HAMMERSLEY, J. M. AND HANDSCOMB, D. C., *Monte Carlo Methods*. Methuen & Co., London, and John Wiley & Sons, Inc., New York, 1964. vii + 178 pp. 25s. or $4.75

and

SHREIDER, YU. A. (ed.), *Methods of Statistical Testing/Monte Carlo Method*, Elsevier Publishing Co., Amsterdam-London-New York, 1964, ix + 303 pp., $15.00.

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For a reader interested in Monte Carlo Methods, or for one interested in learning about them, the appearance of these two books should help fill a previous void. The concepts underlying the use of a Monte Carlo method, that is, using a mechanical device to create simulated observations, may be considered as old as the subjects of probability and statistics. However, the term "Monte Carlo Method" and the associated development of formal techniques came into use in the 1940's. The term had been used initially to identify the application of statistical computational procedures for obtaining numerical estimates for problems in nuclear physics. The initial Monte Carlo procedures could be considered similar in nature to those which could be employed to estimate probabilities of events in a gambling game; namely, one could play a game many times and use the relative frequency of occurrences of the events of interest to estimate the unknown probabilities. The term now usually signifies that estimates to solutions to problems are to be obtained by creating, by computational methods on a digital computer, a set of observations or experiments so that numerical characteristics of the computational observations or experiments can be the basis of the estimates.

Interest in the theory and application of Monte Carlo techniques has fluctuated. The introduction of high-speed digital computers caused many to predict great things for Monte Carlo techniques. Part of this prediction rested on the conjecture that the computational work associated with using a Monte Carlo technique ought to increase as a linear function of the dimension of the problem whereas classical numerical techniques would increase exponentially. In spite of significant improvements in the size and speed of computers, many Monte Carlo techniques are still not economically feasible or reliable on today's computers.

The appearance of these two fine books on Monte Carlo offer their reader, for essentially the first time, both a global picture of the techniques and literature of this relatively young subject and considerable detail about known results related to theory and application. Statisticians and numerical analysts should find many challenging and open questions by reading these two books. For example, they may be motivated to consider the implications and potential advantages of using nonrandom observations.

Although at first glance it may appear that there is considerable overlap in the topics covered by the two books, the tone or emphasis in the two presentations is significantly distinct. Perhaps because of the many contributions that have been made to the theory of Monte Carlo by Hammersley, the book by him and Handscomb is much more conservative (perhaps more realistic) in its presentation with regard to the potential usefulness of Monte Carlo than the other book being reviewed. As a result, one may find the book edited by Shreider more motivating with regard to encouraging one to consider applications of the techniques discussed. However, upon reflecting on some of the claims, or enthusiasm, for the Monte Carlo Method noted in Chapters IV and V of the book by Shreider, "Investigation of Mass Service or Congestion Processes, Including Queueing" (Chapter IV) and "Information Theory" (Chapter V), one may feel that some of the enthusiasm is unjustified. The book edited by Shreider is more complete in the sense of giving considerable attention to computational considerations.

The background required to read either book is comparable, although the authors specify their requirements differently. In 'S', "It is assumed that the reader is familiar with the fundamentals of random events and random variables, and with their probability characteristics. . . . Furthermore, the reader must have a grasp of the concept of normal distribution, Lyapunov's theorem and, for certain chapters, of the elements of mathematical statistics. An understanding of Markov processes would also be desirable." In 'H&H', "We assume that the reader has what might roughly be described as a 'graduate' knowledge of mathematics. The actual mathematical techniques are, with few exceptions, quite elementary, but we have freely used vectors, matrices, and similar mathematical language for the sake of conciseness." It will be noted that H&H specify in more detail the mathematical level required but do not require any background in statistics since they outline relevant statistical terms, (except for Markov Chains—4 pages in chapter 9) in 15 pages in Chapter 2. In parts either book will be challenging, especially if one does not have some feeling for numerical analysis. One very nice feature provided in S which is not covered in H&H is an ample discussion and illustration of the use of Chebyshev's inequality both to assure the reader that a Monte Carlo-type estimator does converge in the probability sense and to explain the implications of the sample size.

Both books provide an excellent and extensive list of references on Monte Carlo, S providing a list of 282 articles and books, while H&H provide both a list of over 150 entries associated by chapter plus many more additional useful references by subject area not specifically mentioned in the text of the book. The authors are to be complimented on having so diligently cited the available literature, and thus providing a useful guided tour of the literature.

With a subject as young as Monte Carlo, and with as many diverse applications to be considered to illustrate and explain techniques, it is pleasing to find that the authors of both books have provided some sense of continuity among the chapters. This unity is especially pleasing in the book edited by Shreider since five authors contributed to the seven chapters of their book.

The specific contents of each book will be considered next. The book by

Hammersley and Handscomb consists of 12 chapters and is impressive on several accounts, one being how much material is covered well in a relatively few pages. For a first edition the book is surprisingly free of serious errors. In a private communication with the reviewer, Hammersley noted that the trivial errors will be corrected in a new edition about to be released. He points out that the reader may wish to note the following three errors:

1. p. 73. eq. (5.8.17) For $g(0)$—read $g(0)+$
2. p. 73, eq. (5.8.20) For $(1-2\eta)g(0)$—read $(1-2\eta)g(0)+$
3. p. 84. 1.11 For 47000 read 470000.]

Chapter 1, "The General Nature of Monte Carlo Methods" is an interesting presentation which stresses among other things that mathematics ought to be viewed in terms of being theoretical or experimental. There is much philosophical food for thought here. (An interesting finding of Dr. Stephen Brush is cited in that a paper by Lord Kelvin in 1901 employed a Monte Carlo technique in studying the Boltzmann equation.) For the uninitiated the terse initial definition of Monte Carlo "that branch of experimental mathematics which is concerned with experiments on random numbers" may be troublesome: see page 2.

In Chapter 2, "Short Resumé of Statistical Terms," the authors achieve their objective, but it may be too concise to be of much value to many readers. The reviewer would have preferred to see more attention given to how the material presented relates to the other chapters. This would have required a much more comprehensive treatment, especially the treatment of sampling methods. Less than half a page is devoted to sampling.

In Chapter 3, "Random, Pseudorandom, and Quasirandom Numbers" one will find a survey of the theory and analytical analysis of methods for generating numbers which are used to approximate random numbers. The reader will find a presentation which distinguishes between pseudorandom numbers and quasirandom numbers, quasirandom numbers being those which may violate specific statistical tests for randomness but pass those tests which are relevant for a specific application. In passing, it is not clear how one can make this judgment about relevant tests which may depend upon the unknown desired solution. Although the topic of this chapter is fundamental for the rest of the book, it may cause the reader anguish to realize how much more research is needed here to assure a user that he is safe in using a specific technique to generate pseudorandom or quasirandom numbers. In addition, much of the theory now available does not treat the case when numbers can have only a limited amount of precision on a digital computer. On a somewhat trivial note, the authors infer, on page 26, that the physical processes used to generate the Rand tables "were random in the strict sense," actually the numbers generated by the physical process failed certain tests and had to be subjected to arithmetic processing before they were found acceptable.

"Direct Simulation" is the title and topic of Chapter 4. (Direct simulation is defined in Chapter 1 on page 2 as, "in case of a probabilistic problem the simplest Monte Carlo approach is to observe random numbers, chosen in such a way that they directly simulate the physical random processes of the original problem.") According to the authors, direct simulation "possesses little theoretical interest and so will not occupy us much in this book, but it remains one of the principal forms of Monte Carlo. . . , characterized by a fairly simple structure overlaid with a mass of small and rather particular details. These problems are beyond the reach of general theory on account of the details. . . ." It is a pity the authors had to be so negative and did not even suggest that techniques of statistical experimental design might aid these problems, especially the applications to Operations Research. (By comparison see Chapters IV and V of Shreider.) Chapter 4 of H&H does contain some interesting material and in Section 4.3, "Quantification of qualitative mathematical theory," the reader may be delighted to find a stimulating example which considers aspects of the travelling salesman problem in terms of dimensional analysis, fortified by Monte Carlo estimation.

Chapter 5, "General Principles of the Monte Carlo Method" contains a wealth of information on techniques for reducing the variance of a Monte Carlo estimator, such as stratified sampling, importance sampling, control variates, antithetic variates, regression methods, and the use of orthonormal functions. The material is presented with the aid of illuminating examples. "Conditional Monte Carlo" is the title of Chapter 6 and, as the title might indicate, it deals with the use of conditional distributions in forming estimators. The theory of conditional Monte Carlo is presented in a clear and concise manner, the difficult but classic example of Arnold, Bucher, Trotter and Tukey is given. Chapters 5 and 6 are two of the finest in the book.

Chapters 7 through 11 may be considered to represent treatment of special areas of significant applications. By chapter, the areas covered are: Chapter 7, "Solution of Linear Operator Equations" consisting of simultaneous linear equations, sequential Monte Carlo, Fredholm integrals of the second kind, the Dirichlet problem, and Eigenvalue problems, Chapter 8, "Radiation Shielding and Reactor Criticality," Chapter 9, "Problems in Statistical Mechanics," Chapter 10, "Long Polymer Molecules," Chapter 11, "Percolation Processes" (deterministic flow in a random medium). Chapters 7 through 11, while covering topics of specialized interest, do contain valuable information and references about techniques which can have application in other areas. By chapter, the following methods provide such examples: Chapter 8 "The semi-analytic method," a "matrix method," and the use of a maximum likelihood approach, Chapter 9 Markov chains whose states are sequences (page 125), Chapter 10 "inversely restricted sampling" and "enrichment techniques," Chapter 11 "Ratio estimates."

Chapter 12, "Multivariable Problems," while brief, considers an area of great potential for Monte Carlo since other more classical numerical techniques have been so limited in treating multivariable analysis. In this chapter the reader is invited to imagine sampling from the population of terms in a mathematical formula. Specific examples are interpolation of functions of many variables and calculations with large matrices.

The book edited by Shreider consists of seven chapters and in general is well coordinated and well translated, although the translator is not identified. It is perhaps unfortunate that the title of the book, "Method of Statistical Testing: Monte Carlo Method" may be somewhat misleading, possibly as a result of translation. However, within the book the more meaningful term, *statistical trials*, is used instead of *statistical testing*. A brief review of each chapter will be considered.

Chapter I—"Fundamentals of the Monte Carlo Method," by the editor Yu. A. Shreider, consists of the following six sections: (1) Definition and simple examples of the application of the Monte Carlo Method, (2) Accuracy of the Monte Carlo Method and its main characteristics, (3) Generation of random numbers, (4) Solution of systems of linear algebraic equations, (5) Random walk and the solution of boundary value problems, and (6) The Monte Carlo Method and the simulation of Markov Processes in a computer. In thirty-eight pages, a wealth of information related to estimating solutions to mathematical problems by Monte Carlo procedures is covered, but if the reader is without the background specified in the Preface and without a background in matrices and partial differential equations, he may find this material difficult in spots. However, with the necessary background this chapter is not only an excellent introduction to the subsequent chapters, but also a useful introduction to the field of Monte Carlo techniques. The reviewer does take exception to the remark on page 14 to the effect that to approximate the Gaussian law it is sufficient to use a sum of five (5) uniform deviates.

Chapter II—"Evaluation of Definite Integrals," by I. M. Sobol, one finds the following five sections: (1) Simple application of the Monte Carlo Method, (2) Some methods for reducing the variance, (3) Evaluation of multidimensional integrals, (4) Evaluation of Wiener integrals, and (5) Application of quasirandom points to the Monte Carlo scheme. Section two contains a review of some of the variance reduction techniques which are also cited in Chapter 5 in Hammersley and Handscomb. Attention is also given here to the advantages to be gained from symmetrization of an integrand. This chapter is rich with interesting examples and it contains a rewarding presentation of the various known techniques for estimating one dimensional definite integrals, an area of much larger application than is obvious. General-purpose powerful Monte Carlo techniques for estimating multidimensional integrals are still needed, since deterministic techniques have inadequacies here too. However, the reader will find food for thought in this chapter, for example, the recent work by J. M. Hammersley, J. H. Halton, and the extension by the Russians of the use of quasirandom numbers to estimate multidimensional integrals.

Chapter III is also by I. M. Sobol on—"Applications of the Monte Carlo Method in Neutron Physics." This chapter is essentially for one interested in techniques developed for neutron physics and radio engineering. However, for the reader studying complex processes other than of the type cited here which do not lend themselves directly to analytical analysis, there is much in this chapter that is worth studying. This is especially true if the process can be viewed as a set of connected branch or decision points. Attention is given to how to organize a simulation model so as to take advantage of knowledge about the physical process and to arrange the calculations depending upon the amount of ancestor branching. (This aspect is not given comparable attention by H&H in Chapter 8.)

Chapter IV is by N. P. Buslenko titled "Application of the Monte Carlo Method to the Investigation of Mass Service or Congestion Processes, Including Queueing." As the author notes, the object of mass-service theory is the study of the time-patterns (such as delays) arising in the process of serving a stream of calls forming an input into a system. This class of problems can include production scheduling, say of inspection, and automatic control of complex assemblies. This is a relatively new area of application of Monte Carlo and simulation techniques. It should hold great promise to those interested in quality control. Some attention is given to how to generate the input stream of calls for a few general problems and some attention is given to a block diagram to indicate how one advances the clock to study the behavior of a system to the calls as a function of the service algorithm. Here is an area where the authors' enthusiasm may not be justified because much more work needs to be done in this application area. This is one of the few areas where one of the authors appears to be unaware of some of the work done in the U. S. and England relating to the development of generalized simulation languages to simplify the specification and programming of models of time dependent processes (e.g. Gordon, Markowitz, Tocher.)

Chapter V—"Application of the Monte Carlo Method to Information Theory" by V. G. Sragovich. This chapter is primarily a brief introduction to noise theory, the use of a Neyman-Pearson likelihood ratio test, and a Wald sequential likelihood ratio test to detect a signal in the presence of noise. From this material it is then seen that the detection problems, as stated in this chapter, reduce to the evaluation of multiple integrals, an area where Monte Carlo techniques might help. There is little mention of specific Monte Carlo techniques or of the effectiveness of Monte Carlo techniques in this area of application. In general, the reader may find this a stimulating chapter, but it may be too compact and of questionable value, especially for anyone who is not already working in this area.

Chapter VI—"Generating Uniformly Distributed Random Quantities by Means of Electronic Computers," is by D. L. Golenko. A review is given of some of the early analytical methods for generating pseudorandom deviates on digital computers. Several mixed-congruential methods that use specific instructions found on several Russian computers are noted. Some intuitive arguments which have not convinced the reviewer, are advanced for using two generators alternately, say every fifteenth number. Attention is given to criteria to test the quality of the analytical methods and some useful advice about the pitfalls of these tests are cited. Several physical devices which can be attached to computers for generating numbers are mentioned, for example, radiation of radioactive substances and generation of noise of electron tubes. The presentation in this chapter deserves careful reading for those interested in generation methods and for those interested in tests of pseudorandom numbers.

Chapter VII—"Transformation of Random Numbers," is by N. P. Buslenko and V. G. Sragovich. Attention is given here to various methods to obtain observations from arbitrary distributions through use of random numbers. (The only topic demanding special comment is that the authors give special attention to the errors and problems introduced by the fact that a number in a digital computer is represented by only a limited number of digits. A comparable warning is not cited by H&H in their book.)

In summary, one may find these demanding books to read, however much interesting and useful material is presented. The two books complement each other. One cannot help but be impressed with how aware the Russians are of the work done in Monte Carlo in this country and in England. It is also very impressive to see how much original and imaginative work they have done in this young field in the last ten years. These fine books contain enough differences to recommend them both to one desiring more knowledge in the theory and application of Monte Carlo techniques.