BOOK REVIEW

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EMANUEL PARZEN, *Stochastic Processes*, Holden-Day, Inc., San Francisco, 1962. \$10.95, xi + 324 pp.

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The writing of an introductory text-book about such an intrinsically sophisticated subject as probability theory or, more particularly, stochastic processes, presents the would-be author with a fundamental difficulty. The reader who is supposed to benefit from the book may not be assumed to have mathematical maturity, nor much willingness to develop interest in technical mathematical points. We are thinking here of economists, sociologists, biologists, and so on, who are engrossed in their "practical" problems, have never studied much mathematics, yet feel (correctly) that some appreciation of the theory of stochastic processes would be a blessing. Engineers and physicists generally have a good mathematical training, but even these are seldom prepared or inclined to appreciate the occasional purely mathematical difficulty which can arise in this theory. Yet the author of a book on stochastic processes is usually an expert who has been rubbing shoulders with sets of measure zero, Fubini's theorem, the Radon-Nikodym theorem, separability, and so on, for years; he would rather die than write a non-rigorous thing (I cannot say "proof"). Moreover, the author of an introductory text may have other objectives in mind. He may be involved in training tyros in his own specialty, budding analysts of stochastic processes for whom everything must be done "properly"; he may also wish that the book will become a valuable reference work. Both of these objectives call for the shunning of any imprecision; all conditions under which results hold are to be clearly stated; the use of "hand-waving" intuitive arguments, favored especially by certain British authors, is to be avoided. I do not deprecate these gymnastics, by the way, because I believe strongly that a major role to be played by an *introduction* is to inculcate enthusiasm; the mind of the tyro is a fire to be kindled, not a vessel to be filled. Nothing can dampen the enthusiasm or destroy the beautiful vision like a long string of conditioning subordinate clauses. I am reminded that Hermann Weyl once said, half-joking, "My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful."

Thus the author is driven by a number of motives which can, unfortunately, militate against each other. It is interesting to see by what compromises he resolves his difficulties. Broadly speaking there are three basic strategies available.

The first method is that of restriction of subject matter and this must clearly be practised by all authors to some extent. Thus Professor Feller has produced an absolutely excellent introduction to probability theory by restricting his book to discrete random variables and thereby avoiding most of the technical difficulties which would otherwise obstruct his smooth progress. The second method is the "hand-waving" one already mentioned. The existence of technical difficulties is admitted (usually *sotto voce*) but no attempt is made to get the beginning student to grips with them. Instead the results are "established" by plausible physical arguments or by simple proofs based on massive restrictions ("let us suppose $f(x)$ to be differentiable") which will not always be valid in later applications. There remains what we might non-committally, and with a nod to the B. B. C., call the "third method", favored by Professor Parzen. In this method every effort is made to describe most clearly and explicitly the technical problems encountered; there is no systematic attempt to avoid difficulties by restricting artificially the domain of discussion. On the other hand, since anything done must be rigorously done, there is heavy reliance on the phrase "it can be shown that", usually coupled with a reference to some scholarly memoir; by this means the tyro is spared a number of difficult arguments, though clearly aware of what they are about (an exception to all this being Section 1–2, describing the probability law of a stochastic process; one feels that a more conscientious treatment would be more compatible with the general tenor of the book). However there are those beginners, one cannot help feeling, who might nevertheless welcome at these places a little intuitive discussion.

An excellent feature of the book is that it bristles with "practical" examples; nearly every development in the theoretical exposition is followed by an abundance of clearly described applications and complements of that development. Moreover, there is also a plenitude of exercises for the reader; they vary considerably in difficulty, which is a good thing, and are mainly very interesting (if sometimes facetious). At this point I would complain at a minor irritation I found in this work, stressing that it is a minor matter. In one or two places the author indulges in a needless display of erudition at the ignorant reader's expense; I was made to feel very much one down over my ignorance of "... Holtzmark's treatment of the force acting on a star.... (p. 152)", "Bistable symmetric vibrators (p. 37)", and sundry similar mysteries; clearly it was expected that I would have known all about them.

Chapter 1, entitled "Random variables and stochastic processes" is concerned with establishing the basic ideas of probability theory and with describing a few basic stochastic processes. In setting up shop the author leans heavily on his book *Modern Probability Theory and Its Applications* (New York, Wiley, 1960). A would-be student of stochastic processes must not be too deceived by the "completeness" of the present book; he is strongly urged to have gained familiarity with basic probability theory first. On the other hand he is urged to take little notice of the long and scary footnote on p. 10. A beginner might easily be put off going any further by the threat that "in this book the word function (unless qualified) will mean a Borel function and the word set (of real numbers) will mean a Borel set." In fact it is not necessary to have heard of Borel to get a great deal from this book and the reviewer feels that it could profitably be read by any interested student whose mathematical training includes a year of "advanced calculus" and a short course on basic probability theory. Incidentally, the reviewer admits to being a little puzzled as to what background the author imagines his reader to have; whilst he wants to assume his reader knows about Borel sets, he takes pains to explain at an early stage (footnote to p. 14) what a gamma function is!

The Wiener process is amongst the processes discussed in Chapter I. It is introduced by stating four conditions it must satisfy; no proof is given that a process exists and satisfies these conditions; but how can one give such a proof in an introductory book? Poisson processes are also introduced, not only as counts on the real line but, with astronomical references, as points in space.

Chapter 2 continues the preparatory groundwork of the first chapter; it is concerned with conditional probabilities and expectations, a subject which is difficult to present in both an elementary and a completely rigorous way. This chapter comes off very well, and even though a student may have studied the subject of conditional expectations previously he will benefit from a reading of it, and from its numerous exercises. Predictably and unavoidably, the Radon-Nikodym theorem is mentioned (p. 52), in a paragraph the beginner will do well to skip.

In Chapter 3 the study proper of stochastic processes begins; this chapter is about normal processes and covariance stationary processes. The author has contributed notably to this subject and is clearly enthusiastic about it. The remark half-way down p. 71 seems very important and might profitably be italicised; it is to the effect that the mean value function of a covariance stationary process need not be constant. It should also be noted that the use of the word *ergodic* (introduced on p. 73) is rather more special than customary. The chapter begins with some definitions and examples and then in its second section discusses conditions which will make a process ergodic, and the distinction between stationary and evolutionary processes. Any process which is not stationary is termed evolutionary. This, fairly widespread, terminology is occasionally misleading; surely a name should be coined for those processes which, while not stationary, tend towards a stationary condition? In the third section the integration and differentiation of stochastic processes is discussed, with some interesting applications to Brownian motion, the integrated Wiener process, and Chebyshev inequalities for processes. Unfortunately the reader is asked to make many acts of faith here; the phrase "it may be shown that" occurs often (why *may* and not *can*?). One feels the reader could be helped more than he is. A study of the time integral of a Poisson process, for example, would show that a process can be stochastically differentiable whilst its realizations are not. Surely some intuitive explanation is possible why a process is mean-square differentiable if its covariance function is differentiable at the origin? In the treatment of the Chebyshev inequality various manipulations in the mean-square calculus are performed without comment or justification.

Normal stationary processes are of course the most important class of stationary processes and Sections 4 and 5 of Chapter 3 are concerned with them. In a few pages the author obtains a number of useful results about, for example, nonlinear operations on normal processes and on normal processes as limits of other stochastic processes. In accordance with the level of rigor at which the author aims in this work, it would seem that an appeal to the continuity theorem is necessary on p. 97, where the approximate normality of a Poisson variable is proved; similarly on p. 99 when Brownian motion is discussed. The example 5B on goodness-of-fit tests for empirical distributions is very interesting. Unfortunately the introductory explanation is confusing because $F(x)$ seems to be both known and unknown at the same time; after a reader has got his ideas straight on this point he is required (with no help) to do a horrible amount of algebra to obtain the formulae in the middle of p. 100. All the same, this is a very thought-provoking example and it is a pity it fizzles out somewhat in the end with "It seems reasonable $\cdots$" and "It may be shown $\cdots$"; in a book of this level, however, occasional let-downs like this must be unavoidable.

The last section of Chapter 3, on the harmonic analysis of stochastic processes, is surely the most disappointing part of the whole book. From the author's great and acknowledged expertise in this area we had anticipated something more exciting. The section discusses the simplest ideas connected with the spectrum of a stationary process and with the filtering of such a process. But very little is proved; the reader is told that much of the theory is beyond the scope of this book and the phrase "it may be shown $\cdots$" is used frequently. Years ago, we recall, Professor H. E. Daniels was presenting in lectures at Cambridge a derivation of the main Wiener-Khinchine result which, if not a proof, gave his audience valuable intuitive understanding of the spectral analysis of stationary processes. There was nothing terrifying about Daniels' arguments and some such discussion would have been possible in the present book. Similarly some attempt could be made to show in what way variations in the spectral distribution function may be interpreted in terms of the make-up of the process.

Chapter 4 is entitled "Counting processes and Poisson processes." The author begins by listing ways (including the usual ones) of defining a Poisson process in one dimension and then chooses to adopt the one which, whilst of considerable interest to the theoretician must surely be the most off-putting to a beginner. The Poisson process is defined by a set of axioms which characterize it as a certain kind of integer-valued process with stationary independent increments and unit jumps. This approach has the advantage that, by dropping various axioms, one is led to either a generalised Poisson process or a non-homogeneous Poisson process. In this chapter the Poisson and associated processes are discussed in great detail and, in contradistinction to the previous chapter, nearly every quoted theorem is proved. One even is provided with a detailed discussion of the famous Hamel functional equation. A drawback to the author's way of introducing the

Poisson process is that it seems to be quite difficult to establish in Theorem 3A that the successive time-intervals between events of the process are independent and negative exponential; indeed the "proof" given is not quite a proof, in that it relies on the crutch "it seems intuitively plausible that." Such a phrase is quite to be expected, we have insisted, in a first course on stochastic processes; we draw attention to it merely because Professor Parzen so rarely adopts this tack and because a more mundane approach to the Poisson process might have avoided various difficulties encountered. Incidentally the lines of an alternative, rigorous, proof of Theorem 3A are indicated a little later in the chapter! The final section of this chapter, which abounds in practical examples, concerns the "filtered Poisson process" and includes discussions of the number of busy channels in a telephone system, the number of pulses locking a paralyzable counter, and Campbell's theorem about shot noise processes.

Chapter 5, a short one, is concerned with what Professor Parzen very sensibly names a renewal counting process; namely the process $\{N(t),\ t \geqq 0\}$ which records the number of renewals to have occurred by time t, since the initial installation at time zero. The discussion is clear, with more references to practical problems, especially to Type I and Type II counters. However, the discussion of a renewal process with gamma distributed interarrival times could undoubtedly be simplified by reference to an underlying Poisson process. When we come to renewal theorems of a fairly general character references are again substituted for proofs; this is not surprising in view of the difficulty of some theorems here. The result $E[N(t)] \sim t/\mu$, where μ is the average renewal life-time, can be quite easily proved, however, and it is a pity that at least this one result should be quoted without proof. But this is a minor criticism.

Chapter 6, a big chapter, is devoted to Markov chains in discrete time. In the course of some examples at the start of this chapter one learns, amongst other more prosaic things, the interesting fact that the Samoan language provides a representation of a certain very simple Markov chain! In developing the general theory of Markov chains, the vector $p(n)$ in (2.18) is defined as a column vector when, judging from its later use, it should evidently be a row. This is a most unfortunate error which could cause a great deal of trouble to the beginner.

Amongst the examples given of Markov chains are included certain queueing problems and branching processes. The latter processes are given a clear treatment except that early on the author assumes "$p_0 + p_1 < 1$" without explaining why, and much of what he does has no need of this assumption.

The general theoretical treatment of Markov chains is very full and generally clear; if anything the treatment is fuller, and with more details, than that in Feller's book. For instance, attention is directed to the existence of Césaro limits to the probability of finding the system in a specific state. Generating functions play quite a part in the arguments and reliance is placed on a limit theorem (what might be called the Littlewood-Karamata-Feller theorem) which is not proved (reference given). The Césaro limits have much in common with the "elementary" renewal theorem mentioned earlier, however, and could be proved in a similar elementary way. On page 212, line 8b, the word "since" should surely be "if"? Later on there is a most interesting discussion of the gambler's ruin problem in which various general questions are touched upon, not found in other introductory books.

Some years ago, it was pointed out by M. R. Leadbetter to the reviewer, under slightly embarrassing circumstances, that part of the treatment in the book by Professor Feller of the "mean times to absorption" (in a Markov chain) is fallacious. The fallacy is perpetuated by Professor Parzen, so it seems that a few detailed remarks are needed here, on this point. Feller derived certain linear equations satisfied by the quantities m_j, say, the expected number of transitions from an initial transient state E_j to the first state of an absorbing set; in familiar notation:

$$(*) \qquad m_j = 1 + \sum_{k \varepsilon T} p_{jk} m_k .$$

In the course of arguments about the probability of being absorbed it was, in effect, shown that the null solution is the only *bounded* solution of the homogeneous equations

$$y_j = \sum_{k \varepsilon T} p_{jk} y_k ,$$

if it is known that the probability of remaining forever in the transient states is zero (regardless of the initial state). The word *bounded* was overlooked in discussing the uniqueness of solutions to $(*)$; there is, in fact, no reason why two solutions to $(*)$, $\{m_j'\}$ and $\{m_j''\}$, should not be such that $\{(m_j' - m_j'')\}$ is unbounded. Indeed, consider the following example. Let all even-numbered states be absorbing. Let the only transitions from an odd-numbered state E_{2k+1}, say, be to the absorbing state E_{2k+2} with probability q_{2k+1} or to the transient state E_{2k+3} with probability $p_{2k+1} = 1 - q_{2k+1}$. Provided $\sum_k q_{2k+1} = \infty$ the probability of remaining forever in the transient states is zero, as required. But the most general solution to $(*)$ is given by $(k = 1, 2, \cdots)$,

$$m_{2k+1} = 1 + \sum_{r=1}^{\infty} \{\prod_{j=1}^{r} p_{2k+j}\} + C/p_1 p_3 p_5 \cdots p_{2k-1} ,$$

where C is an arbitrary constant and $m_1 = 1 + p_1 m_3$. In this particular example it is easy to see that the correct value for the "duration of play" is obtained by letting $C = 0$; in the general theory, however, it is not clear how to characterize the required solution. Thus in Professor Parzen's book the paragraph following equation (7.8) on p. 239 needs amendment, as does Theorem 7A on p. 243. In addition, Example 7C, essentially the "gambler's ruin problem, with an infinitely rich adversary" needs further discussion, since reliance is placed on the uniqueness of solutions to $(*)$.

The final chapter of the book is a short one introducing the reader to Markov chains in continuous time. Once again we find a wealth of interesting examples; the treatment is clear, and a surprising amount is covered in a few pages.

Let us end by attempting to remove any false impressions we may have created; this book is a boon to anyone charged with enlightening beginning students about the subject of stochastic processes; its value is enhanced even more by the paucity of suitable books for this purpose. The excellent book by Professor Rosenblatt, *Random Processes* (Oxford University Press) seems to me the only serious rival, but that book calls for students with substantial mathematical competence. The present book will be intelligible and useful to a much wider range of readers. The reviewer takes his hat off to Professor Parzen for his success in a ticklish task.