate to his memory this volume of the *Annals of Mathematical Statistics*, the journal which he did so much to bring to its present eminence.

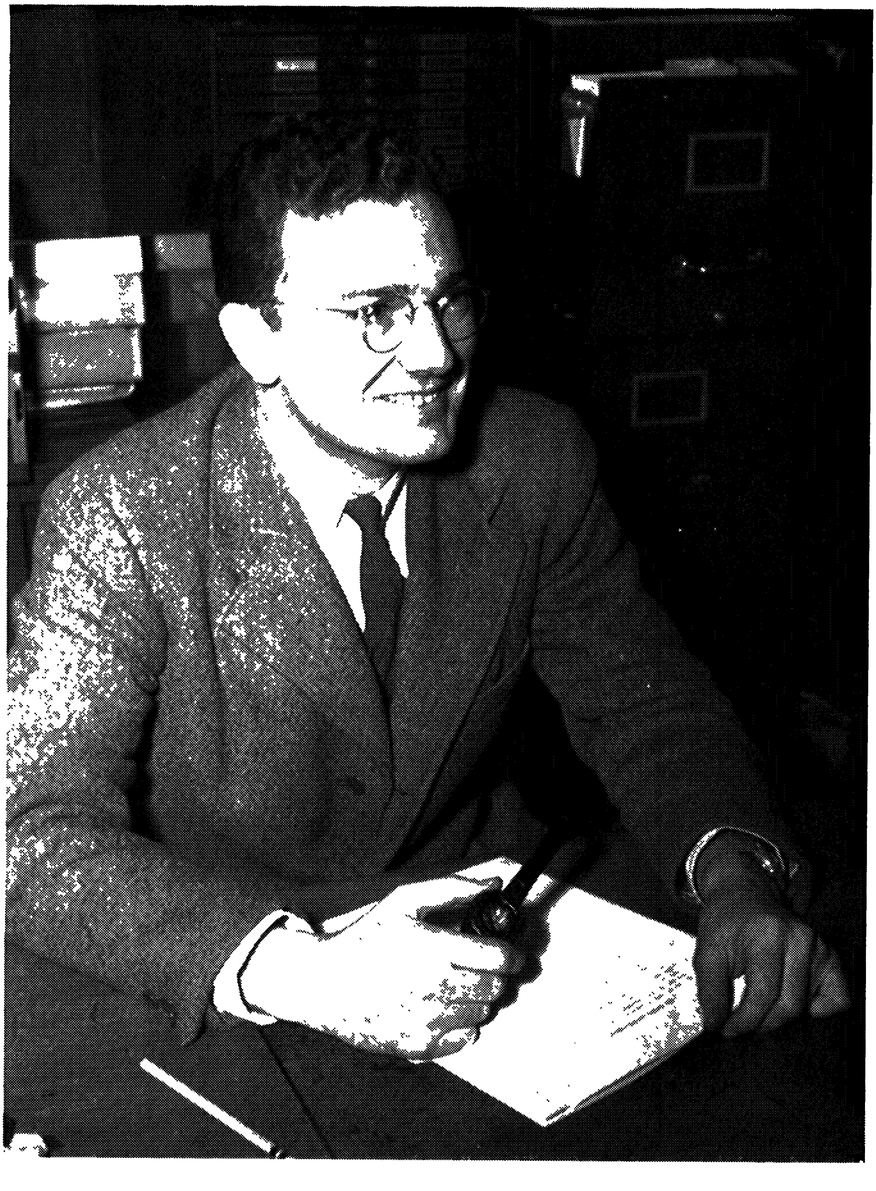
**1. Life and career.** Samuel Stanley Wilks was born in Little Elm, Texas, on June 17, 1906, the first child of Chance C. Wilks and Bertha May Gammon Wilks. His brother, Syrel Singleton, less than two years younger, was his boyhood chum; William Weldon, eight years younger, was his "baby brother." (The names came by Chance, not by chance.) His father trained for banking, but made his livelihood by farming his 250 acre ranch outside Little Elm. His mother, whose formal education did not go beyond high school, had artistic and musical talent; her lively curiosity was transmitted to her children. Syrel pursued his studies to obtain a Ph.D. in physiology, taught at San Marcus (Texas) College, and is now at the School of Aerospace Medicine, Brooks Air Force Base, Texas. William took a B.S. degree and is research advisor to Bell Aircraft Company, Fort Worth, Texas.

Wilks' early education took place in a typical rural white one-room schoolhouse. His instructor during seventh grade was W. M. Whyburn who later became president of Texas Technological College and chairman of the Department of Mathematics at the University of North Carolina. While the Wilks boys benefited by the instruction, some of these frontier lads made discipline the real challenge; in fact, one afternoon one of them took off after Syrel with a shotgun.

To attend high school Sam roomed in nearby Denton and went home on weekends, if necessary walking the fifteen miles. His ambition and interest in mathematics showed up early. During his final high school year the authorities found that he was not in attendance at study hall. Investigation indicated that instead of going to study hall he was taking a mathematics course at North Texas State Teachers College.

After high school graduation he continued his education at North Texas State Teachers College with particular attention to industrial arts and mathematics. His degree of A.B. was granted in 1926 in architecture. Had he considered his eyesight adequate for an architect, he might have made his career in that profession.

In 1926–27 he taught manual training in a high school in Austin, Texas, and studied mathematics at the University of Texas. During the next two years he



*By action of the Council of the Institute of Mathematical Statistics, the 1965 volume of the Annals of Mathematical Statistics is dedicated to the memory of*

SAMUEL STANLEY WILKS

*in recognition of his many contributions to statistical theory and to the development of statistics as an independent discipline, and in appreciation of his pioneering service as Editor of the Annals, 1938–1949.*

was instructor (part time 1927–28) in mathematics at the University of Texas, receiving his M.A. in mathematics in 1928. He studied topology with R. L. Moore, but statistics, taught by E. L. Dodd, interested him more.

Wilks accepted a fellowship at the University of Iowa, and studied statistics, 1929–1931, under Henry L. Rietz, a leading American statistician. Allen T. Craig and John L. Curtiss were concurrently graduate students at this center of statistical training. Wilks continued an early interest in psychology by attending lectures of E. F. Lindquist, professor of education and director of Iowa Testing Programs. Wilks' doctoral dissertation was in the field of multivariate statistical analysis and was entitled "On the distributions of statistics in samples from a normal population of two variables with matched sampling of one variable." This paper[1] [2] will be studied in more detail in Section 3. The Ph.D. was granted in 1931.

Wilks married Gena Orr of Denton September 1, 1931. Gena had attended Texas Women's University, receiving her baccalaureate the same year as Sam, and she did graduate work in English at the University of Texas, receiving her Master's degree in 1929. They were acquainted earlier than college days, however, since the Wilks and the Orr families were friends. Many statisticians have come to know Gena at home and in travels with Sam; her graciousness and hospitality are appreciated by students, professors, and Sam's many associates.

Stanley Neal Wilks was born in England in October, 1932. Stanley also graduated from North Texas State College ("Teachers" having been dropped from the name), received a Master's degree in Applied Mathematics at Columbia University, and is technical research director with the Systems Development Corporation, Falls Church, Virginia. He married Jocelyn Wilkins, daughter of Eugene Wilkins, a classmate of Sam's at North Texas State Teachers College; Stanley has three daughters.

Granted a National Research Fellowship by the National Research Council, Wilks went to Columbia University to work in 1931–1932 with Harold Hotelling, who had just been appointed Professor of Economics. Hotelling had taken his doctorate in mathematics (topology) at Princeton University; but his interests had shifted to mathematical economics and statistics while he was associated with the Food Research Institute at Stanford University. This was a productive year for Wilks, who wrote or completed four papers. These were in the area of multivariate analysis, an area in which Hotelling was active, having recently published his paper on the generalized $T^2$ and soon to publish his work on principal components. Wilks also sat in on lectures of C. Spearman, a pioneer in factor analysis, at Teachers College; one of Wilks' papers written at Columbia dealt with the standard error of a tetrad, a useful contribution to factor analysis. During the year the Wilkses became acquainted with the Shewharts; Walter Shewhart of Bell Telephone Laboratories interested Wilks in quality control, inspection sampling, and other applications of statistics in industry.

[1] Following this obituary article is "The Publications of S. S. Wilks." Books are numbered ⟨1⟩, ⋯ ; articles are numbered [1], ⋯ ; and other writings are numbered {1}, ⋯ .

The National Research Council renewed Wilks' fellowship for 1932–33 with the appropriate change in title to International Research Fellowship. Wilks spent the fall of 1932 at the Department of Applied Statistics, University College, London, of which Karl Pearson was director. Wilks collaborated with E. S. Pearson on a paper [7] on the likelihood ratio criteria of equality of means and/or covariance matrices in several normal populations. In January, 1933, Wilks moved on to Cambridge to work with John Wishart. M. S. Bartlett and W. G. Cochran were completing their studies with Wishart. Before the age of 27 Wilks had worked or studied with many of the outstanding statisticians of the time.

Although he had an excellent training, had published six research papers, and was one of the most promising young men in statistics and applied mathematics generally, the end of his second fellowship year approached with no prospect of a job in the United States, which was then in the depths of the depression. Finally, in May, 1933, he was offered a position at Princeton University, which he accepted.

Wilks was an instructor in mathematics at Princeton, 1933–36. He continued his research, primarily in multivariate analysis. During that first year he worked with C. C. Brigham, a psychologist, on the scaling of achievement tests given by the College Entrance Examination Board; the pressing problem was that the percentage of students passing the College Board's examination, in a given subject with a certain required numerical score, varied considerably from year to year. From that time until his death Wilks worked with the College Board and later the Educational Testing Service, completing a comprehensive report on this same subject in 1961.

Wilks was promoted to an assistant professorship in 1936 and an associate professorship in 1938. In his letter of recommendation for the assistant professorship, Dean L. P. Eisenhart, Chairman of the Department of Mathematics, noted that Wilks had just received an appointment as a Collaborator in the United States Soil Conservation Erosion program. Little was it realized that an active and broad career of government service was beginning. In 1936–37 Wilks taught his first statistics courses, a graduate course in the fall, leading to his lecture notes, *Statistical Inference* ⟨1⟩, and an undergraduate course in the spring. A third course (also one semester in length) was added in 1939–1940. Soon a number of graduate students were working under Wilks; in 1939 the first of them received his Ph.D. writing a dissertation under Wilks' direction. In Section 2 of this paper we shall take a closer look at Wilks' work in education and training in mathematics and statistics, not only at Princeton, but throughout the United States and at every level.

The Second World War began in 1939; the United States stepped up its defense activities; the draft started in the fall of 1940. Wilks was quick to realize the problems of war and the importance of mathematics and statistics in the war effort. In July, 1941, he wrote to Dean Eisenhart "After working for about five weeks and making numerous trips to Washington, New York and New

London getting data and having conferences on the problem, I can see the whole thing pretty well now and believe that with a man like McCarthy I can go along with the work next year on the basis . . . of being free the last two or three days of the week from classes." Wilks thus started research work for the National Defense Research Committee with the assistance of P. J. McCarthy, then a graduate student. He wrote further "I feel that the N.D.R.C. work is rather important at this time [a typical Wilks understatement]. The arrangement would mean a shift in my own research primarily to defense research work of a secret nature, except for research suggestions and guidance for graduate students." This was a turning point in Wilks' career. Till this time he had been very active in mathematical statistical research, but after this point he did not return to sustained academic research. I remember him telling me during the war that he thought a mathematician should not go through life concentrating on research, but he should take on broader responsibilities. I think Wilks made a deliberate choice to give up mathematical research in favor of taking on other duties of import in defense, government, mathematics generally, natural and social sciences, and education.

As the war progressed, Wilks devoted more and more of his time to the war effort. After Pearl Harbor, he gave up plans to take leave for an exchange prefessorship of one semester at Santiago, Chile. In due course he was released entirely from academic duties for war research work. He directed the Princeton Statistical Research Group under contract between Princeton University and the Applied Mathematics Panel of the Office of Scientific Research and Development. The technical staff of the research group, located both in Princeton and New York, consisted of his students, such as McCarthy, F. Mosteller, D. F. Votaw and myself, former students, such as A. M. Mood, W. J. Dixon, J. D. Williams, and other statisticians, such as R. L. Anderson, W. G. Cochran and Charles P. Winsor. A variety of problems were studied, such as the efficiency of various methods of long-range weather forecasting for the Air Force, tactical problems for the Navy, and sensitivity testing of explosives for the Army. In 1947 Wilks was awarded the Presidential Certificate of Merit for his contributions toward antisubmarine warfare and the solution of convoy problems. As a member of the Applied Mathematics Panel, 1942–45, he helped to guide the mathematical aspects of the war research activities on a broad scale.

After he returned to his university duties, he continued to serve defense activities. He was chairman of the Mathematics Panel, Research and Development Board, Department of Defense, 1948–50, member of the Scientific Advisory Committee, Selective Service System, 1948–53, member of the Scientific Advisory Board, National Security Agency, 1953 to his death (chairman 1958–60), and member of the National Academy of Sciences Advisory Committee for Air Force Systems Command from 1962. In addition, he advised various defense agencies in many different ways.

In 1944 he was promoted to Professor of Mathematics, effective on his return to academic duties. At this same time, plans were made for a Section of Mathematical Statistics in the Department of Mathematics, with Wilks as Director. These plans provided for additional officers of instruction, visiting lecturers, research associates and assistants, and office staff, with the university responsible for the part of the budget for teaching. Although many other leading universities have subsequently set up separate departments of statistics, at Princeton mathematical statistics has been retained as part of the mathematics department. Wilks continued as professor at Princeton and Director of the Section of Mathematical Statistics until his death, turning down invitations from other universities for professorships, departmental chairmanships, deanships, and the presidency of a major institution.

In the spring of 1951 Sam at long last managed to take his first sabbatical leave of absence. He and Gena returned to the University of Cambridge where Wilks, as a Fulbright research scholar, began work which eventually developed into his encyclopedic *Mathematical Statistics* ⟨4⟩. He continued this authorship mainly in the summers. In the spring and summer of 1956 the Wilkses travelled around the world on a lecture tour sponsored by the Ford Foundation and the Carnegie Fund. Wilks lectured in India, Australia, New Zealand, and Japan. In the fall term of 1961–62 he was a Fellow at the Center for Advanced Study in the Behavioral Sciences in Palo Alto, California, putting the finishing touches on his book.

Wilks was prominent in organizing the Institute of Mathematical Statistics, and from its inception on September 12, 1935, he was an active and leading member. H. C. Carver, who had founded the *Annals of Mathematical Statistics* on his own initiative and from his own funds, offered to turn the *Annals* over to the Institute. The Institute began publishing the *Annals* with the June, 1938, issue. Wilks was the first editor appointed by the Institute, and he continued the editorship through the December, 1949, issue. During these twelve years the character of the journal was formed. As the journal of a professional organization, each contribution offered to it was given serious and impartial consideration. Research of a theoretical and mathematical sort was welcome; at the same time Wilks was concerned with encouraging papers describing statistical methods of direct applicability. This was a period of considerable expansion in mathematical statistics. The number of people in the field was increasing; the level of sophistication and knowledge was rising; and the scope of statistical application and interest was broadening. The wisdom and judgment of Wilks was critical to the *Annals* during this time. When his editorship ended, the *Annals* had become the foremost publication in mathematical statistics. The *Annals* was then and is now probably the single most important and influential activity of the Institute. [See "Our Silver Anniversary," by Allen T. Craig (1960).]

Wilks contributed in other ways to the Institute. In 1940 he was president. He was a member of the Council for many years and served on innumerable committees. He gave the Rietz lecture, "The Problem of Two Samples from Continuous Distributions," to the Annual Meeting of the IMS, December 29, 1959, at Washington, D. C.

Wilks was, of course, active in other statistical circles. In particular, he was a member of the American Statistical Association and served it in many ways, being president in 1950. Wilks encouraged cooperation among the various statistical societies active in the United States. He helped plan procedures for bringing together the various societies, which resulted in the Committee of Presidents of Statistical Societies (COPSS). He was also a member of the International Statistical Institute, the American Society for Quality Control, the American Society for Human Genetics, the Biometric Society, the Market Research Council, the Operations Research Society of America and the Psychometric Society and a Fellow of the Econometric Society and the Royal Statistical Society. At the time of his death he was a vice-president of the American Association for the Advancement of Science and chairman of its Section U on Statistics.

Mathematics and its development in a wide sense received increasing attention from Wilks. From all his professional viewpoints he saw mathematics as a whole; he was anxious to keep mathematics unified. At Princeton, he carried on statistics within the mathematics department. In the war effort and later in the various national and international activities he tried to get mathematicians of all types to cooperate in scientific and social advancement. He was a member of the Division of Physical Sciences of the National Research Council 1947–49 (when mathematics was included in that division) and of the Division of Mathematics 1951–60, being chairman 1958–60. He was a member of the Division's Committee on Applied Statistics and then the Committee on Statistics 1943–61. One of the most important vehicles for cooperation among mathematicians in the United States is the Conference Board of the Mathematical Sciences. Wilks helped to create the Conference Board and was chairman in 1960. Characteristic of his appreciation of the growing function of mathematics was his talk given at a dinner meeting of the National Science Foundation Summer Institute for High School Mathematics Teachers at Teachers College, Columbia University, July 31, 1957, entitled "New Fields and Organizations in the Mathematical Sciences" [38]. He was a member of the Divisional Committee for the Mathematical, Physical and Engineering Sciences of the National Science Foundation, 1952–56, and a member of the United States National Committee on Mathematics, 1951–63, which constitutes the American delegation to the International Mathematical Union.

Another broad sphere of Wilks' interests was the social sciences. He was associated with the Social Science Research Council for many years. He was a member of the Board of Directors from 1947 to 1956 and from 1961 to his death; he was chairman of the Board 1954–55 and chairman of the Executive Committee from 1961. He served on the Board of Trustees of the Russell Sage Foundation from 1953 and on its Executive Committee from 1955. He was a member of what became the Divisional Committee for the Social Sciences of the National Science Foundation 1957–62. He wrote a brief survey "Mathematics and the Social Sciences" [44].

Wilks was a member of the U. S. National Commission for UNESCO 1960–62.

He received a Centennial Alumni Award of the University of Iowa in 1947. He was a member of the American Philosophical Society and of the American Academy of Arts and Sciences. Among his other services to the United States government were membership on the Task Group on Manpower Statistics of the Panel on Scientific and Technical Manpower of the President's Science Advisory Committee, membership on the Advisory Committee of the Office of Statistical Standards of the Bureau of the Budget since 1951 and membership on the National Academy of Sciences Committee on Battery Additives, 1953.

**2. Education and training of statisticians and mathematicians.** A major aspect of Wilks' professional life was his devotion to education in statistics and mathematics more generally. His university activities were entirely at Princeton, but his other activities were much broader and had their impact through text books that he wrote, articles on education and training, and serving on various committees and organizations.

Wilks began his teaching career at Princeton in the manner typical of the young instructor by teaching three or four courses of elementary mathematics each term, such as algebra, trigonometry, analytic geometry and calculus. Although he did not teach statistics, he did confer with students on the subject; one of these conferees was Churchill Eisenhart, who went on to take his Ph.D. in mathematical statistics at the University of London in 1937. After three years Wilks introduced two statistics courses, each of one semester's duration. One was designed for undergraduates with a mathematics background. The other was designed for graduate students in mathematics.

*Statistical Inference* ⟨1⟩, Wilks' first book, came out of the graduate course, Wilks' first statistics course. The book was planographed (typed and photographed) and issued by Edwards Brothers. Wilks was reluctant to consider these lecture notes as a bonafide publication. The edition must have been tiny for already in 1939 graduate students could not obtain copies. This book is significant because it is the first devoted to modern mathematical statistics. It includes the standard sampling distributions, as well as point estimation (maximum likelihood), interval estimation (called "fiducial inference"), and testing hypotheses. The level of exposition is uneven, as one would expect from lecture notes of a course the first time it is given; the idea of probability was introduced in terms of "aggregates" of von Mises (1931), which was awkward and which Wilks dropped from his lectures later.

Wilks gave the graduate course several more times before the war, though not every year. For a number of years the undergraduate statistics course with some mathematics prerequisite was given to students in all fields in the second half of the sophomore year. Another course was introduced as an upperclass course for students of mathematics who wanted to specialize in statistics; this course was also taken by beginning graduate students. It consisted of a rather thorough mathematical treatment of statistical inference and included a laboratory section devoted to examples and computations.

At the end of the war Wilks resumed teaching. He managed to continue his many public services by arranging his teaching schedule so that by Wednesday noon his classes were over for the week. Nevertheless he devoted much attention to the university. He saw the need for good training in statistics for students throughout the university, and he thought that the elementary training should be common to the different disciplines. To achieve these goals he felt that a university-wide course should be given by mathematical statisticians. He initiated this one-semester course and developed the curriculum by writing a syllabus. Sections were taught by other statisticians in the mathematics department. Wilks also recruited instructors from other departments; not only did this recruitment augment the teaching staff, it also increased cooperation with the different departments. While the course was kept at a high level of rigor and demanded much of the student, it served the university broadly; perhaps 20% of Princeton undergraduates take the course.

Wilks' syllabus went through several dittoed versions before it was published by Princeton University Press as *Elementary Statistical Analysis* ⟨3⟩. As earlier, Wilks was unwilling to give his manuscript the formality of letter-press printing and cloth binding. First, frequency distributions and the sample mean and standard deviation are introduced. After an elementary discussion of probability, distributions are studied in general terms and specific terms. Then some elements of statistical inference are discussed. The course is planned to be given without a laboratory session and without the use of calculating machines.

The one-semester course in statistics for students with mathematics background continued. Since it was intended primarily for science and engineering students, a comparable course for students in the social sciences was added. Together with Irwin Guttman, Wilks worked up the syllabus for the former course to a manuscript, entitled *Introductory Engineering Statistics* ⟨5⟩. At present Guttman is readying the manuscript for publication.

Soon after Wilks came to Princeton in 1933 he started attracting graduate students. His first doctoral student, Joseph F. Daly, received his Ph.D. in 1939, and George W. Brown and Alexander M. Mood followed in 1940. After a pause due to the war, the flow of doctorates resumed. Princeton has granted Ph.D.'s to approximately 40 men in mathematical statistics and probability. All these studied to some extent with Wilks; the dissertations of about half of them were supervised by Wilks. These alumni include several department chairmen, professors in a number of universities, and statisticians in government and industry. Wilks usually had two or three undergraduates writing senior theses under his direction; several worked out suggestions of Wilks and the results were published. Postdoctoral students and other associates benefited from his advice and stimulation.

After the war, John W. Tukey shared the responsibility of teaching statistics; later Francis J. Anscombe joined the staff. William Feller came to carry the major load in probability. A number of statisticians visited Princeton University, some to lecture and some for research. The Statistical Research Group, broadened to include basic research and supported by the Office of Naval Research, continued after the war with Wilks as project director.

Wilks made his contributions to Princeton University in many ways. He even brought statistics to bear on that great American game of football. On his advice movies of each game were analyzed; a score was assigned to each player on each play. This assisted the coach to evaluate players under different conditions, and hence enabled him to optimize the use of his manpower. It is said that this technique contributed substantially to Princeton's success in the sport.

Wilks' concern about instruction and training in statistics had broad scope, and he influenced education in many areas. In 1947 the Committee on Applied Mathematical Statistics of the National Research Council prepared a report entitled "Personnel and Training Problems Created by the Recent Growth of Applied Statistics in the United States" [National Research Council (1947)]. This report presented the growing use of statistical methods, the demand for personnel, problems of training, and recommendations. As secretary of the committee, Wilks wrote a summary [30] of the report.

In the same year Wilks published "Statistical Training for Industry" [31] in *Analytical Chemistry*. He surveyed some of the uses of statistics in industry, such as operations research and consumer research. He noted that the Committee on Applied Mathematical Statistics had considered the need for personnel in quality control so critical in 1942 that it recommended a nationwide program of short intensive courses; Wilks had been active in following out the recommendation with 34 such courses in the latter part of the war.

Wilks' Presidential Address at the annual meeting of the American Statistical Association, December 28, 1950, was entitled "Undergraduate Statistical Education" [35]. Drawing on his experience at Princeton, he urged teaching of the principles of statistical inference to freshmen and sophomores. He further proposed revamping the high school curriculum in mathematics to provide for topics in probability, statistics, logic, and other modern mathematical subjects. In an invited address at the Thirty-eighth Summer Meeting of the Mathematical Association of America, August 28, 1957, he continued his discussion of "Teaching Statistical Inference in Elementary Mathematics Courses" [39].

In furtherance of his ideas on the revision of mathematical instruction in secondary schools, Wilks served on the Commission on Mathematics of the College Entrance Examination Board, 1955–58, which prepared a report issued by the College Entrance Examination Board (1959), recommending major changes in the teaching of mathematics in secondary schools and suggesting inclusion of an option of Introductory Probability with Statistical Applications in Grade 12. He collaborated with Edwin C. Douglass, Frederick Mosteller, Richard S. Pieters, Donald E. Richmond, Robert E. K. Rourke, and George B. Thomas on an experimental text, *Introductory Probability and Statistical Inference for Sec-*

*ondary Schools* (1957). Later he was a member of the Advisory Board of the School Mathematics Study Group (SMSG).

During the academic year 1958–59 he gave 26 talks in 7 colleges in Oklahoma and Texas as a Visiting Lecturer for the Mathematical Assoiation of America. In 1963–64 he was a Visiting Lecturer in Statistics for the South and Southwest in the program arranged by COPSS. The development of statistics in that part of the country was a recurring interest of his.

Throughout his career at Princeton, Wilks worked with the College Entrance Examination Board and the Educational Testing Service. He was Assistant to the Director of Reading and then Associate Director of Reading 1934–41. He was chairman of the Graduate Record Examination Mathematics Test Committee from 1951 on. For many years he was mathematics consultant to the Committee for the Preliminary Actuarial Examination. A Study Group met with him from 1959 to 1961 and came out with report {12} which included 30 pages of summary and conclusions written by Wilks (and almost 200 pages of appendices by members of the Study Group). There is a popular Princeton story of how Sam persuaded the College Board to face up to the uncertainties of grading essay questions. When the second reader knew the grade given by the first reader, the two grades agreed pretty well; when—at Wilks' suggestion—the grade of the first reader was concealed from the second reader, the discrepancies in grades were so dramatic the need for a change was unquestioned.

Wilks neglected no phase of mathematical education. In his last few years he worked with an experimental program in Miss Mason's school in Princeton, which involved new mathematics at the elementary level, down to kindergarten.

Another significant contribution to education was his editing, with Walter A. Shewhart, of the Wiley Publications in Statistics (John Wiley and Sons, Inc.). With approximately 30 titles in Mathematical Statistics and an equal number in Applied Statistics listed, it has been the leading series of books in statistics.

In yet another direction—Wilks was a director of the University of the Andes, Bogota, Colombia.

## 3. Research and other writings.

*3.1. Research in multivariate statistical analysis.* The first ten papers published by Wilks were in the field of multivariate statistical analysis, and this was the branch of statistical research to which he made the greatest contributions. At the time that Wilks was getting into statistics many of the leading statisticians were active in this area. Fisher (1928) had derived the exact distribution of the multiple correlation coefficient in samples from a multivariate normal distribution; Wishart (1928) had found the distribution of the sample covariance matrix; and Hotelling (1931) had published the distribution of the multivariate generalization of Student's $t$-statistic. The early approach to problems of multivariate analysis of Karl Pearson was to find standard errors of statistics, such as the correlation coefficient. This approach had given way to finding the exact sampling distributions. Student had argued the distribution of the $t$-statistic (1908a) and conjectured the distribution of the correlation coefficient in the case of independence (1908b); Fisher (1915) had derived the distribution in general. A good deal of Wilks' work followed along in this direction—exact sampling distributions of statistics for the multivariate normal distribution. Many of these statistics he developed to test various null hypotheses.

Wilks' first research was reported in his doctoral dissertation entitled "On the distributions of statistics in samples from a normal population of two variables with matched sampling of one variable," published as [2], preceded by a brief note [1]. Let $x$ and $y$ have a bivariate normal distribution. Suppose $N$ observations are drawn so that the $N$ values of the $x$ variables are $N$ specified numbers, say $x_1, \cdots, x_N$. Then the corresponding $y$ values, say $y_1, \cdots, y_N$, are independently normally distributed with variance $\sigma_y^2(1 - \rho_{xy}^2)$ and expected values $\mu_y + \beta(x_i - \mu_x)$, $i = 1, \cdots, N$. The model is one of regression and can be obtained without assuming the bivariate normal distribution, but I classify this work as multivariate since Wilks considered it in that fashion. Then Wilks found the distribution of the mean $\bar{y}$ and of any linear combination of the $y$'s (normal), of the sample variance of $y$ (noncentral $\chi^2$), of the correlation between $y$ and $x$, and functions of these quantities. The method used was that of characteristic functions, though the terminology was different; Wilks called inversion of the characteristic function "solving an integral equation," referring to Romanowsky (1925), and argued the uniqueness of the solution by the theory of closure of Stekloff (1914).

This problem was suggested to Wilks by Lindquist, who had used the technique of matched samples in experimental work in educational psychology. Several of the mathematical statistical problems that Wilks studied arose similarly out of questions raised by people using statistical methods. Wilks had considerable skill in abstracting a fairly simple mathematical problem from some substantive question. In this paper as in many others Wilks introduced the mathematical study by describing a practical problem.

The next paper [3] was devoted to finding maximum likelihood estimates of the means, variances, and covariance of a bivariate normal distribution when some observations come in pairs ($x$ and $y$) and some on each variable alone. Since the equations obtained by setting the derivatives of the likelihood equal to zero are in general very complicated, Wilks considered estimating some of the parameters when the others are known, such as estimating the means when the variances and covariance are known. Wilks pointed out that the covariances of the asymptotic normal distribution of the maximum likelihood estimates was the inverse of the information matrix (consisting of the expected values of the second derivatives of the likelihood function). Wilks proposed the determinant of the inverse of the asymptotic covariance matrix of any set of estimates as a measure of the information in those estimates and proposed the ratio of this determinant to the determinant of the information matrix as a measure of the joint efficiency of those estimates. He proved that the efficiency of any set of estimates cannot be greater than one. His proof is based on the demonstration of Fisher (1925) for one parameter and is subject to the corresponding limitations. The problem of missing observations is still present; recent papers on this subject are by Anderson (1957) and Trawinski and Bargmann (1964).

In a series of papers ([4], [6], [7], [8], [9], [28], and [29]) Wilks found the likelihood ratio criteria for testing various hypotheses about multivariate normal distributions, derived the moments of the criteria under the respective null hypotheses, and characterized the corresponding distributions. The criteria are powers of ratios of products of determinants of sample covariance matrices, and the moments are products of beta functions. The methods used in the papers were somewhat similar; these ideas will be sketched here.

A vector $x' = (x_1, \cdots, x_p)$ of $p$ random variables is distributed according to the multivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with mean vector $\boldsymbol{\mu}' = (\mu_1, \cdots, \mu_p)$ and covariance matrix $\boldsymbol{\Sigma} = (\sigma_{ij})$, $i, j = 1, \cdots, p$, if its density function is

$$n(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{1}{2}p}|\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]. \tag{1}$$

For a sample $\mathbf{x}_1, \cdots, \mathbf{x}_N$ from $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$,

$$N\bar{\mathbf{x}} = \textstyle\sum_{\alpha=1}^{N} \mathbf{x}_\alpha, \qquad \mathbf{A} = \textstyle\sum_{\alpha=1}^{N} (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})' \tag{2}$$

constitute a sufficient set of statistics, $\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}}$ and $\hat{\boldsymbol{\Sigma}} = (1/N)\mathbf{A}$ are the maximum likelihood estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, and $\bar{\mathbf{x}}$ and $\mathbf{S} = (1/n)\mathbf{A}$, where $n = N - 1$, are unbiased estimates. $\bar{\mathbf{x}}$ and $\mathbf{A}$ are independently distributed, $\bar{\mathbf{x}}$ having the distribution $N[\boldsymbol{\mu}, (1/N)\boldsymbol{\Sigma}]$ and $\mathbf{A}$ having the Wishart distribution $W(\boldsymbol{\Sigma}, n)$ [Wishart (1928)] with density (where $\mathbf{A}$ is positive definite)

$$w(\mathbf{A} \mid \boldsymbol{\Sigma}, n) = K(\boldsymbol{\Sigma}, n)|\mathbf{A}|^{\frac{1}{2}(n-p-1)} \exp\left(-\tfrac{1}{2} \operatorname{tr} \boldsymbol{\Sigma}^{-1}\mathbf{A}\right), \tag{3}$$

where

$$\begin{aligned} K^{-1}(\boldsymbol{\Sigma}, n) &= \int \cdots \int |\mathbf{A}|^{\frac{1}{2}(n-p-1)} \exp\left(-\tfrac{1}{2} \operatorname{tr} \boldsymbol{\Sigma}^{-1}\mathbf{A}\right) d\mathbf{A} \\ &= 2^{\frac{1}{2}np}\pi^{p(p-1)/4}|\boldsymbol{\Sigma}|^{\frac{1}{2}np}\textstyle\prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(n + 1 - i)] \end{aligned} \tag{4}$$

and $\int \cdots \int \cdots d\mathbf{A}$ indicates integration over the $\frac{1}{2}p(p + 1)$-dimensional space of elements of positive definite matrices $\mathbf{A}$.

In [6] Wilks defined $|\hat{\boldsymbol{\Sigma}}|$ as the *generalized variance* of the sample and suggested it as a measure of scatter of the $N$ points in $p$-dimensional space; it is a generalization of the one-dimensional variance and has some of its properties. Considerably later, Wilks wrote an expository paper [43], which developed many likelihood ratio criteria in these terms of scatter.

Wilks initiated his methods of finding moments in [4] for the multiple correlation coefficient, but possibly the simplest case is finding the moments of the generalized variance $|\hat{\mathbf{\Sigma}}| = |\mathbf{A}|/N^p$. The $h$th moment of $|\mathbf{A}|$ is

$$
\begin{aligned}
\mathcal{E}|\mathbf{A}|^h &= \int \cdots \int |\mathbf{A}|^h w(\mathbf{A} \mid \mathbf{\Sigma}, n)\, d\mathbf{A} \\
&= \int \cdots \int K(\mathbf{\Sigma}, n)|\mathbf{A}|^{h+\frac{1}{2}(n-p-1)} \exp\left(-\tfrac{1}{2} \operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{A}\right) d\mathbf{A} \\
&= \frac{K(\mathbf{\Sigma}, n)}{K(\mathbf{\Sigma}, n + 2h)} \qquad (5) \\
&\quad \cdot \int \cdots \int K(\mathbf{\Sigma}, n + 2h)|\mathbf{A}|^{\frac{1}{2}[(n+2h)-p-1]} \exp\left(-\tfrac{1}{2} \operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{A}\right) d\mathbf{A} \\
&= \frac{K(\mathbf{\Sigma}, n)}{K(\mathbf{\Sigma}, n + 2h)} \int \cdots \int w(\mathbf{A} \mid \mathbf{\Sigma}, n + 2h)\, d\mathbf{A},
\end{aligned}
$$

which is

$$
\frac{K(\mathbf{\Sigma}, n)}{K(\mathbf{\Sigma}, n + 2h)} = 2^{hp}|\mathbf{\Sigma}|^h \prod_{i=1}^{p} \frac{\Gamma[h + \frac{1}{2}(n + 1 - i)]}{\Gamma[\frac{1}{2}(n + 1 - i)]}. \tag{6}
$$

The $h$th moment of $\chi^2_{n+1-i}$, having the $\chi^2$-distribution with $n + 1 - i$ degrees of freedom, is

$$
\begin{aligned}
\mathcal{E}(\chi^2_{n+1-i})^h &= \int_0^\infty 2^{-\frac{1}{2}(n+1-i)}\Gamma^{-1}[\tfrac{1}{2}(n + 1 - i)]u^{h+\frac{1}{2}(n+1-i)-1}e^{-\frac{1}{2}u}\, du \\
&= 2^h\Gamma[h + \tfrac{1}{2}(n + 1 - i)]/\Gamma[\tfrac{1}{2}(n + 1 - i)].
\end{aligned} \tag{7}
$$

Hence, $|\mathbf{A}|$ has the same moments as $|\mathbf{\Sigma}|$ times $\prod_{i=1}^p \chi^2_{n+1-i}$.

In this same paper Wilks found the likelihood ratio criterion for testing the hypothesis

$$
H_0 : \boldsymbol{\mu}^{(1)} = \cdots = \boldsymbol{\mu}^{(q)} \tag{8}
$$

on the basis of samples $\mathbf{x}_1^{(1)}, \cdots \mathbf{x}_{N_1}^{(1)}, \cdots, \mathbf{x}_1^{(q)}, \cdots, \mathbf{x}_{N_q}^{(q)}$ from $N(\boldsymbol{\mu}^{(1)}, \mathbf{\Sigma}), \cdots, N(\boldsymbol{\mu}^{(q)}, \mathbf{\Sigma})$, respectively. Here let

$$
\begin{aligned}
\mathbf{A} &= \textstyle\sum_{i=1}^q \sum_{\alpha=1}^{N_i} (\mathbf{x}_\alpha^{(i)} - \bar{\mathbf{x}}^{(i)})(\mathbf{x}_\alpha^{(i)} - \bar{\mathbf{x}}^{(i)})', \\
\mathbf{B} &= \textstyle\sum_{i=1}^q N_i(\bar{\mathbf{x}}^{(i)} - \bar{\mathbf{x}})(\bar{\mathbf{x}}^{(i)} - \bar{\mathbf{x}})',
\end{aligned} \tag{9}
$$

where $\bar{\mathbf{x}}^{(i)}$ is the mean of the $i$th sample and $\bar{\mathbf{x}}$ is the pooled mean. Under the null hypothesis the $N = N_1 + \cdots + N_q$ observations are from a common normal distribution and the maximum likelihood estimate of $\mathbf{\Sigma}$ is $(1/N)(\mathbf{A} + \mathbf{B})$; when the null hypothesis is not assumed true, the maximum likelihood estimate of $\mathbf{\Sigma}$ is the weighted average of the estimate in the $q$ samples, namely $(1/N)\ \mathbf{A}$. The likelihood ratio criterion for testing the null hypothesis is the $N/2$ power of

$$
\Lambda = \frac{|\mathbf{A}|}{|\mathbf{A} + \mathbf{B}|}. \tag{10}
$$

**A** and **B** are multivariate analogs of the "within" and "between" sums of squares in the one-way analysis of variance, and $\Lambda$ corresponds to $1/(1 + mF/n)$, where $F$ is the $F$-ratio with $m = q - 1$ and $n = N - q$ degrees of freedom.

Wilks' procedure for finding the moments of $\Lambda$ uses the fact that **A** has the distribution $W(\mathbf{\Sigma}, n)$ and **B** has a distribution, say $F(\mathbf{B})$ such that $\mathbf{C} + \mathbf{B}$ has the Wishart distribution $W(\mathbf{\Sigma}, r + q - 1)$ if **C** has a distribution $W(\mathbf{\Sigma}, r)$. Thus

$$
\begin{aligned}
\mathcal{E}\Lambda^h &= \mathcal{E}\, |\mathbf{A}|^h |\mathbf{A} + \mathbf{B}|^{-h} \\
&= \int \cdots \int\int \cdots \int |\mathbf{A}|^h |\mathbf{A} + \mathbf{B}|^{-h} w(\mathbf{A} \mid \mathbf{\Sigma}, n)\, d\mathbf{A}\, dF(\mathbf{B}) \\
&= \int \cdots \int\int \cdots \int K(\mathbf{\Sigma}, n)\, |\mathbf{A}|^{\frac{1}{2}[(n+2h)-p-1]} \\
&\qquad\qquad \cdot \exp\left(-\tfrac{1}{2} \operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{A}\right) |\mathbf{A} + \mathbf{B}|^{-h}\, d\mathbf{A}\, dF(\mathbf{B}) \qquad (11) \\
&= \frac{K(\mathbf{\Sigma}, n)}{K(\mathbf{\Sigma}, n + 2h)} \int \cdots \int\int \cdots \int |\mathbf{A} + \mathbf{B}|^{-h} w(\mathbf{A} \mid \mathbf{\Sigma}, n + 2h)\, dF(\mathbf{B}) d\mathbf{A} \\
&= \frac{K(\mathbf{\Sigma}, n)}{K(\mathbf{\Sigma}, n + 2h)}\, \mathcal{E}\, |\mathbf{A} + \mathbf{B}|^{-h},
\end{aligned}
$$

where $\mathcal{E}|\mathbf{A} + \mathbf{B}|^{-h}$ is computed on the basis of **A** (formally) having the density $w(\mathbf{A} \mid \mathbf{\Sigma}, n + 2h)$; that is, $\mathbf{A} + \mathbf{B}$ having the distribution $W(\mathbf{\Sigma}, n + 2h + q - 1)$. Since (5) holds for negative moments to a certain order,

$$
\begin{aligned}
\mathcal{E}\Lambda^h &= \frac{K(\mathbf{\Sigma}, n)}{K(\mathbf{\Sigma}, n + 2h)} \frac{K(\mathbf{\Sigma}, n + 2h + q - 1)}{K(\mathbf{\Sigma}, n + q - 1)} \\
&= \prod_{i=1}^{p} \frac{\Gamma[\frac{1}{2}(n + 1 - i) + h]\Gamma[\frac{1}{2}(n + q - i)]}{\Gamma[\frac{1}{2}(n + 1 - i)]\Gamma[\frac{1}{2}(n + q - i) + h]} \qquad (12) \\
&= \prod_{i=1}^{p} B[\tfrac{1}{2}(n + 1 - i) + h, \tfrac{1}{2}(q - 1)]/B[\tfrac{1}{2}(n + 1 - i), \tfrac{1}{2}(q - 1)],
\end{aligned}
$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the beta function. Then $\Lambda$ is distributed as $\prod_{i=1}^{p} v_i$, where $v_1, \cdots, v_p$ are independent and $v_i$ has the density

$$v^{\frac{1}{2}(n+1-i)-1}(1 - v)^{\frac{1}{2}(q-1)-1}/B[\tfrac{1}{2}(n + 1 - i), \tfrac{1}{2}(q - 1)].$$

In this same paper Wilks found the likelihood ratio criterion for testing the null hypothesis

$$H_1 : \mathbf{\Sigma}_1 = \cdots \mathbf{\Sigma}_q \qquad (13)$$

on the basis of samples of size $N_1, \cdots, N_q$ from $N(\boldsymbol{\mu}^{(1)}, \mathbf{\Sigma}_1), \cdots, N(\boldsymbol{\mu}^{(q)}, \mathbf{\Sigma}_q)$, respectively. It is a constant times the square root of

$$\prod_{i=1}^{q} |\hat{\mathbf{\Sigma}}_i|^{N_i}/|\hat{\mathbf{\Sigma}}|^{N}, \qquad (14)$$

where $\hat{\mathbf{\Sigma}} = (1/N)\mathbf{A}$ and $\hat{\mathbf{\Sigma}}_i$ is the maximum likelihood estimate of $\mathbf{\Sigma}_i$ based on the

$i$th sample. The moments of this criterion under the null hypothesis were found by a further development of the technique based on Wishart distributions; they are again ratios of products of gamma functions. The likelihood ratio criterion for testing $H_0$ and $H_1$ simultaneously (that is, that the $q$ populations are identical) is the product of the two criteria for the separate hypotheses and under $H_1$ they are independently distributed. These statistical methods were developed in greater detail for the bivariate case ($p = 2$) by E. S. Pearson and Wilks [7]. Neyman and Pearson (1928), (1931) had earlier proposed likelihood ratio tests in general and had derived them in particular for the univariate normal distribution.

In [6] Wilks also found moments for some other statistics such as the ratios of two independent generalized variances and the ratio of a generalized variance to one of its principal minors (that is, $|\hat{\mathbf{\Sigma}}|/|\hat{\mathbf{\Sigma}}^*|$, where $\hat{\mathbf{\Sigma}}^*$ is formed from $\hat{\mathbf{\Sigma}}$ by deleting columns and corresponding rows). In [4] Wilks had found the moments of $1 - R^2$, where $R$ is the multiple correlation coefficient between $x_1$ and $x_2, \cdots, x_p$; since

$$(15) \qquad 1 - R^2 = |\hat{\mathbf{\Sigma}}|/(\hat{\sigma}_{11}|\hat{\mathbf{\Sigma}}_{22}|),$$

where $|\hat{\mathbf{\Sigma}}_{22}|$ is the cofactor of $\hat{\sigma}_{11}$ in $\hat{\mathbf{\Sigma}}$, it will be seen that [6] exploits the idea put forth in [4]. In [8] Wilks obtained a method of deriving such moments directly from the multivariate normal distribution without using the Wishart distribution.

Let $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(k)}$ be subvectors consisting of $p_1, \cdots, p_k$ components of $\mathbf{x}$, respectively. In [9] Wilks found the likelihood ratio criterion for testing the hypothesis that the subvectors are independently distributed. If the population covariance matrix is partitioned into submatrices similarly

$$(16) \qquad \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} & \cdots & \mathbf{\Sigma}_{1k} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} & \cdots & \mathbf{\Sigma}_{2k} \\ \vdots & \vdots & & \vdots \\ \mathbf{\Sigma}_{k1} & \mathbf{\Sigma}_{k2} & \cdots & \mathbf{\Sigma}_{kk} \end{pmatrix},$$

the null hypothesis is $\mathbf{\Sigma}_{ij} = \mathbf{0}$, $i \neq j$, and the likelihood ratio criterion is the $N/2$ power of

$$(17) \qquad \frac{|\hat{\mathbf{\Sigma}}|}{\begin{vmatrix} \hat{\mathbf{\Sigma}}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{\Sigma}}_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \hat{\mathbf{\Sigma}}_{kk} \end{vmatrix}} = \frac{|\hat{\mathbf{\Sigma}}|}{\prod_{i=1}^k |\hat{\mathbf{\Sigma}}_{ii}|}.$$

Wilks found the moments, expressed the criterion as a product of random variables, and found explicit expressions for the distributions in special cases.

In [28] Wilks found the likelihood ratio criteria for testing the hypotheses $H_{vc}$ that all $\sigma_{ii}$ are equal and all $\sigma_{ij}$, $i \neq j$, are equal, $H_m$ that all $\mu_i$ are equal given $H_{vc}$ is true, and that $H_{vc}$ and $H_m$ are true. The criteria are ratios of determinants, and the moments are ratios of gamma functions. In [29] Tukey and Wilks approximate the distributions of the criteria by $F$-distributions. Votaw (1948), in his doctoral dissertation under Wilks' direction, generalized this study to partitioned vectors and matrices. These hypotheses are hypotheses of symmetry. Suppose $x_1, \cdots, x_p$ are scores on $p$ psychological tests; the idea that they are parallel tests is made explicit by $H_{vc}$ and $H_m$. It was this substantive problem that Wilks abstracted to the mathematical problem. [Many of the details of these results are given in Wilks' book ⟨4⟩ and in Anderson (1958).]

These papers initiated a considerable development of test procedures in multivariate analysis. Hotelling (1931) had generalized Student's $t$-test; Wilks now generalized a number of univariate tests, particularly in the analysis of variance. The distribution of these statistics have been studied further. Bartlett (1938) suggested a multiple of $-\log \Lambda$ as a $\chi^2$-variable in large samples under the null hypothesis; Wald and Brookner (1941) found an asymptotic expansion of the distribution of the criterion for independence of subvectors; Rao (1948) modified this expansion and showed Bartlett's approximation was of order $n^{-2}$; Box (1949) found a more general asymptotic expansion of the distribution of the logarithm of these likelihood ratio criteria. Other criteria have been suggested for testing these hypotheses. For the general hypothesis of equality of means, Lawley (1938) and Hotelling (1947) have suggested $\operatorname{tr} \mathbf{BA}^{-1}$ and Roy (1953) has suggested the largest characteristic root of $\mathbf{BA}^{-1}$. Properties of these procedures have been studied by Roy (1957) and others.

Five other papers ([5], [10], [12], [16], and [48]) fall in the province of multivariate analysis. In [5] Wilks found the first and second moments of the tetrad $t_{1234} = r_{12}r_{34} - r_{13}r_{24}$, where $r_{ij} = s_{ij}/(s_{ii}s_{jj})^{\frac{1}{2}}$, in case the components of $\mathbf{x}$ are independent; tetrads are used in factor analysis to test whether the part of $\mathbf{\Sigma}$ not including the main diagonal is of rank one. [10] is an expository paper on the likelihood ratio tests discussed previously.

In [12] Wilks studied a multivariate version of an intraclass correlation model which is a slight generalization of a components of variance model in the analysis of variance (Model II). The random vector $\mathbf{x}$ composed of subvectors $\mathbf{x}^{(1)}$, $\cdots$, $\mathbf{x}^{(k)}$ is assumed to have a multivariate normal distribution; the expected value of each subvector is $\boldsymbol{\mu}$; the covariance matrix of $\mathbf{x}^{(i)}$ is $\mathbf{\Sigma}$; and the covariance between $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$, $i \neq j$, is $\mathbf{\Sigma}^*$. $N$ observations are made on $x$. Wilks finds the likelihood ratio test of $\mathbf{\Sigma}^* = \mathbf{0}$, shows that certain matrices involved in the criteria have independent Wishart distributions, and develops a general method based on characteristic functions.

In [16] Wilks showed that the correlation between two linear functions $L_1 = \sum_{i=1}^n k_i x_i$ and $L_2 = \sum_{i=1}^n l_i x_i$ goes to 1 as $n$ increases under certain conditions on the constants $k_1, k_2, \cdots$ and $l_1, l_2, \cdots$ and the joint distributions of $x_1, \cdots, x_n$, $n = 1, 2, \cdots$. Each variable $x_i$ may be interpreted as the score of an individual on the $i$th item of a test (for example, $x_i = 1$ for a correct answer to a question and $x_i = 0$ for an incorrect answer); $k_1, \cdots, k_n$ and $l_1, \cdots, l_n$ are alternative sets of weights for the items and $L_1$ and $L_2$ are the test scores of the individual under the alternative scoring methods. The first result in the paper is a solace to every instructor, for if the test is long enough it does not matter much how much weight is given to each part in the sense that two scores based on two different weighting schemes would be highly correlated.

In [48] Wilks makes use of the generalized variance again. From a set of $N$ vector observations one calculates the generalized variance $|\hat{\mathbf{\Sigma}}|$ of the entire sample and the generalized variance $|\hat{\mathbf{\Sigma}}_\alpha|$ of the sample omitting the $\alpha$th observation, $\alpha = 1, \cdots, N$. If $\min_\alpha|\hat{\mathbf{\Sigma}}_\alpha|/|\hat{\mathbf{\Sigma}}|$ is sufficiently small, the observation corresponding to the minimum is called an "outlier" because eliminating it substantially reduces the scatter. Wilks considered several outliers and treated the associated probability problems.

*3.2. Other research.* From studying likelihood ratio tests for multivariate normal distributions, Wilks went on to investigating likelihood ratio tests and related confidence interval procedures in other contexts. In [11] he found the likelihood ratio criteria for (1) testing the hypothesis that a multinomial distribution is a given one, (2) testing the hypothesis that several multinomial distributions are the same and (3) testing the hypothesis of independence in a contingency table; in each case he showed that a function of the criterion ($-2 \log \lambda$) had the appropriate asymptotic $\chi^2$-distribution and that it was the same as the $\chi^2$-criterion to terms of order $N^{-\frac{1}{2}}$. In [14] he considered sampling from a population with density $f(x, \theta_1, \cdots, \theta_h)$ and testing the hypothesis $\theta_{m+1} = \theta_{0,m+1}, \cdots, \theta_h = \theta_{0h}$; Wilks showed that if the maximum likelihood estimates $\hat{\theta}_1, \cdots, \hat{\theta}_h$ have an asymptotic multivariate normal distribution [for example, Doob (1935)], then when the null hypothesis is true $-2 \log \lambda$ has a limiting $\chi^2$-distribution with $h - m$ degrees of freedom, where $\lambda$ is the likelihood ratio criterion for testing the hypothesis. In [17] Wilks considered sampling from a population with density $f(x, \theta)$ and basing confidence intervals on

$$\psi^* = n^{-\frac{1}{2}} A^{*-1} \sum_{i=1}^{n} h(x_i, \theta), \tag{18}$$

where $\mathcal{E}h(x, \theta) = 0$ and $\mathcal{E}h^2(x, \theta) = A^{*2}$. If $\mathcal{E}h^3(x, \theta) < \infty$, the limiting distribution of $\psi^*$ is normal with mean 0 and variance 1, and asymptotic confidence intervals can be based on this fact. Wilks showed that the shortest average confidence limits based on statistics of the form (18) came about when $h(x, \theta) = \partial \log f(x, \theta)/\partial\theta$ in the sense that the absolute value of $\mathcal{E}(\partial\psi^*/\partial\theta)$ was maximized. [18] was an expository paper on confidence intervals; Wilks used "confidence limits" and "fiducial limits" interchangeably. [20] (written with Daly) generalizes the results to the case of several parameters; if $\theta$ above is replaced by $\theta_1, \cdots, \theta_h$, $h(x, \theta)$ by $h_1(x, \theta_1, \cdots, \theta_h), \cdots, h_h(x, \theta_1, \cdots, \theta_h)$, then $\partial\psi^*/\partial\theta$ is replaced by the Jacobian $|\partial\psi_i^*/\partial\theta_i|$ in the result. These papers cleared up questions in the use of the maximum likelihood technique in large samples. Several of these results were generalized by Wald (1942), (1943a).

Some other papers dealt with topics in analysis of variance and regression. At the request of E. S. Pearson, Wilks [13] found the moments of the likelihood ratio criterion for testing equality of variances in several univariate normal populations (generalized in [6]) when the hypothesis is not necessarily true and suggested a Pearson Type III distribution to approximate the distribution of the criterion. The second part of this *Biometrika* contribution was "An investigation into the adequacy of Dr Wilks's curves," by Catherine M. Thompson (1937), who judged the proposed approximate distributions sufficiently accurate on the basis of a random sampling experiment. In [15] Wilks set up a general analysis of variance model in terms of regression by using dummy variates (being 1 and 0 to denote presence and absence of a parameter); he obtained simplified computations for the sums of squares used in testing various hypotheses. In [34] Gulliksen and Wilks considered a model where $y_{ki}$ is distributed according to $N(\alpha_k + \beta_k x_{ki}, \sigma_k^2)$, $i = 1, \cdots, n_k$, $k = 1, \cdots, K$. Procedures were found for testing the following null hypotheses:

$$(19) \qquad \begin{aligned} H_A:&\quad \sigma_1^2 = \cdots = \sigma_K^2, \\ H_B:&\quad \beta_1 = \cdots = \beta_K \quad \text{given } H_A, \\ H_C:&\quad \alpha_1 = \cdots = \alpha_K \quad \text{given } H_A \text{ and } H_B. \end{aligned}$$

The motivation for this problem came from educational testing; individuals may have been selected on the basis of a test score $x$ and tested later on $y$; the questions have to do with whether the $K$ tests for $y$ are equivalent. In two-stage sampling of a finite population a sample of primary units is drawn at the first stage, and at the second stage a sample of objects is drawn from each primary unit drawn at the first stage. In [42] Wilks studied methods of determining the second-stage samples so as not to specify a sample requiring more units than in that primary unit.

An important area of Wilks' research was that of nonparametric or distribution-free methods. Some of his students in doctoral dissertations and in senior theses tackled problems in this area. Wilks' first published work was on tolerance limits [23]. Tolerance limits for a characteristic of a manufactured item, say the diameter of a ball bearing, would be the lower and upper limits of acceptable values of the characteristic. Shewhart (1939) asked for a sample size $n$ and two tolerance limits $L_1(x_1, \cdots, x_n)$ and $L_2(x_1, \cdots, x_n)$, where $x_1, \cdots, x_n$ are the sample values from a population with arbitrary continuous cumulative distribution function $F(x)$, so that the probability is at least $\alpha$ that the proportion of the population covered by $(L_1, L_2)$,

$$(20) \qquad P = F(L_2) - F(L_1),$$

is at least $\beta$. Wilks refined the problem, requiring $\mathcal{E}P = a$ (specified) and $\Pr\{\beta < P < \gamma\} \geqq \alpha$. Wilks showed that $\mathcal{E}P = a$ if $L_1$ is the $r$th smallest observation and $L_2$ is the $r$th largest observation and $r = \frac{1}{2}(n+1)(1-a)$. (Here $a$ and $n$ must be values so $r$ is an integer.) Then $P$ has the density $P^{n-2r}(1-P)^{2r-1}/$

$B(n - 2r + 1, 2r)$, and $n$ can be found to satisfy the other condition. In [26] Wilks focused attention on the proportion $N_0/N$ of observations in a second sample of size $N$ that will fall between $L_1$ and $L_2$, instead of the proportion $P$ of the population; he obtains a formula for the probability of $N_0$ in terms of factorials. Wald (1943b) extended Wilks' ideas to the multivariate case; Tukey (1947) showed how the procedure could be applied to various types of regions. Since then a number of authors have generalized these results.

In [40] Wilks found the probability that it takes a second sample of size $N$ to obtain $k$ observations that exceed the $r$th largest observation in the first sample of size $n$. This problem has some similarity to the one-sided tolerance problem, but here the random variable is the second sample size. In [45] Wilks considered the problem of using a sample $x_1, \cdots, x_n$ from $F(x)$ and a sample $y_1, \cdots, y_m$ from $G(y)$ to test the hypothesis that $F(x) \equiv G(x)$. Let $r_j$ be the number of $y$'s that fall between the two $x$'s that are $j - 1$ and $j$ in order, $j = 1, \cdots, n + 1$. [If the null hypothesis is true, we would expect the $r_j$'s to be about $m/(n + 1)$.] Let $s_i$ be the number of $r_j$'s equal to $i$, $i = 0, 1, \cdots, m$. The asymptotic distribution of these is normal as $n, m \to \infty$ with $m/n \to \lambda > 0$. Wilks proposed the corresponding quadratic form as a test statistic; it has a limiting $\chi^2$-distribution.

"Order Statistics" [32] was an elaboration of an address to the Summer meeting of the American Mathematical Society in New Haven, Connecticut, September 4, 1947, by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings. This was an extensive review (90 references cited) of nonparametric statistical inference, including distributions of ordered observations (order statistics), distributions of coverage, tests for one and two samples based on randomization, and tests of independence in two or more dimensions. [41] was a more recent survey of nonparametric statistics. Wilks' Rietz Lecture in 1959 was an exposition of nonparametric tests of whether two continuous distributions are the same.

*3.3. Other published papers.* Wilks' broad interests and versatility are demonstrated by the many expository papers he wrote over the years. "The Rise of Modern Statistical Science" [19] delivered at the Conference on Engineering and Industrial Statistics held at the Massachusetts Institute of Technology, September 8–9, 1938, began with a brief history of statistics and concluded with a survey of its uses in engineering and industry. The application of statistics to polling was the subject of [21] and [22]. The former paper described representative sampling; the latter was essentially an appendix that explains confidence interval procedures for the parameter of a binomial distribution. [25] was presented to the 49th Annual Meeting of the Society for the Promotion of Engineering Education at the University of Michigan, June 23–27, 1941. [27] was a 48 page introduction to mathematical statistics, including probability, sampling distributions, multivariate statistics, tests of hypotheses, and large-sample theory. [33] was a review of statistics in engineering and industry. [37] was a survey of the design of experiments according to R. A. Fisher, which was presented to the American Philosophical Society, November 11, 1954. [46] was a discussion of measurement from the point of view of a mathematical statistician; it initiated a conference on quantification which included contributions in the physical and social sciences. The last of Wilks' expository papers, "Statistical Inference in Geology" [47], was a study of the uses of statistics in geology, presented to a conference on the earth sciences in connection with the semicentennial celebration of Rice Institute. Although Wilks was not versed in the field of geology, he learned about the methods and data of the discipline; perhaps he was persuaded to accept the invitation even though it involved extensive preparation because it came from Texas. (The footnote to the paper giving the author's credentials starts "Dr. Wilks, a native Texan.") This brief glance at papers telling about statistics to scientists in many fields shows again Wilks' untiring efforts to make statistics known and useful.

Shortly after the death of Karl Pearson, Wilks wrote a sketch of his life entitled "Karl Pearson: Founder of the Science of Statistics" [24]. In view of Wilks own broad interests and extreme industriousness and drive, it is intriguing to read here his admiration of these characteristics in Pearson. Membership in the American Philosophical Society is by election, and the number of members is held to about 500. In [36] Wilks made a study of the age distribution and mortality of resident membership in order to predict total membership over the years 1950 to 2000 if alternative policies with respect to admission of new members was followed. During the period 1937–49 the average age at election was 54.5, and the average age at death was 75.7. Regrettably, Wilks was atypical; with respect to age at death he fell in the lower second or third centile.

*3.4. Some other writings.* Wilks wrote many book reviews. Those that have come to my attention are listed in the bibliography, but there may well be others because Wilks did not keep a record of them and I have not searched carefully for them. He wrote innumerable reviews of papers for *Mathematical Reviews*, some of them appearing posthumously. It is impressive that with all Wilks had to do he continued to write these reviews.

A number of things he wrote were never published. One of these was "Statistical Aspects of Experiments in Telepathy" {7}, a lecture delivered to the Galois Institute of Mathematics, Long Island University, December 4, 1937. Wilks closed the mimeographed paper with "the purpose of this paper is neither to support nor condemn the ESP Hypothesis, but to discuss the main principles of the statistical method, which appears as the only method now available for arguing scientifically about the hypothesis." Wilks pointed out the cardmatching model for the typical extrasensory perception experiment with the experimenter taking each card from a deck and looking at it and the subject simultaneously calling the card without seeing the experimenter or having any sensory cue of the card held.

Wilks must have written hundreds of committee reports. One committee of which he was chairman was the Committee on the Analysis of Pre-election Polls and Forecasts of the Social Science Research Council. This committee and its staff was set up directly after Harry Truman won the 1948 presidential election, contrary to the predictions of the pollsters. Within 6 weeks the committee and its staff (including two former students) had prepared a report; Wilks wrote Chapter 1 {11}.

*3.5. Other books.* Wilks wrote two books entitled *Mathematical Statistics.* Each covered a substantial part of the accepted mathematical theory of statistics at the time of its publication, the first ⟨2⟩ in 1943 and the second ⟨4⟩ in 1962. In each the treatment is essentially mathematics without applications or examples. Although Wilks constantly had applications in mind, he deliberately chose this style so as to make the wide coverage feasible. The first book was brought out quickly. Henry Scheffé, David Votaw and I prepared a draft of most of it in the summer and fall of 1942; by April, 1943, Wilks had the manuscript ready for publication, and it came out in lithoprinted form. The pressure of war work made it impossible to give it more attention. The writing of the second book was begun in 1951, though some of the material in different forms had appeared in earlier books and papers of Wilks as well as in his class lectures. After 1951 he could only spend summers at writing. That he could get it out at all was due to his dedication and tirelessness, since he was carrying on so many other activities and the field of statistics was growing so rapidly.

At the time the first book came out there was no other book that gave an adequate treatment of a large part of the theory and methodology of statistics. Volume I of Kendall's *The Advanced Theory of Statistics* appeared in 1943 and the second volume in 1946; Cramér's *Mathematical Methods of Statistics* was published in 1945. These books then came to serve a similar function. The first *Mathematical Statistics* was hardly a textbook; it did not have exercises or examples with data. However, in and out of class situations, many mathematicians and others learned a solid part of statistics from that book.

The second *Mathematical Statistics* is encyclopedic in the sense that it covers so much of modern statistics. Its terse and direct mathematical style permit the exposition of some probability and a great deal of inference in its 18 chapters and 644 pages. Problems are used to expand the coverage. The book has been reviewed so recently and extensively that it seems unnecessary to describe it thoroughly here. It can be expected to be a standard text and reference for a good many years.

**4. Conclusion.** As this article has indicated, S. S. Wilks had a broad career and contributed significantly to education, science and society in many ways. He was interested in many fields—mathematical statistics, the applications of mathematics and statistics especially in psychological testing, industry, and government, the natural sciences, the social sciences, and education. His activities were at many levels, such as direct participation—in research, teaching and writing—planning and advising, organizing and leading, and studying and appraising. Through his career he moved from one discipline to another and from one activity to another, usually from the more specific to the more general, and always in a significant direction.

The achievements of Wilks as reported here are impressive, yet not all can be put down on paper. Indeed, not all have become known. Wilks worked modestly and quietly. Although he was an organizer in the sense of helping to create a number of organizations, he was responding to the needs he thought could be served by the organizations. He wrote and spoke frequently, not because he liked to attract attention but because he had ideas he felt should be expressed. His style was informal and unostentatious, but effective.

A good deal of Sam Wilks' work and influence was not public; they took place in committees, seminars and conversations. His advice was sought after, for he was sympathetic and had good common sense as well as wide knowledge and experience. He was willing to face problems directly and take responsibility.

Although Sam carried a heavy load of responsibilities, he had a friendly and personal interest in his students, colleagues and associates. He was informed and sociable, with a sense of humor; one enjoyed his company. Sam will be missed in many respects, not least as counsellor, companion, and friend.

Samuel Stanley Wilks was buried at Little Elm Cemetery near Little Elm, Texas, March 11, following funeral services at Denton. A memorial service was held at the Princeton University Chapel, April 19. A session of the Institute of Mathematical Statistics and the American Statistical Association was devoted to his memory, December 27.

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