EXACT MOMENTS AND PERCENTAGE POINTS OF THE ORDER STATISTICS AND THE DISTRIBUTION OF THE RANGE FROM THE LOGISTIC DISTRIBUTION[1]

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1. Introduction and Summary. The logistic curve $y = k/(1 + \alpha e^{-\beta t})$ has been used in studies pertaining to population growth by Verhulst [17] and by Pearl and Reed [14] and by several later authors. The logistic function $P = 1/(1 + e^{-(\alpha+\beta x)})$ has been very widely used by Berkson [1], Berkson and Hodges [2] as a model for analyzing bioassay and other experiments involving quantal response. Gumbel [8] has shown that the asymptotic distribution of the midrange of exponential type initial distributions is logistic. In connection with problems involving censored data, Plackett [15], [16] has considered the use of the logistic distribution.

In this paper a random variable Y is said to follow a logistic distribution (denoted by $L(\mu, \sigma^2)$) if its cumulative distribution function (c.d.f.) is

$$F(y; \mu, \sigma) = 1/[1 + e^{-[(y-\mu)/\sigma]\cdot(\pi/3^{\frac{1}{2}})}]. \tag{1.1}$$

The probability density function (p.d.f.) corresponding to (1.1) is

$$f(y; \mu, \sigma) = (\pi/\sigma 3^{\frac{1}{2}})e^{-\pi(y-\mu)/3^{\frac{1}{2}}\sigma}/[1 + e^{-\pi(y-\mu)/3^{\frac{1}{2}}\sigma}]^2, \tag{1.2}$$

where $-\infty < y < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$.

It should be noted that the distribution (1.2) is symmetrical with mean μ and variance σ^2. The moment generating function of $X = (Y - \mu)/\sigma$ is easy to derive (see for example, Gumbel [8]) and is

$$M_X(t) = \Gamma(1 + t/g)\Gamma(1 - t/g), \qquad g = \pi/3^{\frac{1}{2}}. \tag{1.3}$$

In this paper order statistics from the standard logistic distribution $L(0, 1)$ are studied. If $X_1, X_2, \cdots, X_n$ are n independent and identically distributed logistic random variables with density function,

$$f(x) = (\pi/3^{\frac{1}{2}})(e^{-x\pi/3^{\frac{1}{2}}})/(1 + e^{-x\pi/3^{\frac{1}{2}}})^2, \qquad -\infty < x < \infty, \tag{1.4}$$

then we are concerned with the moments, the distribution and some estimation problems using the ordered random variables $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ where

$$X_{(1)} \leqq X_{(2)} \leqq \cdots \leqq X_{(k)} \leqq \cdots \leqq X_{(n)}. \tag{1.5}$$

In the sequel, we shall call $X_{(k)}$, the kth order statistic in a sample of size n from the logistic distribution $L(0, 1)$.

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TABLE I

*Exact moments of the kth order statistic in a sample of size n from a standard logistic distribution**

n	k	μ_1'	μ_2'	μ_3'	μ_4'
1	1	0	1	0	$21/5$
2	1	$-a$	1	$-3a$	$21/5$
3	1	$-3a/2$	$b(1 + \pi^2/3)$	$-9a/2$	$d(2\pi^2 + 7\pi^4/15)$
	2	0	$b(-2 + \pi^2/3)$	0	$d(-4\pi^2 + 7\pi^4/15)$
4	1	$-11a/6$	$b(2 + \pi^2/3)$	$c(3 + 11\pi^2/2)$	$d(4\pi^2 + 7\pi^4/15)$
	2	$-a/2$	$b(-2 + \pi^2/3)$	$c(9 - 3\pi^2/2)$	$d(-4\pi^2 + 7\pi^4/15)$
5	1	$-25a/12$	$b(35/12 + \pi^2/3)$	$-c(15/2 + 25\pi^2/4)$	$d(1 + 35\pi^2/6 + 7\pi^4/15)$
	2	$-5a/6$	$b(-5/3 + \pi^2/3)$	$c(15-5\pi^2/2)$	$d(- 4- 10\pi^2/3 + 7\pi^4/15)$
	3	0	$b(-5/2 + \pi^2/3)$	0	$d(6 - 5\pi^2 + 7\pi^4/15)$
6	1	$-137a/60$	$b(15/4 + \pi^2/3)$	$-c(51/4 + 137\pi^2/20)$	$d(3 + 15\pi^2/2 + 7\pi^4/15)$
	2	$-13a/12$	$b(-5/4 + \pi^2/3)$	$c(75/4 - 13\pi^2/4)$	$d(-9 - 5\pi^2/2 + 7\pi^4/15)$
	3	$-a/3$	$b(-5/2 + \pi^2/3)$	$c(15/2 - \pi^2)$	$d(6 - 5\pi^2 + 7\pi^4/15)$
7	1	$-147a/60$	$b(203/45 + \pi^2/3)$	$c(441/24 + 147\pi^2/20)$	$d(35/6 + 406\pi^2/45 + 7\pi^4/15)$
	2	$-77a/60$	$b(-49/60 + \pi^2/3)$	$c(21 - 77\pi^2/20)$	$d(-14 - 49\pi^2/30 + 7\pi^4/15)$
	3	$-7a/12$	$b(--7/3 + \pi^2/3)$	$c(105/8 - 7\pi^2/4)$	$d(7/2 - 14\pi^2/3 + 7\pi^4/15)$
	4	0	$b(-49/18 + \pi^2/3)$	0	$d(28/3 - 49\pi^2/9 + 7\pi^4/15)$
8	1	$-363a/140$	$b(469/90 + \pi^2/3)$	$-c(967/40 + 1089\pi^2/140)$	$d(28/3 + 469\pi^2/45 + 7\pi^4/15)$
	2	$-29a/20$	$b(-7/18 + \pi^2/3)$	$c(889/40 - 87\pi^2/20)$	$d(-56/3 - 7\pi^2/9 + 7\pi^4/15)$
	3	$-47a/60$	$b(-21/10 + \pi^2/3)$	$c(693/40 - 47\pi^2/20)$	$d(-21\pi^2/5 + 7\pi^4/15)$
	4	$-a/4$	$b(-49/18 + \pi^2/3)$	$c(49/8 - 3\pi^2/4)$	$d(28/3 - 49\pi^2/9 + 7\pi^4/15)$

9	1	$-761a/280$	$b(29531/5040 + \pi^2/3)$	$c(2403/80 - 2283\pi^2/280)$	$d(1069/80 + 29531\pi^2/2520 + 7\pi^4/15)$
	2	$-223a$	$b(81315 + \pi^2/3)$	$c(909/40 - 669\pi^2/140)$	$d(-229/10 + 16\pi^2/315 + 7\pi^4/15)$
	3	$-19a/20$	$b(-331/180 + \pi^2/3)$	$c(819/40 - 57\pi^2/20)$	$d(-77/20 - 331\pi^2/90 + 7\pi^4/15)$
	4	$-9a/20$	$b(-118/45 + \pi^2/3)$	$c(441/40 - 27\pi^2/20)$	$d(181/30 - 236\pi^2/45 + 7\pi^4/15)$
	5	0	$b(-205/72 + \pi^2/3)$	0	$d(273/24 - 205\pi^2/36 + 7\pi^4/15)$
10	1	$-7129a/2520$	$b(6515/1008 + \pi^2/3)$	$c(4523/126 - 7129\pi^2/840)$	$d(285/16 + 6515\pi^2/1008 + 7\pi^4/15)$
	2	$-481a/280$	$b(61/144 + \pi^2/3)$	$c(1271/56 - 1443\pi^2/280)$	$d(-427/16 + 61\pi^2/72 + 7\pi^4/15)$
	3	$-1377a/1260$	$b(85/84 + \pi^2/3)$	$c(1271/56 - 1377\pi^2/420)$	$d(-31/4 + 85\pi^2/42 + 7\pi^4/15)$
	4	$-37a/60$	$b(-89/36 + \pi^2/3)$	$c(359/24 - 37\pi^2/20)$	$d(21/4 - 89\pi^2/18 + 7/15\pi^4)$
	5	$-a/5$	$b(-205/72 + \pi^2/3)$	$c(41/8 - 3\pi^2/5)$	$d(91/8 - 205\pi^2/36 + 7/15\pi^4)$

* Here $a = 3^{\frac{1}{2}}/\pi = 0.5513{,}2889$, $b = 3/\pi^2 = 0.3039{,}6355$, $c = 3^{\frac{1}{2}}/\pi^3 = 0.0558{,}6130$, $d = 9/\pi^4 = 0.0923{,}9384$.

In this paper the exact expressions for the moments of $X_{(k)}$ have been derived. The values of the first four exact moments for all sample sizes n from 1 to 10 have been tabulated (Table I). More generally, the moments of $X_{(k)}$ have been expressed in terms of expressions involving Bernoulli and Stirling numbers of 1st kind. These derivations are obtained from the moment generating function which has been derived. The cumulants of $X_{(k)}$ are expressed in terms of polygamma functions, as was pointed out by Plackett [15]. Birnbaum and Dudman [3] have tabulated expected values and standard deviations from the logistic distribution using tabulated values of the digamma and trigamma functions. Table III of the present paper gives the percentage points of $X_{(k)}$ (i) for all $k(k \leqq n)$ and all n from 1 to 10 (ii) for $k = 1, n$ and $\frac{1}{2}n$ and $\frac{1}{2}(n+2)$ (n even) or $\frac{1}{2}(n+1)$ (n odd) for $n = 11(1)25$. In Section 3, we obtain series expansions for the joint moment generating function and covariance of the two order statistics. In Section 5, the use of one and two order statistics for estimating μ and σ in $L(\mu, \sigma^2)$ is shown. In Section 6, expressions (closed form) are derived for the cumulative distribution function and the density function of the sample range. Using the results of Section 6, a short table (Table II) of the sample range of the logistic is given for $n = 2$ and 3. Section 7 gives a description of the tables in this paper.

2. Moments and cumulants of the kth order statistic.

Moments. The moment generating function $M(t)$ of $X_{(k)}$ is

$$\begin{aligned} (2.1)\qquad M(t) &= E(e^{tX_{(k)}}) \\ &= [g/B(k, n-k+1)] \int_{-\infty}^{\infty} [e^{xt} e^{-xg(n-k+1)}/(1+e^{-xg})^{n+1}]\, dx \end{aligned}$$

$$(2.2)\qquad M(t) = B(k + t/g, n - k + 1 - t/g)/B(k, n-k+1)$$

where $B(p, q)$ and $\Gamma(x)$ are the usual beta and gamma functions.

After some algebraic simplification (2.2) can be written as

$$(2.3)\qquad M(t) = [(-1)^{n-k}\pi \operatorname{cosec}(\pi t/g)/(k-1)!(n-k)!](k-1+t/g)_n$$

where $(x)_n = x(x-1) \cdots (x-n+1)$.

By expanding $\operatorname{cosec}(\pi t/g)$ in powers of t and by writing $(x)_n$ in terms of Stirling numbers of first kind, we obtain

$$\begin{aligned} M(t) &= [(-1)^{n-k}\pi/((k-1)!\,(n-k)!)] \\ (2.4)\qquad & [(g/\pi t) + (\pi t/6g) + (7/360)(\pi^3 t^3/g^3) + \cdots + (2(2^{2p-1}-1) \\ &\cdot(-1)^{p-1}/(2p)!)B_{2p}(\pi t/g)^{2p-1}]\,[\textstyle\sum_{i=1}^{n} (k-1+t/g)^i s(i, n)] \end{aligned}$$

where B_n and $s(i, n)$ denote the Bernoulli numbers and Stirling numbers of first kind, respectively. Now (2.4) can be expressed as

$$\begin{aligned} M(t) &= [(-1)^{n-k}/(k-1)!(n-k)!] \\ (2.5)\qquad &\cdot[\textstyle\sum_{i=1}^{n} \sum_{j=0}^{i} s(i, n)\binom{i}{j}(t/g)^{j-1}(k-1)^{i-j} \\ &+ 2\textstyle\sum_{p=1}^{\infty} \sum_{i=1}^{n} \sum_{j=0}^{i} [(2^{2p-1}-1)(-1)^{p-1}/(2p)!] \\ &\cdot\pi^{2p}\binom{i}{j}(k-1)^{i-j}B_{2p}s(i, n)(t/g)^{j+2p-1}]. \end{aligned}$$

From (2.5) collecting the coefficient of t^{2r-1} and t^{2r} we obtain

$$(2.6)\quad \mu'_{2r-1}(k, n) = (1/g^{2r-1})[(-1)^{n-k}(2r-1)!/((k-1)!(n-k)!)] \cdot[\textstyle\sum_{i=2r}^{n} b_{i(2r)}s(i, n) + 2\sum_{j=1}^{r}\sum_{i=2(r-j)}^{n} a_j b_{i,2(r-j)}s(i, n)]$$

$$(2.7)\quad \mu'_{2r}(k, n) = (1/g^{2r})[(-1)^{n-k}(2r)!/((k-1)!(n-k)!)] \cdot[\textstyle\sum_{i=2r+1}^{n} b_{i(2r+1)}s(i, n) + 2\sum_{j=1}^{r}\sum_{i=2(r-j)+1}^{n} a_j b_{i,2r-2j+1}s(i, n)]$$

where $a_p = [(-1)^{p-1}(2^{2p-1}-1)/(2p)!]B_{2p}\pi^{2p}$ and $b_{ij} = \binom{i}{j}(k-1)^{i-j}$, $0 \leqq j \leqq i$. Since $\mathcal{K}_1 = \mu_1' = \psi(k-1) - \psi(n-k)$, we obtain the identity

$$(2.8)\quad \psi(k-1) - \psi(n-k) = [(-1)^{n-k}/((k-1)!(n-k)!)] \cdot[\textstyle\sum_{i=2}^{n}\binom{i}{2}(k-1)^{i-2}s(i, n) + (\pi^2/18)\sum_{i=0}^{n}(k-1)^i s(i, n)].$$

Similar identities can be obtained by equating, $\mathcal{K}_r = (1/g^r)[\psi^{(r-1)}(k-1) - \psi^{(r-1)}(n-k)]$ to the expression available for $\mathcal{K}_r$ in terms of μ'_{2r-1} and μ'_{2r} and using (2.6) and (2.7).

It should be pointed out that for small n, the computation of exact moments is simpler if we collect the coefficient of the appropriate power of t in the right hand side of (2.3). This procedure was followed to obtain the exact values of moments that are given in Table I.

Cumulants of $X_{(k)}$. We rewrite (2.2) as

$$(2.9)\qquad M(t) = [\Gamma(k + t/g)\Gamma(n-k+1-t/g)]/[\Gamma_{(k)}\Gamma_{(n-k+1)}].$$

From (2.9) we obtain the rth cumulant $\mathcal{K}_r(k, n)$ as

$$(2.10)\qquad \mathcal{K}_r(k, n) = (d^r/dt^r)\log_e\Gamma(k + t/g)\,|_{t=0} + (d^r/dt^r)\log_e\Gamma(n-k+1-t/g)\,|_{t=0}$$

$$(2.11)\qquad \mathcal{K}_r(k, n) = (1/g^r)[\Psi^{(r-1)}(k-1) + (-1)^r\Psi^{(r-1)}(n-k)]$$

where $\Psi^{(r-1)}(x) = (d^r/dx^r)\log_e\Gamma(1+x) = (d^{r-1}/dx^{r-1})[\Gamma'(1+x)/\Gamma(1+x)]$ and $\Psi^{(0)}(x) = \Psi(x) = \Gamma'(1+x)/\Gamma(1+x)$.

It is clear from (2.11) that

$$(2.12)\qquad \mathcal{K}_{2r-1}(k, n) = -\mathcal{K}_{2r-1}(n-k+1, n)$$

$$(2.13)\qquad \mathcal{K}_{2r}(k, n) = \mathcal{K}_{2r}(n-k+1, n).$$

Using the series expansions for $\Psi^{(r-1)}(x)$ and $\Psi(x)$

$$(2.14)\quad \Psi^{(r-1)}(x) = (r-1)!\,(-1)^r\textstyle\sum_{j=1}^{\infty} 1/(j+x)^r, \qquad r \geqq 2$$

$$(2.15)\qquad \Psi(x) = \textstyle\sum_{\nu=1}^{\infty}[(1/\nu) - 1/(\nu+x)]$$

$$(2.16)\quad \mathcal{K}_r(k, n) = [(r-1)!\,(-1)^r/g^r][\textstyle\sum_{j=1}^{\infty}(1/(j+k-1)^r) + (-1)^r\sum_{j=1}^{\infty}(1/(j+n-k)^r)], \qquad r \geqq 2$$

$$(2.17)\quad \mathcal{K}_1(k, n) = -(1/g)[1/k + 1/(k+1) + \cdots + 1/(n-k)] \qquad \text{if } n-k > k-1.$$

From (2.17), we obtain for the special cases

$$\mathcal{K}_1(m, 2m) = -1/gm, \tag{2.18}$$

$$\mathcal{K}_1(1, n) = -(1/g)[1 + \tfrac{1}{2} + \cdots + 1/(n-1)]. \tag{2.19}$$

Formula (2.19) is given in [8], p. 128. It should be mentioned that Plackett [15] gave the $\mathcal{K}_r(k, n)$ $(r = 1, 2, 3, 4)$ in the form (2.16) and (2.17) for the case $n - k < k - 1$.

3. Covariance between the pth and mth order statistics $(m > p)$. The joint moment generating function of $X_{(p)}$ and $X_{(m)}$, $(m > p)$, is

$$\begin{aligned} M(t_1, t_2) &= E[\exp(t_1 x + t_2 y)] \\ &= C \int_{-\infty}^{\infty} dy \int_{-\infty}^{y} \exp(t_1 x + t_2 y) F^{p-1}(x)[F(y) - F(x)]^{m-p-1} \\ &\quad \cdot [1 - F(y)]^{n-m} f(x) f(y)\, dx, \end{aligned} \tag{3.1}$$

where $C = n!/[(p-1)!\,(m-p-1)!\,(n-m)!]$ and where $f(x)$ and $F(x)$ are defined in Section 1. After substituting $p_1 = 1/(1 + \exp(-gx))$ and $p_2 = 1/(1 + \exp(-gy))$ we obtain

$$\begin{aligned} M(t_1, t_2) &= C \int_0^1 dp_2 \int_0^{p_2} [p_1/(1-p_1)]^{t_1/g} [p_2/(1-p_2)]^{t_2/g} \\ &\quad \cdot p_1^{p-1} (p_2 - p_1)^{m-p-1} (1 - p_2)^{n-m}\, dp_1 . \end{aligned} \tag{3.2}$$

$$\begin{aligned} &= C \sum_{\alpha=0}^{m-p-1} \sum_{\nu=0}^{\infty} (-1)^{\alpha} \binom{m-p-1}{\alpha} \\ &\quad \cdot \{[(\nu - 1 + t_1/g)_{\nu} B(m + \nu + (t_1 + t_2)/g,\, n - m + 1 - t_2/g)] \\ &\quad \cdot [\nu!\,(\nu + m - \alpha - 1 + t_1/g)]^{-1}\}. \end{aligned} \tag{3.3}$$

From the above expression for the joint m.g.f. of the pth and mth order statistics, one can obtain the bivariate moments as follows.

$$E(X_{(p)}^r X_{(m)}^s) = \mu'_{rs}(X_{(p)}, X_{(m)}) = (\partial^{r+s}/\partial t_1^r \partial t_2^s) M(t_1, t_2) \big|_{t_1 = t_2 = 0} . \tag{3.4}$$

The case $r = 1$ and $s = 1$ is important. In this case (details omitted) we obtain]

$$\begin{aligned} \mu'_{11}(X_{(p)}, X_{(m)}) &= (C/g^2) \sum_{i=0}^{m-p-1} \sum_{c=0}^{n-m} (-1)^{i+1} \binom{m-p-i}{i} \binom{n-m}{c} \\ &\quad \cdot [(1/(m - i - 1))\{[2/(m+c)^3] \\ &\quad - \sum_{r=1}^{\infty} [1/r(m + c + r)^2]\} + [1/(m - i - 1)^2 \\ &\quad \cdot \{[1/(m+c)^2] - [1/(m+c)] \textstyle\sum_{x=1}^{m+c} 1/x\} \\ &\quad - \sum_{r=1}^{\infty} [1/(m + r - i - 1)]\{[1/(m + r + c)^2] \\ &\quad - [1/(m + r + c)] \textstyle\sum_{x=1}^{m+r+c} 1/x\}]. \end{aligned} \tag{3.5}$$

For $n = 2$, the covariance of the two order statistics is $3/\pi^2$.

4. Percentage points, modes and some remarks on the distribution of $X_{(k)}$. The density function $h_{k,n}(x)$ and the c.d.f. $H_{k,n}(x)$ of the kth order statistic in a sample of size n from $L(0, 1)$ are,

$$(4.1)\quad h_{k,n}(x) = [g/B(k, n-k+1)]e^{-xg(n-k+1)}/(1+e^{-xg})^{n+1}, \quad -\infty < x < \infty.$$

By differentiating the above expression for $h_{k,n}(x)$ with respect to x, we find that the mode $\breve{x}(k, n)$ of the kth order statistic is $\breve{x}(k, n) = (1/g)\log_e[k/(n-k+1)$. Also the expression for the c.d.f. $H_{k,n}(x)$ is

$$(4.2)\quad H_{k,n}(x) = I_{1/(1+\exp(-xg))}(k, n-k+1)$$

where $I_x(p, q)$ is the incomplete beta function,

$$(4.3)\quad H_{k,n}(x) = [1/B(k, n-k+1)]\textstyle\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}[1/(j+k)(1+e^{-xg})^{j+k}]$$

$$(4.4)\quad H_{k,n}(x) = [1/B(k, n-k+1)]\textstyle\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(1/(n-k+j+1)) \cdot\{[e^{-xg}/(1+e^{-xg})]^{n-k+j+1} - 1\}.$$

From (4.2) we see that the $(100)\alpha$-percentage points $x_\alpha(k, n)$ of $X_{(k)}$ is the solution of

$$(4.5)\quad x_\alpha(k, n) = (1/g)\log_e[B_\alpha(k, n-k+1)/(1-B_\alpha(k, n-k+1))].$$

Using values of $B_\alpha(k, n-k+1)$, the 100α percentage point of the Beta distribution, from [7] [14], we solved for $x_\alpha(k, n)$. These are given in Table III. Note that for $k = 1$ and $k = n$, we see from (4.3), (4.4) or otherwise that

$$(4.6)\quad -x_{1-\alpha}(1, n) = x_\alpha(n, n) = (1/g)\log_e(\alpha^{1/n}/(1-\alpha^{1/n})).$$

Note that relations (4.1)–(4.5) are true in general for any continuous distribution if $1/(1+e^{-xg})$ is replaced by the c.d.f. of the given distribution. This is by virtue of the well-known result that the c.d.f. of the kth order statistic is a beta variable. Further, from the symmetric relation satisfied by the incomplete beta functions, it follows that the 100α percentage of $X_{(k)}$ is $-100(1-\alpha)$ percentage point of $X_{(n-k+1)}$. This relation can be verified, mathematically, for the $X_{(k)}$ of the logistic distribution by using (4.5).

REMARK. The distribution of the sum of two symmetrical order statistics $V = X_{(k)} + X_{(n-k+1)}$ is of interest in some problems. In this connection, the following remark is relevant. From (4.2), we see that

$$(4.7)\quad H_{k,n}(x) + H_{n-k+1,n}(-x) = 1$$

$$(4.8)\quad h_{k,n}(x) = h_{n-k+1,n}(-x).$$

The above expressions are true for the order statistics from any continuous symmetric distributions.

5. Estimation of μ and σ based on one and two order statistics. In some problems it may be desirable to obtain estimators of μ and σ using only a single order statistic. Mosteller [11] first described and studied such "inefficient" estimators.

The unbiased minimum variance point estimator $\hat{\mu}$ of μ based on a single order statistic $y_{(k)}$ in a sample of size n from $L(\mu, \sigma^2)$, with σ assumed known, is

$$\hat{\mu} = \sigma\mu_1'\{[\tfrac{1}{2}(n+1)], n\} + y_{([\frac{1}{2}(n+1)])} \tag{5.1}$$

where $[x]$ denotes the largest integer $\leqq x$.

$$\operatorname{Var}(\hat{\mu}) = \sigma^2\mathcal{K}_2\{[\tfrac{1}{2}(n+1)], n\}. \tag{5.2}$$

Comparing the above variance with the Cramér-Rao lower bound, which is $3\sigma^2/(g^2n)$, the efficiency of $\hat{\mu}$ is

$$e(\hat{\mu}) = 3n^{-1}[\tfrac{1}{3}\pi^2 - \textstyle\sum_{x=1}^{[\frac{1}{2}(n+1)]-1}(1/x^2) - \sum_{x=1}^{n-[\frac{1}{2}(n+1)]}(1/x^2)]^{-1}. \tag{5.3}$$

The following line of table gives the values of $e(\hat{\mu})$ for selected values of n(odd):

$n =$	3	5	7	9
$e(\hat{\mu}) =$	.78	.76	.75	.75

Shortest confidence limits for μ (σ known) based on a single order statistic can be obtained by using the percentage points given in Table III of this paper.

Now, an unbiased estimator of σ based on the range is

$$\hat{\sigma} = (y_{(n)} - y_{(1)})/2\mathcal{K}_1(n, n). \tag{5.4}$$

Now using the distribution of range given in the next section, one can compute the variance of $\hat{\sigma}$ without evaluating the covariance of $y_{(1)}$ and $y_{(n)}$, which are hard to evaluate except in special cases, and compare it with the corresponding Cramér-Rao lower bound.

6. Distribution of the range. Let us define the sample range W_n by

$$W_n = (Y_{(n)} - Y_{(1)})/\sigma = X_{(n)} - X_{(1)} \tag{6.1}$$

where $Y_{(k)}$ and $X_{(k)}$ denote the kth order statistics from the logistic distributions $L(\mu, \sigma^2)$ and $L(0, 1)$, respectively.

The cumulative distribution of W_n can be written as

$$P(W_n \leqq w) = n\textstyle\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\int_{-\infty}^{\infty}[F(x+w)]^{n-1-j}[F(x)]^jf(x)\,dx. \tag{6.2}$$

Now we shall derive the c.d.f. of the range W_n from $L(\mu, \sigma^2)$. We start with (6.2) and obtain

$$\begin{aligned} P(W_n \leqq n) \\ &= n\textstyle\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\int_{-\infty}^{\infty}ge^{-gx}/[(1+ae^{-gx})^{n-1-j}(1+e^{-gx})^{2+j}]\,dx \end{aligned} \tag{6.3}$$

where $a = \exp(-gw)$. From (6.3) by substituting $t = 1/[1 + a\exp(-gx)]$, we obtain

$$P(W_n \leqq w) = n\textstyle\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^ja^{j+1}A(j, n) \tag{6.4}$$

where

$$\begin{aligned} A(j, n) &= \textstyle\int_0^1 t^{n-1}(1+ct)^{-j-2}\,dt, \qquad c = a - 1 \\ &= [1/(-c)^{n-1}]\textstyle\sum_{\alpha=0}^{n-1}\binom{n-1}{\alpha}(-1)^\alpha\int_0^1(1+ct)^{\alpha-j-2}\,dt \end{aligned} \tag{6.5}$$

or

$$(6.6)\quad A(j, n) = [-1/(1-a)^n][\tbinom{n-1}{j+1}(-1)^{j+1}\log a + \textstyle\sum_{\alpha=0,\alpha\neq j+1}^{n-1}\binom{n-1}{\alpha}(-1)^\alpha(a^{\alpha-j-1}-1)/(\alpha-j-1)].$$

Hence, from (6.4) and (6.6), we obtain the c.d.f. of W_n as

$$(6.7)\quad P(W_n \leqq w) = [n/(1-a)^n]\textstyle\sum_{j=0}^{n-1}\binom{n-1}{j}(-a)^{j+1}\{\binom{n-1}{j+1}(-1)^j\log(1/a) + \textstyle\sum_{\alpha=0,\alpha\neq j+1}^{n-1}\binom{n-1}{\alpha}(-1)^\alpha(a^{\alpha-j-1}-1)/(\alpha-j-1)\},$$

where in the first term inside the braces $\binom{n-1}{j+1}$ is to be put equal to zero for $j > n-2$.

By differentiating (6.4) with respect to w one can obtain the density function $p(w)$ as follows

$$(6.8)\quad p(w) = n\textstyle\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j g a^{j+1} \cdot[(j+2)aA(j+1, n+1) - (j+1)A(j, n)].$$

For $n = 2$ and 3 we obtain from (6.7)

$$(6.9)\quad P(W_2 \leqq w) = [1 - a^2 - 2(gwa)]/(1-a)^2$$

$$(6.10)\quad P(W_3 \leqq w) = [1 + 9a - 9a^2 - a^3 - 6(gwa)(1+a)]/(1-a)^3.$$

Using (6.9) and (6.10), the probability integral of the range has been computed for $n = 2$ and 3. These values are given in Table II along with the values of the probability integral of the sample range of the normal distribution $N(\mu, 1)$ which have been taken from [14]. Some detailed computations on the probability integral and percentage points of the range are in progress and these will be published later.

It is interesting to note that for $-\infty < w < \infty$ or $0 < a < 1$, we obtain from (6.9) and (6.10)

$$(6.11)\quad \begin{gathered}(1-a)^2 + 2(gwa) > 1 - a^2 > 2(gwa),\\ (1-a)^3 + 6(gwa)(1+a) > 1 + 9a(1-a) - a^3 > 6(gwa)(1+a).\end{gathered}$$

TABLE II

Probability Integrals of the Sample Range from $L(\mu, 1)$ (top) and from $N(\mu, 1)$ (bottom)

n	w = .20	.40	.60	.80	1.00	1.50	2.00	2.50	3.00	3.50	4.00
2	.12039 .1125	.23768 .2227	.34902 .3286	.45212 .4284	.54213 .5205	.73047 .7112	.85109 .8427	.92224 .9229	.96113 .9661	.98121 .9867	.99115 .9953
3	.01306 .0110	.05138 .0431	.11084 .0944	.18655 .1616	.28725 .2407	.50027 .4614	.69272 .6665	.82676 .8195	.93300 .9145	.95340 .9870	.97765 .9988

TABLE III

*Percentage points of the kth order statistic in a sample of size n from a standard logistic distribution**

n	k	α					
		0.500	0.750	0.900	0.950	0.975	0.990
1	1	0.0000	0.6057	1.2114	1.6234	2.0198	2.5334
2	1	−0.4859	0.0000	0.4251	0.6863	0.9220	1.2114
	2	0.4859	1.0289	1.6083	2.0126	2.4054	2.9170
3	1	−0.7428	−0.2933	0.0792	0.2972	0.4872	0.7126
	2	0.0000	0.3996	0.7789	1.0224	1.2472	1.5278
	3	0.7428	1.2659	1.8367	2.2385	2.6305	3.1410
4	1	−0.9179	−0.4859	−0.1382	0.0598	0.2290	0.4251
	2	−0.2565	0.0966	0.4144	0.6098	0.7848	0.9968
	3	0.2565	0.6277	0.9892	1.2262	1.4469	1.7241
	4	0.9179	1.4312	1.9977	2.3983	2.7893	3.2999
5	1	−1.0507	−0.6290	−0.2957	−0.1090	0.0482	0.2279
	2	−0.4313	−0.1013	0.1868	0.3593	0.5109	0.6912
	3	0.0000	0.3186	0.6156	0.8021	0.9710	1.1777
	4	0.4313	0.7861	1.1402	1.3738	1.5922	1.8677
	5	1.0507	1.5583	2.1222	2.5221	2.9128	3.4230
6	1	−1.1578	−0.7428	−0.4188	−0.2396	−0.0900	0.0792
	2	−0.5640	−0.2478	0.0227	0.1820	0.3202	0.4822
	3	−0.1748	0.1178	0.3825	0.5446	0.6889	0.8623
	4	0.1748	0.4753	0.7612	0.9428	1.1084	1.3120
	5	0.5640	0.9094	1.2583	1.4897	1.7070	1.9813
	6	1.1578	1.6615	2.2237	2.6230	3.0136	3.5236
7	1	−1.2474	−0.8373	−0.5199	−0.3457	−0.2015	−0.0396
	2	−0.6709	−0.3639	−0.1049	0.0456	0.1751	0.3253
	3	−0.3074	−0.0307	0.2148	0.3626	0.4925	0.6497
	4	0.0000	0.2726	0.5245	0.6808	0.8210	0.9905
	5	0.3074	0.5967	0.8756	1.0541	1.2176	1.4193
	6	0.6709	1.0102	1.3554	1.5854	1.8017	2.0752
	7	1.2474	1.7485	2.3095	2.7084	3.0987	3.6087
8	1	−1.3245	−0.9179	−0.6054	−0.4351	−0.2948	−0.1382
	2	−0.7604	−0.4601	−0.2092	−0.0647	0.0585	0.2005
	3	−0.4143	−0.1482	0.0844	0.2228	0.3434	0.4850
	4	−0.1326	0.1229	0.3544	0.4957	0.6209	0.7705
	5	0.1326	0.3927	0.6365	0.7891	0.9266	1.0935
	6	0.4143	0.6959	0.9701	1.1465	1.3085	1.5088
	7	0.7604	1.0953	1.4378	1.6668	1.8824	2.1555
	8	1.3245	1.8236	2.3835	2.7824	3.1723	3.6825

TABLE III—*Continued*

n	k	α 0.500	0.750	0.900	0.950	0.975	0.990
9	1	−1.3921	−0.9883	−0.6795	−0.5122	−0.3748	−0.2223
	2	−0.8374	−0.5420	−0.2973	−0.1573	−0.0386	0.0973
	3	−0.5037	−0.2453	−0.0219	0.1098	0.2238	0.3566
	4	−0.2395	0.0044	0.2222	0.3536	0.4690	0.6056
	5	0.0000	0.2421	0.4646	0.6017	0.7238	0.8704
	6	0.2395	0.4910	0.7291	0.8791	1.0147	1.1799
	7	0.5037	0.7799	1.0506	1.2254	1.3863	1.5857
	8	0.8374	1.1689	1.5094	1.7377	2.0961	2.2255
	9	1.3921	1.8899	2.4493	2.8478	3.2373	3.7478
10	1	−1.4523	−1.0507	−0.7450	−0.7069	−0.4450	−0.2957
	2	−0.9050	−0.6135	−0.3734	−0.2370	−0.1217	0.0096
	3	−0.5808	−0.3282	−0.1117	0.0152	0.1243	0.2506
	4	−0.3290	−0.0935	0.1145	0.2388	0.3472	0.4747
	5	−0.1069	0.1228	0.3311	0.4579	0.5699	0.7031
	6	0.1069	0.3266	0.5559	0.6900	0.8099	0.9544
	7	0.3290	0.5742	0.8082	0.9562	1.0904	1.2543
	8	0.5808	0.8528	1.1206	1.2943	1.4543	1.6531
	9	0.9050	1.2340	1.5729	1.8007	2.0151	2.2876
	10	1.4523	1.9484	2.5073	2.9056	3.2953	3.8051
11	1	−1.5066	−1.1068	−0.8035	−0.6403	−0.5073	−0.3606
	6	0.0000	0.2199	0.4214	0.5449	0.6544	0.7853
	11	1.5066	2.0017	2.5601	2.9583	3.3482	3.8580
12	1	−1.5561	−1.1578	−0.8564	−0.6948	−0.5634	−0.4189
	6	−0.0895	0.1210	0.3118	0.4277	0.5298	0.6511
	7	0.0895	0.3022	0.4985	0.6196	0.7273	0.8564
	12	1.5561	2.0503	2.6082	3.0064	3.3962	3.9060
13	1	−1.6014	−1.2044	−0.9047	−0.7445	−0.6144	−0.4716
	7	0.0000	0.2029	0.3883	0.5016	0.6017	0.7210
	13	1.6014	2.1216	2.6526	3.0506	3.4404	3.9501
14	1	−1.6434	−1.2474	−0.9492	−0.7900	−0.6611	−0.5198
	7	−0.0770	0.1185	0.2955	0.4029	0.4974	0.6092
	8	0.0770	0.2740	0.4552	0.5664	0.6651	0.7827
	14	1.6434	2.1362	2.6936	3.0915	3.4813	3.9910
15	1	−1.6823	−1.2873	−0.9904	−0.8322	−0.7042	−0.5642
	8	0.0000	0.1893	0.3620	0.4672	0.5606	0.6701
	15	1.6823	2.1746	2.7318	3.1296	3.5194	4.0290

TABLE III—*Continued*

n	k	α					
		0.500	0.750	0.900	0.950	0.975	0.990
16	1	−1.7187	−1.3245	−1.0286	−0.8713	−0.7442	−0.6054
	8	−0.0675	0.1156	0.2815	0.3820	0.4703	0.5747
	9	0.0675	0.2519	0.4211	0.5245	0.6160	0.7248
	16	1.7187	2.2105	2.7675	3.1653	3.5550	4.0646
17	1	−1.7528	−1.3593	−1.0644	−0.9078	−0.7815	−0.6437
	9	0.0000	0.1781	0.3403	0.4390	0.5259	0.6287
	17	1.7528	2.2443	2.8010	3.1987	3.5884	4.0981
18	1	−1.7850	−1.3921	−1.0981	−0.9421	−0.8164	−0.6795
	9	−0.0602	0.1128	0.2694	0.3642	0.4474	0.5455
	10	0.0602	0.2341	0.3933	0.4905	0.5761	0.6778
	18	1.7850	2.2760	2.8326	3.2303	3.6200	4.1296
19	1	−1.8153	−1.4230	−1.1298	−0.9744	−0.8493	−0.7132
	10	0.0000	0.1687	0.3222	0.4153	0.4973	0.5941
	19	1.8153	2.3061	2.8625	3.2602	3.6498	4.1594
20	1	−1.8441	−1.4523	−1.1598	−1.0049	−0.8803	−0.7450
	10	−0.0542	0.1101	0.2588	0.3487	0.4276	0.5203
	11	0.0542	0.2194	0.3702	0.4620	0.5429	0.6387
	20	1.8441	2.3346	2.8909	3.2885	3.6781	4.1877
21	1	−1.8715	−1.4802	−1.1882	−1.0338	−0.9097	−0.7750
	11	0.0000	0.1606	0.3066	0.3951	0.4729	0.5647
	21	1.8715	2.3617	2.9179	3.3154	3.7050	4.2146
22	1	−1.8975	−1.5066	−1.2152	−1.0613	−0.9377	−0.8035
	11	−0.0494	0.1075	0.2494	0.3351	0.4103	0.4987
	12	0.0494	0.2069	0.3506	0.4379	0.5147	0.6055
	22	1.8975	2.3875	2.9436	3.3411	3.7307	4.2402
23	1	−1.9224	−1.5319	−1.2411	−1.0875	−0.9642	−0.8306
	12	0.0000	0.1536	0.2931	0.3776	0.4518	0.5392
	23	1.9224	2.4121	2.9680	3.3656	3.7552	4.2648
24	1	−1.9462	−1.5561	−1.2657	−1.1125	−0.9896	−0.8564
	12	−0.0453	0.1050	0.2409	0.3231	0.3949	0.4794
	13	0.0453	0.1962	0.3336	0.4171	0.4903	0.5768
	24	1.9462	2.4357	2.9916	3.3891	3.7787	4.2882
25	1	−1.9691	−1.5792	−1.2892	−1.1364	−1.0138	−0.8811
	13	0.0000	0.1475	0.2813	0.3623	0.4333	0.5169
	25	1.9691	2.4584	3.0161	3.4116	3.8012	4.3107

* For given n, k and α, the above table gives the values of y for which

$$[1/B(k, n-k+1)] \int_0^{F(y)} x^{k-1}(1-x)^{n-k}\, dx = \alpha$$

where $F(y)$ denotes the c.d.f. of a standard logistic random variable.

In general, for $0 < a < 1$, we have

$$(6.12)\quad (1 - a)^n/n > \sum_{j=0}^{n-1} \binom{n-1}{j}(-a)^{j+1}\{\binom{n-1}{j+1}(-1)^j \log (1/a) + \sum_{\alpha=0,\alpha\neq j+1}^{n-1} \binom{n-1}{\alpha}(-1)^{\alpha}(a^{\alpha-j-1} - 1)/(\alpha - j - 1)\} > 0$$

and

$$(6.13)\quad \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j g a^{j+1}[(j+2)aA(j+1, n+1) - (j+1)A(j, n)] > 0$$

where $A(j, n)$ is given by (6.6).

7. Description of tables and comparison with normal order statistics. Birnbaum and Dudman [3] have tables and graphs comparing normal and logistic order statistics. A comparison of this type is omitted in this paper. It may be recalled that Plackett [16] observed that the standard logistic and standard normal are similar in shape between the range of logistic probability levels .05 and .95. A comparison with the tables of Owen ([12] p. 254) and Pearson and Hartley's tables ([14] p. 104) shows that the two c.d.f.'s agree to within 2 units in the second decimal place. The density function curve of the logistic crosses the density curve of the normal between 0.68 and 0.69. The inflection points of the standard logistic are ± 0.53 (approx.) whereas the inflection points of standard normal are ± 1.00.

Table I of this paper gives the exact expressions for the moments about the origin of the kth order statistic in a sample of size n from the $L(0. 1)$.

Since $\mu_r'(k, n) = (-1)^r \mu_r'(n - k + 1, n)$, we give only the values of $\mu_r'(k, n)$ for $k = 1, 2, \cdots, n/2$ (n even), $(n + 1)/2$ (n odd). The range of values of n is $n = 1(1)10$.

Table III gives the percentage points of the kth order statistic in a sample of size n from the $L(0, 1)$ distribution. These computations are similar to the computations described in [6], [9], [10]. The percentiles of the distributions were obtained from [14], [7]. The values are given to four decimal places. Independent checks have revealed no errors. However, the fourth decimal place may be off by one unit. The table contains values for $k = 1(1)n$, $n = 1(1)10$ and $k = 1$, $n/2$ (n even), $(n + 1)/2$ (n odd) and n, for $n = 11(1)25$. The 100α percentage points are listed for $\alpha = .50, .75, .90, .95, .975$ and $.99$. The $100(1 - \alpha)$ percentage point of kth order statistic is the negative of the 100α percentage point of the $(n - k + 1)$th order statistic.

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