ON SECOND MOMENTS OF STOPPING RULES[1]

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0. Summary. The current investigation is a natural outgrowth of [2], being concerned with the variance of stopping rules and the effect of non-zero means on the variance of a randomly stopped sum. Some martingale generalizations of applications of [2] also appear.

1. Introduction. A *stopping rule* or *stopping variable* of a sequence $\{X_n, n \geqq 1\}$ of random variables defined on a probability space $(\Omega, \mathfrak{F}, P)$ is a positive integer-valued random variable t such that for every $n \geqq 1$ the event $\{t = n\} \varepsilon \mathfrak{F}_n$, the Borel field generated by $X_1 \cdots X_n$. In contradistinction, a *stopping time*[2] (likewise of a sequence $\{X_n\}$) will be defined as a positive integer or $+\infty$ valued function on Ω subject to the same proviso that $\{t = n\} \varepsilon \mathfrak{F}_n$, $n \geqq 1$. Thus, a stopping time t is a stopping *variable* or stopping *rule* if and only if $P\{t < \infty\} = 1$. In numerous problems of probability theory and statistics it is necessary to demonstrate that what is obviously a stopping time is further a stopping variable and even to obtain detailed information about the latter.

2. Comparison of stopping rules. Let the basic process $\{X_n, n \geqq 1\}$ consist of independent random variables with $EX_n = 0, EX_n^2 = 1, P\{|X_n| \leqq a < \infty\} = 1$ for $n \geqq 1$. Set $S_n = \sum_{i=1}^n X_i$ and define $t_m(c)$ to be the smallest positive index $n \geqq m$ $(m = 1, 2, \cdots)$ for which $S_n^2 > c^2 n$ where c is a positive constant. For the case of coin tossing $(a = 1)$, it was shown in [1] that for all m, $Et_m(c)$ is finite or infinite according as $c < 1$ or $c \geqq 1$ and this was generalized in [2] to the uniformly bounded case. (For $c \geqq 1$, the hypothesis of a uniform bound is superfluous and was not stipulated in [2].) It will be proved in Section 3 that if $c^2 < 3 - 6^{\frac{1}{2}}$, $Et_m^2(c) < \infty$, all $m \geqq 1$ while if $c^2 \geqq 3 - 6^{\frac{1}{2}}$ then $Et_m^2(c) = \infty$ for all sufficiently large (but not necessarily all) m.

It is clear from a comparison technique that there is a non-increasing sequence of non-negative constants $\{c_k, k \geqq 1\}$ such that $Et_m^k(c) < \infty$ for $c < c_k$ (if $c_k > 0$) while $Et_m^k(c) = \infty$ for all sufficiently large m if $c > c_k$. Such comparisons may be formalized by the following

Definition. A stopping time t will be called "more restrictive" than a stopping time s if $\{t = n\} \subset \{s \leqq n\}$ for $n = 1, 2, \cdots$ that is, if $t \geqq s$.

Clearly, if t is more restrictive than s, and t is a bonafide stopping variable, so is s; moreover, the finiteness of Et^α implies that of Es^α for any $\alpha > 0$.

Thus, if $c < c'$, $Et_m^k(c) \leqq Et_m^k(c')$, $(k, m = 1, 2, \cdots)$ corroborating the prior statement about the sequence c_k.

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[2] In [2] the terms are used synonymously but it is clearly desirable to make such a distinction.

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3. Second moments. When $c^2 < 1$, the situation changes in the coin tossing example ($a = 1$) alluded to earlier since now $P\{t_m(c) = 1\} = 1$ for $m = 1$. Thus, to allow the second moment to attain an infinite value, it is necessary to dawdle for a while so as to insure that S_n does not prematurely escape its parabolic bonds. This accounts for the appearance of the phrase "for all sufficiently large m" in

THEOREM 1. *Let $\{X_n\}$ be independent random variables with $P\{|X_n| \leqq a < \infty\} = 1$, $EX_n = 0$, $EX_n^2 = 1$ for $n \geqq 1$ and define t_m = smallest integer $n \geqq m$ for which $S_n^2 > c^2n$ $(n = 1, 2, \cdots)$. If $c^2 < 3 - 6^{\frac{1}{2}}$, then $Et_m^2 < \infty$, all $m \geqq 1$ while if $c^2 \geqq 3 - 6^{\frac{1}{2}}$, $Et_m^2 = \infty$ for all sufficiently large m.*

PROOF. In the case $c^2 < 3 - 6^{\frac{1}{2}}$ we write t for t_m. Set $\gamma_n = EX_n^3$, $\beta_n = EX_n^4$ and $t' = \min(t, k)$ where $k > m$. Since $Et'\sum_{j=1}^{t'}\beta_j \leqq a^4Et'^2 < \infty$, by Theorem 3 of [2],

$$ES_{t'}^4 = 6Et'S_{t'}^2 - 3Et'(t'+1) + 4ES_{t'}\sum_1^{t'}\gamma_j + E\sum_1^{t'}\beta_j, \tag{1}$$

whence

$$E(S_{t'}^2 - c^2t')^2 = (6 - 2c^2)Et'S_{t'}^2 - (3 - c^4)Et'^2 - 3Et' + 4ES_{t'}\sum_1^{t'}\gamma_j + E\sum_1^{t'}\beta_j,$$

implying

$$(3 - c^4)Et'^2 + (2c^2 - 6)Et'S_{t'}^2 \leqq (a^4 - 3)Et' + 4a^3Et'|S_{t'}|. \tag{2}$$

Let $A_k = \{m < t \leqq k\}$. From (2), recalling that $Et' \leqq Et < \infty$ for $c^2 < 1$, [2],

$$(3 - c^4)[\textstyle\int_{[t>k]} k^2 + \int_{A_k} t^2] + (2c^2 - 6)[\int_{[t>k]} c^2k^2 + \int_{A_k} t(ct^{\frac{1}{2}} + a)^2] \leqq 4a^3[\int_{[t>k]} ck^{3/2} + \int_{A_k} t(ct^{\frac{1}{2}} + a)] + O(1).$$

Consequently,

$$(c^4 - 6c^2 + 3)[k^2P\{t > k\} + \textstyle\int_{A_k} t^2] \leqq B[k^{3/2}P\{t > k\} + \int_{A_k} t^{3/2}] + O(1)$$

where $B > 0$ is a constant depending only on c and a. Thus, letting $k \to \infty$, $Et^2 < \infty$ regardless of m.

In the alternative case, we may clearly suppose $3 - 6^{\frac{1}{2}} \leqq c^2 < 1$. Define $u_m(c)$ to be the first index $n \geqq 1$ for which $|S_n| > c(n + m)^{\frac{1}{2}} - 1$ where m is an arbitrary positive integer.

Suppose it has been established for every c^2 in $[3 - 6^{\frac{1}{2}}, 1)$ that $Eu_m^2(c) = \infty$ for all sufficiently large m. Then, if v_m = first $n \geqq m + 1$ such that $|S_n - S_m| > cn^{\frac{1}{2}} - 1$, both $v_m - m$ and $u_m(c)$ have the same distribution and thus $Ev_m^2 = \infty$ for c^2 in $[3 - 6^{\frac{1}{2}}, 1)$ and all sufficiently large m. However, on the set $[t_m = n, |S_m| \leqq 1]$, we have $|S_n - S_m| > cn^{\frac{1}{2}} - 1$ whence

$$E(t_m^2 \mid |S_m| \leqq 1) \geqq E(v_m^2 \mid |S_m| \leqq 1) = Ev_m^2 = \infty$$

implying $Et_m^2 = \infty$ for all sufficiently large m and $c^2 \geqq 3 - 6^{\frac{1}{2}}$.

Thus, it suffices to prove the auxiliary proposition involving $u_m(c)$ and in so doing we denote the latter variable by t.

LEMMA. *For* $0 < c < 1$, $O(m) = Et \geqq mc^2/(1-c^2) + O(m^{\frac{1}{2}})$.

PROOF. Choose $c < c_1 < 1$ and $m_1 > 0$ such that $c_1 n^{\frac{1}{2}} \geqq c(n+m)^{\frac{1}{2}} - 1$ for all $n \geqq m_1$. By the comparison technique and Corollary 2 of [2], $Et < \infty$. By Theorem 2 of [2],

$$Et = ES_t^2 > c^2E(t+m) - 2cE(t+m)^{\frac{1}{2}} + 1 \geqq c^2E(t+m) - 2cE^{\frac{1}{2}}(t+m) + 1,$$

implying $E^{\frac{1}{2}}(t+m)$ exceeds the larger or is dominated by the smaller of the roots $(1-c^2)^{-1}\{-c \pm [c^2 + (1-c^2)(m+1)]^{\frac{1}{2}}\}$. Thus, $Et \geqq mc^2/(1-c^2) + O(m^{\frac{1}{2}})$. On the other hand,

$$Et = ES_t^2 \leqq E[c(t+m-1)^{\frac{1}{2}} + a]^2 \leqq c^2E(t+m-1) + 2acE^{\frac{1}{2}}(t+m-1) + a^2$$

or $(1-c^2)E(t+m-1) - 2acE^{\frac{1}{2}}(t+m-1) - (a^2+m-1) \leqq 0$ whence $Et = O(m)$.

Suppose now that $Et^2 < \infty$ for all m. By Theorem 3 of [2],

$$\begin{aligned} ES_t^4 &= 6EtS_t^2 - 3Et(t+1) + 4ES_t\textstyle\sum_{j=1}^t \gamma_j + E\sum_1^t \beta_j \\ &\geqq 6c^2Et(t+m) - 3Et(t+1) - 4a^3Et|S_t| - 12cEt(t+m)^{\frac{1}{2}} \qquad (3) \\ &\geqq (6c^2-3)Et^2 + (6mc^2-3)Et - 4a^3cE(t+m-1)^{3/2} - 4a^4Et \\ &\quad - 12cEt(t+m)^{\frac{1}{2}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (4)\quad ES_t^4 \leqq E[c(t+m-1)^{\frac{1}{2}} + a]^4 &= c^4E(t+m-1)^2 + 4ac^3E(t+m-1)^{3/2} \\ &\quad + 6c^2a^2E(t+m-1) + 4ca^3E(t+m-1)^{\frac{1}{2}} + a^4 \end{aligned}$$

whence, combining (3) and (4) and recalling that $Et = O(m)$,

$$\begin{aligned} (6c^2 - 3 - c^4)Et^2 \leqq m^2c^4 - 2mc^2(3-c^2)Et \\ + [4ac(a^2+c^2) + 12c]E(t+m)^{3/2} + O(m). \end{aligned}$$

Since $E(t+m)^{3/2} \leqq 2Et^{3/2} + 2m^{3/2} \leqq 2E^{3/4}t^2 + 2m^{3/2}$ and $Et \geqq mc^2(1-c^2)^{-1} + O(m^{\frac{1}{2}})$ (by the lemma),

$$(5)\quad (6c^2 - 3 - c^4)Et^2 \leqq m^2c^4[1 - 2(3-c^2)(1-c^2)^{-1}] + O(E^{3/4}t^2 + m^{3/2}).$$

Employing the lemma again, we have $Et^2 \geqq E^2t \geqq m^2c^4(1-c^2)^{-2} + O(m^{3/2}) \to \infty$ and

$$(6)\qquad 6c^2 - 3 - c^4 \leqq O(E^{-\frac{1}{4}}t^2) + O(m^{-\frac{1}{2}}) + (c^2-5)(1-c^2)^{+1}.$$

Hence $6c^2 - 3 - c^4 < 0$ which is patently false for c^2 in $[3 - 6^{\frac{1}{2}}, 1)$. Thus, $Et^2 = \infty$ for all sufficiently large m and the theorem is proved.

THEOREM 2. *Let* $\{X_n\}$ *be independent random variables with* $P\{|X_n| \leqq a < \infty\} = 1$, $EX_n = 0$, $EX_n^2 = 1$ *for* $n \geqq 1$. *If* t *designates the smallest integer* $n \geqq m$ *such that* $|S_n| > cn^{1/\alpha}$, *then* $Et^2 < \infty$ *for all* $\alpha > 2$, $c > 0$ *and* $m \geqq 1$.

PROOF. For any $c > 0$ and $\alpha > 2$, if m is sufficiently large $cn^{1/\alpha} < 4^{-1}n^{\frac{1}{2}}$ for $n \geqq m$. It follows therefore from the comparison technique and Theorem 1 that

$Et^2 < \infty$ for all sufficiently large m. Consequently, $Et^2 < \infty$ for all $m \geqq 1$, $\alpha > 2$, $c > 0$.

4. Non-zero means. Let the random variables $\{X_n\}$ of the basic process be independent with $EX_n = \mu_n$, $EX_n^2 = 1 + \mu_n^2$, n $\geqq 1$. If $S_n = \sum_{i=1}^n X_i$ and t is a stopping variable with $Et < \infty$, then

$$E(S_t - \textstyle\sum_{i=1}^t \mu_i)^2 = Et \tag{7}$$

by Theorem 2 of [2]. If in addition $\mu_n = 0$, $ES_t = 0$ by Wald's theorem and the left hand side of (7) is just the variance of S_t, say $\sigma_{S_t}^2$. On the other hand if $\mu_n \neq 0$, this is no longer the case and $\sigma_{S_t}^2$ may even be infinite despite the finiteness of (7).

For example, let $P\{X_n = \mu + 1\} = P\{X_n = \mu - 1\} = \frac{1}{2}$, $\mu \neq 0$ and define t as the first index $n \geqq m$ such that $(S_n - n\mu)^2 > 3n/4$. According to Theorem 1 of the preceding section, $Et^2 = \infty$ for all $m \geqq m'$ (and it will now be stipulated that $m \geqq m'$) while according to (7), $E(S_t - t\mu)^2 < \infty$. In view of the elementary inequality $\mu^2 Et^2 \leqq 2E(S_t - t\mu)^2 + 2ES_t^2$, it follows that $ES_t^2 = \infty$. By Wald's theorem, $ES_t = \mu Et < \infty$ and thus $\sigma_{S_t}^2 = \infty$.

Even when both quantities are finite, no general inequality between $E(S_t - \sum_1^t \mu_i)^2$ and $\sigma_{S_t}^2$ obtains. It is not difficult to verify that

$$\text{Cov}\ (2S_t - \textstyle\sum_1^t \mu_i, \sum_1^t \mu_i) \leqq 0$$

is necessary and sufficient for $\sigma_{S_t}^2 \leqq E(S_t - \sum_1^t \mu_i)^2$ if $E(\sum_1^t \mu_i)^2 < \infty$, $E\sum_1^t E \mid X_i \mid < \infty$. When $EX_n = \mu$, $EX_n^2 = 1 + \mu^2$ and t is a stopping variable with $Et^2 < \infty$, the simple condition μ Cov $(t, S_t) \leqq 0$ implies $\sigma_{S_t}^2 \leqq E(S_t - t\mu)^2$. If $P\{X_n = 1\} = p = 1 - P\{X_n = -b\}$, $b > 0$ and t denotes the first $n \geqq 1$ for which $X_n = 1$, then $S_t = -b(t - 1) + 1$. Since t and S_t are negatively correlated and $Et^2 < \infty$, $\sigma_{S_t}^2 \leqq E(S_t - t\mu)^2$ if $\mu \geqq 0$, i.e., if $p \geqq b/(b + 1)$. Here, this condition is necessary as well.

5. Martingale generalizations. In the following, the basic process $\{X_n\}$ will be postulated to satisfy $E|X_n| < \infty$, $E\{X_{n+1} \mid \mathfrak{F}_n\} = 0$, $n \geqq 1$ so that $S_n = \sum_1^n X_i$ is a martingale.

THEOREM 3. *Let* $\{X_n, n \geqq 1\}$ *satisfy* $E\{X_{n+1} \mid \mathfrak{F}_n\} = 0$, $E \sup X_n^2 < \infty$. *If* $u_n^2 = E\{X_n^2 \mid \mathfrak{F}_{n-1}\}$, *define t as the first integer* $n \geqq m$ *for which* $S_n^2 > c^2 \sum_1^n u_j^2$ *where* $0 < c < 1$ *and* $m = 1, 2 \cdots$. *Then* $\int_{[t \leqq n]} \sum_1^t u_j^2 = O(1)$ *and* $\int_{[t>n]} \sum_1^n u_j^2 = O(1)$ *as* $n \to \infty$.

PROOF. For any integer $k \geqq m$, set $t' = \min(t, k)$ and define $z = \sup |X_n|$, $A_k = \{m < t \leqq k\}$. By Theorem 1 of [2],

$$\int_{[t \leqq k]} \textstyle\sum_1^t u_j^2 + \int_{[t>k]} \sum_1^k u_j^2 = E\sum_1^{t'} u_j^2 = ES_{t'}^2 \leqq \int_{A_k} [c(\sum_1^t u_j^2)^{\frac{1}{2}} + z]^2$$
$$+ \textstyle\int_{[t>k]} c^2 \sum_1^k u_j^2 + O(1).$$

Thus,

$$(1 - c^2)[\textstyle\int_{[t>k]} \sum_1^k u_j^2 + \int_{A_k} \sum_1^t u_j^2] \leqq 2c(\int_{A_k} z^2)^{\frac{1}{2}}(\int_{A_k} \sum_1^t u_j^2)^{\frac{1}{2}} + O(1)$$

and the conclusion follows.

Corollary 1. *If further,* $P\{\sum_1^\infty u_j^2 = \infty\} = 1$, $P\{t > k\} = o(1)$ *and* $E\sum_1^t u_j^2 < \infty$.

Corollary 2. *If moreover* $P\{u_j^2 > \delta > 0\} = 1, j \geqq 1$ *then* $Et < \infty$.

Corollary 3. *If* $\{X_n\}$ *are independent with* $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $E(\sup X_n^2) < \infty$, $\sum_1^\infty \sigma_n^2 = \infty$ *and* $t =$ 1st $n \geqq m$ *such that* $S_n^2 > c^2 \sum_1^n \sigma_j^2$, $0 < c < 1$, *then* $P\{t < \infty\} = 1$ *and* $E(\sum_1^t \sigma_j^2) < \infty$. *If* $\sigma_n^2 > \delta > 0$, $Et < \infty$.

Corollary 3 generalizes Corollary 2 of Theorem 2 of [2] wherein $\sigma_n^2 = 1, n \geqq 1$.

Finally, the method of stopping rules will be utilized to generalize a Kolmogoroff inequality and to derive a result of Doob's [3], p. 320, which does not follow from this inequality.

Theorem 4. *Let* $\{X_n, n \geqq 1\}$ *satisfy* $EX_n^2 < \infty$, $E\{X_{n+1} \mid \mathfrak{F}_n\} = 0$ *and set* $u_n^2 = E\{X_n^2 \mid \mathfrak{F}_{n-1}\}$, $z = \sup |X_n|$. *Then, if* $\epsilon > 0$ *for any positive integer* k,

$$\int_{[\max_{n<k} S_n^2 \leqq \epsilon^2]} \sum_1^k u_j^2 \leqq E(\epsilon + z)^2.$$

Proof. Let $t =$ first $n \geqq 1$ such that $S_n^2 > \epsilon^2$. Set $t' = \min(t, k)$. Then

$$E(\epsilon + z)^2 \geqq ES_{t'}^2 = E\sum_1^{t'} u_j^2 \geqq \int_{[t \geqq k]} \sum_1^k u_j^2 = \int_{[\max_{n<k} S_n^2 \leqq \epsilon^2]} \sum_1^k u_j^2.$$

Corollary 1. *If moreover* $Ez^2 < \infty$, S_n *diverges* a.e. *on* $A = [\sum_1^\infty u_j^2 = \infty]$.

Proof. Let $t =$ 1st $n \geqq m$ for which $S_n^2 > \epsilon^2$. Then for $k \geqq m$ it follows from the theorem that

$$E(\epsilon + z)^2 \geqq \int_{[t \geqq k]} \sum_m^k u_j^2 \geqq \int_{A[t \geqq k]} \sum_m^k u_j^2 \geqq \int_{A[t=\infty]} \sum_m^k u_j^2$$

whence $P\{A[t = \infty]\} = 0$, i.e., $\sup_{n \geqq m} |S_n - S_{m-1}| > \epsilon$, a.e. in A. Since m is arbitrary S_n diverges a.e. in A.

Corollary 2. *If further* $Ez^2 < \infty$ *and* $P\{\sum_1^\infty u_n^2 = \infty\} = 1$, t *is a bonafide stopping variable.*

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