# MARKOVIAN DECISION MODELS FOR THE EVALUATION OF A LARGE CLASS OF CONTINUOUS SAMPLING INSPECTION PLANS[1]

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**1. Introduction and summary.** The purpose of this article is to present a uniform method for the evaluation of a large class $S_D$ of *Dodge-type* continuous sampling inspection plans. The class of Dodge-type plans includes, among others, CSP-1, 2, 3, 4, and 5, MLP-1, $r$, and $T$, and H-106 plans. The evaluation of any plan $S \varepsilon S_D$ is in terms of its average outgoing quality limit (AOQL). The AOQL for $S$ may be defined as an upper bound to the long run proportion of defective items that remains in the output after inspection, given certain assumptions about Nature's (the processes') ability to control process quality. In particular, a specific method of evaluation involving linear programming as its computational tool is developed for the case where Nature is assumed to be unrestricted in her ability to produce and submit defectives. The problem of determining *unrestricted AOQL's* for the plans in $S_D$ is viewed in terms of two Markovian decision models where Nature is taken to be the decision maker. These decision models are abstractly described in Section 2. Their relation to the problem of evaluating continuous sampling plans is specified in Section 3. The linear programs corresponding to the two decision models are derived in Sections 4 and 5. In Section 6 the linear programming approach is illustrated with an example, and in the last section a row reduction theorem is given for one of the linear programs.

**2. The Markovian decision models.** Consider the following dynamic system as in Derman [4], [5], [6]. At times $t = 0, 1, \cdots$ the system is observed to be in some state $i$ ($i = 1, \cdots, L$). After each observation the decision maker "controls" the system by making a decision $d_k$ ($k = 0, \cdots, K_i$), where $K_i < \infty$ denotes the number of available decisions when the observed state is $i$. Let $\{Y_t : t = 0, 1, \cdots\}$ denote the sequence of observed states and $\{\Delta_t : t = 0, 1, \cdots\}$ the sequence of observed decisions. We shall assume that

$$P\{Y_{t+1} = j \mid h_{t-1}, Y_t = i, \Delta_t = d_k\} = q_{ij}(k),$$

$i, j = 1, \cdots, L$; $k = 0, \cdots, K_i$; $t = 0, 1, \cdots$, where for each $t$, $h_t$ denotes the history of states and decisions (i.e., $h_t = \{Y_0, \Delta_0, \cdots, Y_t, \Delta_t\}$) and where the $q_{ij}(k)$'s are non-negative numbers which satisfy the equations, $\sum_j q_{ij}(k) = 1$, $i = 1, \cdots, L$; $k = 0, \cdots, K_i$.

The most general decision procedure (strategy for the decision maker) is defined by the collection of functions $\{D_k(h_{t-1}, Y_t = i) : i = 1, \cdots, L; k = 0, \cdots, K_i; t = 0, 1, \cdots\}$ of observations and decisions such that $D_k(\cdot) \geqq 0$,

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for all $k$ and $\sum_k D_k(\cdot) = 1$. The class of *all* such procedures will be denoted by $C$. Any rule $R \varepsilon C$ is used to control the system by setting

$$P\{\Delta_t = d_k \mid h_{t-1}, Y_t\} = D_k(h_{t-1}, Y_t),$$

for all $k$ and every possible set of observations $(h_{t-1}, Y_t)$, $t = 0, 1, \cdots$. Once the initial state probabilities $P\{Y_0 = i\}$, $i = 1, \cdots, L$, are given and a rule $R$ is designated, it follows that the sequence $\{Y_t : t = 0, 1, \cdots\}$ can be described as a stochastic process with state space $\{1, \cdots, L\}$.

Let $C'$ denote the class of decision procedures $R$ such that

$$D_k(h_{t-1}, Y_t = i) = D_{ik}, \qquad i = 1, \cdots, L; k = 0, \cdots, K_i,$$

independent of $h_{t-1}$ and $t$. Then if $R \varepsilon C'$, the sequence $\{Y_t : t = 0, 1, \cdots\}$ is a Markov chain with stationary transition probabilities

$$p_{ij} = \sum_k q_{ij}(k) D_{ik}, \qquad i, j = 1, \cdots, L.$$

In addition, let $C''$ be the subset of $C'$ such that $D_{ik} = 0$ or 1. (Note that this class contains only a finite number of procedures.)

Continuing as in Derman [4], let $w_{jk}(t) \geqq 0, j = 1, \cdots, L; k = 0, \cdots, K_j$; $t = 0, 1, \cdots$ be finite numbers denoting some function of the observed state and decision taken. It will be assumed that $w_{jk}(t) = w_{jk}$ independent of $t$. Furthermore, let the finite numbers $w'_{jk}(t) > 0, j = 1, \cdots, L; k = 0, \cdots, K_j$; $t = 0, 1, \cdots$ be defined in the same way as the $w_{jk}(t)$'s.

For a fixed procedure $R \varepsilon C$, define

$$\tilde{W}_t = \sum_j \sum_k w_{jk} z_{jk}(t), \qquad \tilde{W}_t' = \sum_j \sum_k w'_{jk} z_{jk}(t),$$

where

$$\begin{aligned} z_{jk}(t) &= 1 \text{ if } Y_t = j,\ \Delta_t = d_k \\ &= 0 \text{ otherwise.} \end{aligned}$$

Also, for $Y_0 = i$ and any rule $R \varepsilon C$, let $\tilde{\Phi}_T{}^R(i)$ be the vectors with components

$$\tilde{x}_{jk}(T) = (T + 1)^{-1} \sum_{t=0}^{T} z_{jk}(t), \quad j = 1, \cdots, L; k = 0, \cdots, K_j.$$

In the analysis to follow, $w_{jk}$ is defined as the expected number of defective items that pass by the inspector when the sampling plan is in state $j$ and nature takes decision $d_k$ and $w'_{jk}$ is defined as the expected number of items both defective and non-defective that pass by the inspector. We shall be primarily concerned with the "average" quality functions

$$\tilde{Q}_R(i) = \limsup_{T\to\infty} (T + 1)^{-1} \sum_{t=0}^{T} \tilde{W}_t = \limsup_{T\to\infty} \sum_j \sum_k w_{jk} \tilde{x}_{jk}(T)$$

and

$$\begin{aligned} \tilde{\psi}_R(i) &= \limsup_{T\to\infty} \left( \sum_{t=0}^{T} \tilde{W}_t \Big/ \sum_{t=0}^{T} \tilde{W}_t' \right) \\ &= \limsup_{T\to\infty} \left( \sum_j \sum_k w_{jk} \tilde{x}_{jk}(T) \Big/ \sum_j \sum_k w'_{jk} \tilde{x}_{jk}(T) \right). \end{aligned}$$

Corresponding to these functions, we define $A_1(R, i)$ to be the smallest number $A_1$ such that

$$P\{\tilde{Q}_R(i) \leqq A_1\} = 1,$$

and $A_2(R, i)$ to be the smallest number $A_2$ such that

$$P\{\tilde{\psi}_R(i) \leqq A_2\} = 1.$$

We can now state the two problems that will be of interest in the sequel.

PROBLEM 1. Suppose $P\{Y_0 = i\} = 1$. Find a procedure $R_1$ such that

$$P\{\tilde{Q}_{R_1}(i) = \sup_{R \epsilon C} A_1(R, i)\} = 1.$$

PROBLEM 2. Suppose $P\{Y_0 = i\} = 1$. Find a procedure $R_2$ such that

$$P\{\tilde{\psi}_{R_2}(i) = \sup_{R \epsilon C} A_2(R, i)\} = 1.$$

**3. Problems 1 and 2 and the evaluation of continuous sampling plans.** Our interest in these problems arose in connection with a study of a class $S_D$ of Dodge-type continuous sampling inspection plans and their evaluation, White [19]. Dodge-type continuous sampling plans [8], [9], [10], [11], [17], [21], [19], (hereafter abbreviated as DCSP's) are used to aid in the control of the percent defective in the output from a continuous production process. A typical DCSP allows for a mixture of 100% inspection and partial inspection (e.g. if quality is good—no defectives have been found for sometime—then partial inspection is employed; if quality is bad, then 100% inspection is used). Any plan in $S_D$ is described by specifying the type of sampling during partial inspection, the set of sampling levels, an inspection rule, and a transition rule. The two most common methods of partial inspection are *probability sampling* where successive items are inspected with probability (say) $B$, and *block sampling* where one item is chosen at random for inspection from consecutive blocks of length (say) $\bar{B}$.

The *levels* of a typical DCSP consist of an *initial level* ($n = 1$) at which inspection begins, *basic levels* ($n = 3, 6, 9, \cdots, \lambda$) and possibly *associated levels* ($n = 2, 4, 5, 7, \cdots, \lambda - 1, \lambda + 1$) for some (or all) basic levels. In general, the $n$th level ($n = 1, \cdots, \lambda + 1; \lambda < \infty$) of a DCSP with probability sampling is defined by the pair $(B_n, s_n)$ where $B_n$ is the sampling probability and $s_n$ ($s_n = 1, \cdots, < \infty$) is the *release number* for level $n$. The release number for $n \neq \lambda$ specifies the number of consecutive non-defectives that must be observed at level $n$ in order to reduce the sampling frequency, i.e., switch to a "higher" sampling level. For level $\lambda$ with $B_\lambda \leqq B_n$, $n \neq \lambda$, it is mathematically convenient to let $s_\lambda = 1$ and to consider the switch in levels as fictitious, from $\lambda$ to $\lambda$. The $n$th level of a DSCP with block sampling is defined by the pair $(\bar{B}_n, s_n)$ where $\bar{B}_n$ is the block length associated with level $n$ and $s_n$ is the release number as defined before. In this case it is assumed that the levels are ordered so that $\bar{B}_\lambda \geqq \bar{B}_n$, $n \neq \lambda$, and again $s_\lambda = 1$. For both probability and block sampling plans it is assumed that the 1st level is a 100% inspection level, i.e., $B_1 = \bar{B}_1 = 1$.

The *inspection rule* of a DCSP with probability sampling designates the probability of inspection at each level. If block sampling is used, the inspection rule specifies a block length and an *inspection-disposition procedure* for each level. The inspection-disposition procedure for any level consists of a method of block inspection including a screening procedure if a defective is found. The *transition rule* of any DCSP controls the changes in the amount of inspection to be performed by the inspector. In general, if the inspector finds a defective, the transition rule causes the frequency of inspection to increase immediately. On the other hand, if a sufficient number of successive non-defectives are found, the transition rule causes a reduction in the inspection rate. (For a more complete description of the class of DCSP's see White [19] or [20].)

At any time $t$, or now more precisely, at *inspection point* $t$ ($t = 0, 1, \cdots$) the inspector can specify the *state* of his plan in terms of the sampling level and the number of consecutive non-defectives that he has observed while at that level. Therefore, the states of a DCSP can be defined by the set

$$N_U = \{n_u : u = 0, \cdots, s_n - 1; n = 1, \cdots, \lambda + 1\}.$$

However, to be consistent with the state space notation previously introduced, we shall usually let $N_U = \{i: i = 1, \cdots, L\}$, where the relationship between $i$ and $n_u$ is given by $i = \sum_{\nu=1}^{n-1} s_\nu + u + 1$.

A DCSP is often evaluated by its average outgoing quality limit (AOQL) relative to an assumed production process quality. A precise definition of the AOQL for any DCSP is most easily accomplished in three steps. First, *outgoing quality* ($\mathrm{OQ}_T$) is defined as the proportion of defectives that remains in the total output after the $(T + 1)$st inspection. In general, the $\mathrm{OQ}_T$ is a random variable. Second, the *average outgoing quality* (AOQ) is defined as the smallest number $A$ such that $P\{\limsup_{T\to\infty} \mathrm{OQ}_T \leqq A\} = 1$.

The AOQ can be thought of as a function of the inspector's (fixed) plan and the sequence of defectives and non-defectives "submitted" to the inspector. Suppose now that a possibly malevolent Nature determines this sequence. Assuming that Nature knows the inspector's plan prior to the start of production and that Nature can follow the progress of the plan once inspection begins, we can characterize the class of *submission decision procedures* available to Nature by the class $C$ introduced in Section 1. Then for a procedure $R \varepsilon C$, the *types* of decisions $\{d_k : k = 0, \cdots, K_i\}$ available to Nature when the plan "is in state $i$" can be seen to depend on the state and on the method of sampling. Specifically, for a DCSP with probability sampling, $K_i = 1$, $i = 1, \cdots, L$, and $d_0$ is interpreted as the decision to submit a non-defective, $d_1$ the decision to submit a defective. For a DCSP with block sampling, $K_i = \bar{B}_i$ and $d_k$ corresponds to the decision to submit $k$ defectives. Moreover, the sequence of decisions $\{\Delta_0, \Delta_1, \cdots\}$ which results from Nature's choice of a particular rule $R \varepsilon C$ has the effect of controlling the inspector's sampling frequency. This can be seen by noting that if the plan is in state $i$ at inspection point $t$ and $\Delta_t = d_k$, then the next state (at inspection point $t + 1$) is determined according to the probability vector

$(q_{i1}(k), q_{i2}(k), \cdots q_{iL}(k))$. Therefore, if the AOQ is written as a function of $R$ and fixed DCSP plan $S \varepsilon S_D$, then the *average outgoing quality limit* for $S$ is defined as

$$\text{AOQL}\,(S) = \sup_{R \varepsilon C} \text{AOQ}\,(R, S).$$

Under this definition, Nature is virtually unrestricted in her choice of a decision procedure. Consequently, to differentiate this AOQL from others made under different assumptions about Nature, we shall refer to this AOQL as *unrestricted* and use the abbreviation UAOQL.

The connection between Problems 1 and 2 and the evaluation of DCSP's can now be made. For Problem 1 define

$$\text{OQ}_T = (T+1)^{-1} \textstyle\sum_{t=0}^{T} \tilde{W}_t$$

and let the initial state $Y_0 = 1_0$ with probability 1. (Actually, it is assumed that all DCSP's begin with 100% inspection.) Then for any DCSP with probability sampling $S_p \varepsilon S_D$, the submission decision procedure $R_1$ is such that

$$\text{UAOQL}\,(S_p) = A_1(R_1, 1_0). \tag{3.1}$$

Similarly, for Problem 2 define

$$\text{OQ}_T = \left(\textstyle\sum_{t=0}^{T} \tilde{W}_t / \sum_{t=0}^{T} \tilde{W}_t'\right)$$

and assume that $P\{Y_0 = 1_0\} = 1$. Then for any DCSP with block sampling $S_b \varepsilon S_D$, the submission decision procedure $R_2$ is such that

$$\text{UAOQL}\,(S_b) = A_2(R_2, 1_0). \tag{3.2}$$

Various methods for determining the UAOQL's of some DCSP's have been derived by Lieberman [16], Derman, et al. [8], and Elfving [11]. Our results, presented in the next two sections, make it possible to use linear programming to evaluate UAOQL's for the class $S_D$ of Dodge-type plans.

Previous to this study, Derman [4], [5], has considered problems similar to Problems 1 and 2. In [4] and [5] the functions $\tilde{W}_t$ and $\tilde{W}_t'$ are defined as expectations rather than as random variables. As might be expected this difference is not material. In fact, in both studies it is shown that (i) attention may be restricted to the class $C''$ in looking for optimal rules, and (ii) the problems can be solved using linear programming techniques.

## 4. Reduction to $C''$.

ASSUMPTION A: For any initial state $i$ $(i = 1, \cdots, L)$ and any other state $j \neq i$, there exists a rule $R(i, j) \varepsilon C''$ such that

$$P\{Y_t = j \quad \text{for some} \quad t > 0 \mid Y_0 = i\} = 1$$

when $R(i, j)$ is used.

THEOREM 1. *When assumption* A *holds, there exist procedures* $R_1$ *and* $R_2$ *in* $C''$ *such that for Problem* 1,

$$\tilde{Q}_{R_1}(i) = \max_{R \varepsilon C} A_1(R, i), \qquad i = 1, \cdots, L, \tag{4.1}$$

*with probability* 1, *and for Problem* 2,

$$\tilde{\psi}_{R_2}(i) = \max_{R \varepsilon C} A_2(R, i), \qquad i = 1, \cdots, L, \tag{4.2}$$

*with probability* 1.

PROOF. The proof of both parts of the theorem depends on several results of Derman [6]. A statement of these results and the development of the proof require the following additional definitions. For every $R \varepsilon C$ and $Y_0 = i$, let $\Phi_T^R(i)$ be the vectors with components

$$x_{jk}(T) = (T+1)^{-1} \sum_{t=0}^{T} P\{Y_t = j, \Delta_t = d_k \mid Y_0 = i\}.$$

Let $\Phi^R(i) = \lim_{T\to\infty} \Phi_T^R(i)$ whenever the limit exists. This will occur when $R \varepsilon C'$ (see Chung [2], p. 32). In any case, let $H_R(i)$ denote the set of limit points of $\{\Phi_T^R(i)\}$ as $T \to \infty$, and let

$$H''(i) = \bigcup_{R \varepsilon C''} H_R(i).$$

For any $R \varepsilon C$, let $\omega$ denote a sample sequence of the joint process $\{(Y_t, \Delta_t): t = 0, 1, \cdots\}$ and define $\tilde{U}^R(i, \omega)$ to be the set of limit points of $\{\tilde{\Phi}_T^R(i)\}$ as $T \to \infty$. Finally, let

$$g(\tilde{\Phi}_T^R(i)) = (T+1)^{-1} \sum_{t=0}^{T} \tilde{W}_t, \qquad G(\tilde{\Phi}_T^R(i)) = \left(\sum_{t=0}^{T} \tilde{W}_t / \sum_{t=0}^{T} \tilde{W}_t'\right)$$

and,

$$g(\Phi_T^R(i)) = \sum_j \sum_k w_{jk} x_{jk}(T),$$
$$G(\Phi_T^R(i)) = \left(\sum_j \sum_k w_{jk} x_{jk}(T) / \sum_j \sum_k w_{jk}' x_{jk}(T)\right).$$

LEMMA 1. (Derman [6]). *There exists a procedure $R^* \varepsilon C''$ and an initial state $i^*$ such that for any $R \varepsilon C$,*

$$\tilde{Q}_R(i) = \limsup_{T\to\infty} g(\tilde{\Phi}_T^R(i)) \leqq g(\Phi^{R^*}(i^*)) \tag{4.3}$$

*with probability* 1, *where*

$$g(\Phi^{R^*}(i^*)) = \max_{R \varepsilon C;\ i=1,\cdots,L} \limsup_{T\to\infty} g(\Phi_T^R(i)).$$

LEMMA 2. (Theorem 2 of [6]). *For any procedure $R \varepsilon C$ and initial state $Y_0 = i$, $P\{\tilde{U}^R(i, \omega) \varepsilon \bar{H}''\} = 1$ where $\bar{H}''$ is the convex hull of $\bigcup_i H''(i)$.*

Lemma 1 guarantees that $\tilde{Q}_R(i)$ is bounded from above, while Lemma 2 insures that all the limit points of any sequence $\{\tilde{\Phi}_T^R(i)\}$ lie in the closed and bounded convex set $\bar{H}''$. But now note that $\Phi^{R^*}(i^*) \varepsilon \bar{H}''$ and is in fact an extreme point of this set. Consequently, if we can construct a procedure $R_1 \varepsilon C''$ which leads to equality in (4.3) we then have a proof for the first half of Theorem 1.

To construct $R_1$ we assume without loss of generality that $i^*$ is a member of an ergodic class (there may be more than one) of the Markov chain $\{Y_t : t = 0, 1, \cdots\}$ generated by $R^*$. Then for $Y_0 = i$ we assert that $R_1 = [R(i, i^*), R^*]$, where $R_1$ is interpreted as "use procedure $R(i, i^*)$ until state $i^*$ is first reached, then use procedure $R^*$." By Assumption A, $R_1 \varepsilon C''$. To show formally that $R_1$ leads to equality in (4.3) we prove the following lemma using some well known results from renewal theory.

LEMMA 3. *Suppose $Y_0 = i$ and that assumption A holds. Then if*

$$R_1 = [R(i, i^*), R^*]$$

*it follows that $\tilde{Q}_{R_1}(i) = g(\Phi^{R^*}(i^*))$ with probability* 1.

PROOF. Let $V_0 = \min\{t\colon Y_t = i^*\}$ given that $Y_0 = i$ and rule $R_1$ is being used. Define a cycle for the joint process $\{(Y_t, \Delta_t)\colon t = 0, \cdots\}$ to be any sequence of the form $(Y_{\tau_1} = i^*, \Delta_{\tau_1}), \cdots (Y_{\tau_2} = i^*, \Delta_{\tau_2})$ with $Y_t \neq i^*$, $\tau_1 < t < \tau_2$. Let $V_r = (\tau_2 - \tau_1)$ for the $r$th cycle, $r = 1, 2, \cdots$; then $EV_r < \infty$ for any $r$. Let $z_r(j, k)$ denote the number of occurrences of the pair $(j, k)$ during the $r$th cycle, and let $z_0(j, k)$ denote the number of occurrences of this pair before $i^*$ is first reached.

Now consider the sequence of points $\{T_a : a = 0, 1, \cdots\}$ (dependent on $\omega$) such that $\tilde{Q}_{R_1}(i) = \lim_{a\to\infty} g(\tilde{\Phi}_{T_a}^{R_1}(i))$. For any value of $a$ it is easily seen that there exists a number $M$, either zero or a positive integer, such that $\sum_{r=0}^{M-1} V_r \leqq T_a \leqq \sum_{r=0}^{M} V_r - 1$, and such that

$$(4.4)\quad \left(\sum_{r=0}^{M-1} z_r(j, k)/\sum_{r=0}^{M} V_r\right) < \tilde{x}_{jk}(T_a) < \left(\sum_{r=0}^{M} z_r(j, k)/\sum_{r=0}^{M-1} V_r - 1\right)$$

where $\tilde{x}_{jk}(T_a)$ is an element of $\tilde{\Phi}_{T_a}^{R_1}(i)$. The sequences $\{z_r(j, k)\colon r = 1, 2, \cdots\}$ and $\{V_r : r = 1, 2, \cdots\}$ are both sequences of independent and identically distributed random variables. Therefore, if the numerators and denominators of (4.4) are divided by $M$, then, by the strong law of large numbers,

$$\lim_{a\to\infty} \tilde{x}_{jk}(T_a) = x_{jk}^*$$

with probability 1, where $x_{jk}^*$ is an element of $\Phi^{R^*}(i^*)$. And since the same result holds for all pairs $(j, k)$, it follows from Lemmas 1 and 2 and the continuity of $g$ that $\tilde{Q}_{R_1}(i) = g(\Phi^{R^*}(i^*))$ with probability 1.

Combining Lemmas 1, 2 and 3 we have that $\tilde{Q}_{R_1}(i) = \max_{R\varepsilon C} A_1(R, i)$ with probability 1 which proves the first half of the theorem.

To prove the second half of Theorem 1 we shall need the following well known proposition.

LEMMA 4. *Let $H$ be a closed and bounded convex set. Let $g(x)$ and $g'(x)$ be linear functions defined on $H$ with $g'(x) > 0$ for all $x \varepsilon H$. Then the ratio $G(x) = g(x)/g'(x)$ takes its maximum at an extreme point of $H$.*

Lemma 4 is used in proving the next lemma which is the counterpart to Lemma 1.

LEMMA 5. *There exists a procedure $R^{**} \varepsilon C''$ and an initial state $i^{**}$ (again without loss of generality, a member of an ergodic class) such that for any $R \varepsilon C$,*

$$(4.5)\quad \begin{aligned} \tilde{\psi}_R(i) &= \limsup_{T\to\infty} G(\tilde{\Phi}_T^{R}(i)) \\ &\leqq G(\Phi^{R^{**}}(i^{**})) \end{aligned}$$

*with probability* 1, *where*

$$G(\Phi^{R^{**}}(i^{**})) = \max_{R \varepsilon C;\ i=1,\cdots,L} G(X^R(i)), \tag{4.6}$$

*$X^R(i) \varepsilon H_R(i)$, $X^R(i)$ denoting any limit point of any sequence $\{\Phi_T{}^R(i)\}$.*

PROOF. Consider the closed and bounded set $\bar{H}''$. By a second theorem of Derman [6], $X^R(i) \varepsilon \bar{H}''$ for all $R \varepsilon C$ and $i = 1, \cdots, L$. But then by Lemma 4 and the construction of $\bar{H}''$ it follows that there exists a procedure $R^{**} \varepsilon C''$ and an initial state $i^{**}$ such that the ratio function $G$ is maximized at the extreme point $\Phi^{R^{**}}(i^{**})$. Equation (4.6) denotes this fact. Now consider the sequence of points $\{T_a : a = 0, 1, \cdots\}$ (dependent on the sample function $\omega$) such that

$$\limsup_{T\to\infty} G(\tilde{\Phi}_T{}^R(i)) = \lim_{a\to\infty} G(\tilde{\Phi}^R_{T_a}(i)).$$

Next, recall from Lemma 2 that for some subsequence $\{T_a(\alpha): \alpha = 0, 1, \cdots\}$, $\lim_{a\to\infty} \tilde{\Phi}^R_{T_a(\alpha)}(i) \varepsilon \bar{H}''$ with probability 1. It follows that $\tilde{\psi}_R(i)$ can be considered as a function on $\bar{H}''$. And since $G$ is a continuous function on $\bar{H}''$, (4.5) follows and the proof of the lemma is complete.

Now we define the procedure $R_2 = [R(i, i^{**}), R^{**}]$ and again by Assumption A $R_2 \varepsilon C''$. Then similar to Lemma 3 we have

LEMMA 6. *Suppose $Y_0 = i$ and Assumption A holds. Then if $R_2 = [R(i, i^{**}), R^{**}]$ it follows that $\tilde{\psi}_{R_2}(i) = G(\Phi^{R^{**}}(i^{**}))$ with probability 1.*

Lemmas 4, 5, and 6 combine to prove that $\tilde{\psi}_{R_2}(i) = \max_{R\varepsilon C} A_2(R, i)$ with probability 1. This completes the proof of Theorem 1.

COROLLARY. *Let the elements of $\Phi^{R^*}(i^*)$ and $\Phi^{R^{**}}(i^{**})$ be denoted by $\{x^*_{jk}\}$ and $\{x^{**}_{jk}\}$ respectively. Then*

$$A_1(R_1, i) = \textstyle\sum_j \sum_k w_{jk}x^*_{jk}, \qquad i = 1, \cdots, L' \tag{4.7}$$

*and*

$$A_2(R_2, i) = (\textstyle\sum_j \sum_k w_{jk}x^{**}_{jk} / \sum_j \sum_k w'_{jk}x^{**}_{jk}), \qquad i = 1, \cdots, L. \tag{4.8}$$

PROOF. Equations (4.7) and (4.8) follow directly from Lemmas 3 and 6.

The implication of the theorem and corollary for the evaluation of DCSP's is clear. The interpretation of Theorem 1 is that a malevolent Nature facing any DCSP can choose her optimal submission decision procedure from the class $C''$. Moreover, from the Corollary and Equations (3.1) and (3.2) it follows that the UAOQL for any DCSP is computed either as a linear function or as the ratio of two linear functions.

**5. Linear programming formulations.** The linear programming formulation of Problems 1 and 2 follows from the Corollary to Theorem 1 and Markov chain considerations. The details of such a formulation are discussed in Derman [4], [7]. The method has also been used by Manne [18], Klein [14], [15] and Dantzig and Wolfe [3] among others. (Howard [13] has provided an alternative approach for similar problems.) In this section we state the problems as they apply specifically to the evaluation of DCSP's.

PROBLEM 1. The UAOQL for any DCSP with probability sampling is given by the solution to the following problem:

Maximize: $\sum_j \sum_k w_{jk} x_{jk}$ ;

subject to:

$$(5.1)\qquad \begin{aligned} x_{jk} &\geqq 0, & j &= 1, \cdots, L;\ k = 0, 1,\\ \textstyle\sum_k x_{jk} - \sum_i \sum_k x_{ik} q_{ij}(k) &= 0, & j &= 1, \cdots, L,\\ \textstyle\sum_i \sum_k x_{ik} &= 1. \end{aligned}$$

Nature's optimal procedure is found by setting

$$D_{jk} = (x_{jk}/\textstyle\sum_m x_{jm}) \qquad j = 1, \cdots, L;\ k = 0, 1.$$

If for some state $j$ the denominator is zero then $\{D_{jk} : k = 0, 1\}$ should be set so that $j$ is a transient state. Nature can always accomplish this by setting $D_{j0} = 1$ or 0 depending on $j$.

In formulating Problem 2 we also make use of the fact that the ratio of two linear functions defined on a convex set can be maximized by linear programming techniques. (See Derman [4] or Charnes and Copper [1] for details.)

PROBLEM 2. The UAOQL for any DCSP with block sampling is given by the solution to the following problem:

Maximize: $\sum_j \sum_k w_{jk} y_{jk}$ ;

subject to:

$$(5.2)\qquad \begin{aligned} y_0 \geqq 0,\ y_{jk} &\geqq 0, & j &= 1, \cdots, L;\ k = 0, \cdots, \bar{B}_j,\\ \textstyle\sum_k y_{jk} - \sum_i \sum_k y_{ik} q_{ij}(k) &= 0, & j &= 1, \cdots, L,\\ \textstyle\sum_j \sum_k y_{jk} - y_0 &= 0,\\ \textstyle\sum_j \sum_k w'_{jk} y_{jk} &= 1. \end{aligned}$$

As before, the elements of Nature's optimal rule can be computed from the formula,

$$D_{jk} = y_{jk}/\textstyle\sum_m y_{jm}, \qquad j = 1, \cdots, L;\ k = 0, \cdots, \bar{B}_j.$$

**6. An example: The evaluation of** CSP-1 **by linear programming.** The first continuous sampling inspection plan, CSP-1, was devised by Dodge and reported in 1943 [10]. The object of this plan as stated by Dodge ([10], p. 264) is "to establish a limiting value of average outgoing quality expressed in percent defective which will not be exceeded no matter what quality is submitted to the inspector." The following is a description of CSP-1 with probability sampling:

(a) Begin by inspecting 100% of the output until $s$ units in succession are found to be non-defective.

(b) When $s$ successive non-defective units are produced switch from 100% inspection to partial inspection, inspecting each successive item with probability $B$.

(c) If a sample unit is found to be defective, revert immediately to 100% inspection and continue as in paragraph (a).

(d) Replace all defective items found with non-defectives.

Lieberman [16] using probability theoretic considerations derived a formula to compute the UAOQL for any CSP-1 plan with probability sampling. The linear programming approach leading to the same formula is as follows.

Let the state space for a CSP-1 plan be given by $N_U = \{1_0, 1_1, \cdots, 1_{s-1}, 3_0\}$. Let $w_{n_u k}$ be defined as the expected number of defectives that pass by the inspector when the plan is in state $n_u$ and Nature takes decision $d_k$. Then

$$
\begin{aligned}
w_{n_u k} &= (1 - B), n = 3; u = 0; k = 1 \\
&= 0, \text{ otherwise.}
\end{aligned} \tag{6.1}
$$

The conditional transition probabilities $(q_{ij}(k))$ for CSP-1 are given by,

$$
\begin{aligned}
q_{1_u n_v}(k) &= 1, k = 0; u = 0, \cdots, s - 2; n = 1; v = u + 1 \\
&= 1, k = 0; u = s - 1; n = 3; v = 0 \\
&= 1, k = 1; u = 0, \cdots, s - 1; n = 1; v = 0 \\
&= 0, \text{ otherwise,}
\end{aligned} \tag{6.2}
$$

and

$$
\begin{aligned}
q_{3_0 n_v}(k) &= 1, k = 0; n = 3; v = 0 \\
&= 1 - B, k = 1; n = 3; v = 0 \\
&= B, k = 1; n = 1; v = 0 \\
&= 0, \text{ otherwise.}
\end{aligned} \tag{6.3}
$$

Now by substituting (6.1), (6.2), and (6.3) into the linear programming formulation of Problem 1 we have for CSP-1 with probability sampling the linear program:

Maximize: $(1 - B)x_{3_0 1}$;
subject to:

$$
\begin{aligned}
x_{n_u k} &\geqq 0, && \text{all } n_u \varepsilon N_U; k = 0, 1, \\
x_{1_0 0} + x_{1_0 1} - \textstyle\sum_{u=0}^{s-1} x_{1_u 1} - B x_{3_0 1} &= 0, \\
x_{1_u 0} + x_{1_u 1} - x_{1_{u-1} 0} &= 0, && u = 1, \cdots, s - 1, \\
\textstyle\sum_n \sum_u \sum_k x_{n_u k} &= 1.
\end{aligned} \tag{6.4}
$$

(Note: We have omitted one equation in (6.4) since in the original set of constraints (5.1), the sum of the first $L$ equations equals the $(L + 1)$st.)

This program can be solved by the usual methods. However, because of the form of the objective function and the last constraint equation, any optimal solution is given in part by $x_{1_u 1} = 0$, $u = 0, \cdots, s - 1$, and $x_{3_0 0} = 0$. But then letting $x_{nk} = x_{n_u k}$, we can rewrite the above linear program as

Maximize: $(1 - B)x_{31}$;

subject to:

$$x_{10} - Bx_{31} = 0, \qquad sx_{10} + x_{31} = 1.$$

The solution to this program is easily seen to be,

$$x_{10} = B/(sB + 1), \qquad x_{31} = 1/(sB + 1), \qquad x_{11} = x_{30} = 0,$$

and consequently,

$$\text{UAOQL} = (1 - B)/(sB + 1).$$

It follows, then, that Nature's decision rule is given by $D_{1_u0} = 1, u = 0, \cdots, s - 1$ and $D_{3_01} = 1$.

Derman, et al. [8] have shown that when $B = 1/\bar{B}$, the same decision rule is optimal against a CSP-1 block sampling plan with parameters $\bar{B}$ and $s$. This conclusion has also been arrived at by the linear programming approach to Problem 2 in White [19]. In the linear programming formulation, $w_{n_uk}$ is again defined as the expected number of defectives that pass by the inspector at inspection point $t$ when $Y_t = n_u$ and $\Delta_t = d_k$. In addition, $w'_{n_uk}$ is defined as the expected number of items (both defective and non-defective) that pass by the inspector at inspection point $t$ when $Y_t = n_u$ and $\Delta_t = d_k$. Thus, for a CSP-1 plan with block sampling,

$$\begin{aligned} w_{n_uk} &= 0, n = 1; u = 0, \cdots, s - 1; k = 0, 1 \\ &= k[1 - (k/\bar{B})] + (k - 1)(k/\bar{B}), n = 3; u = 0; k = 0, \cdots, \bar{B} \end{aligned}$$

and

$$\begin{aligned} w'_{n_uk} &= 1, n = 1; u = 0, \cdots, s - 1; k = 0, 1 \\ &= \bar{B}, n = 3; u = 0; k = 0, \cdots, \bar{B}. \end{aligned}$$

The linear programming method of Problem 1 has also been used by White [19] to find UAOQL's for the class of H-106 plans [21].

**7. A row reduction theorem.** In the example of Section 6 it was shown that the original set of $s + 1$ constraint equations could be replaced by a set of two equations. This result for a two level plan can be generalized for a $\lambda + 1$ level DCSP with probability sampling.

THEOREM 2. *Let*

$$\begin{aligned} x_{n_u0} &= x_{n0}, & u &= 0, \cdots, s_n - 1; n = 1, \cdots, \lambda + 1, \\ x_{n_01} &= x_{n1}, & n &= 1, \cdots, \lambda + 1, \\ x_{n_u1} &= 0, & u &= 1, \cdots, s_n - 1; n = 1, \cdots, \lambda + 1. \end{aligned} \tag{7.1}$$

*Then the linear program for Problem* 1 *can be written in the following reduced form*:
*Maximize*: $\sum_{n=1}^{\lambda+1} (1 - B_n)x_{n1}$;

*subject to*:

$$x_{nk} \geqq 0, \qquad n = 1, \cdots, \lambda + 1; k = 0, 1,$$

$$\sum_{k=0}^{1} x_{nk} - \sum_{m=1}^{\lambda+1} \sum_{k=0}^{1} x_{mk} q_{mn}(k) = 0, \qquad n = 1, \cdots, \lambda + 1,$$

$$\sum_{n=1}^{\lambda+1} (s_n x_{n0} + x_{n1}) = 1,$$

*where*,

$$q_{mn}(0) = q_{m_{s-1}n_0}(0), \; m \neq n,$$

$$= q_{n_0 n_0}(0), \; m = n,$$

*and*

$$q_{mn}(1) = q_{m_0 n_0}(1).$$

PROOF. The proof of this theorem follows from the observation that if Nature is going to submit defectives at any level $n$ $(n = 1, \cdots, \lambda + 1)$, then she can always begin submitting them at state $n_0$ without reducing (and possibly increasing) the value of the objective function. Thus, only the class of feasible solutions with this property need be considered. This class is described by the Equations (7.1).

REMARK. It is also possible to make some reductions in the size of the linear program for Problem 2. See White [19] for details.

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