SOME NON-ORTHOGONAL PARTITIONS OF 4×4, 5×5 AND 6×6 LATIN SQUARES

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1. Summary. Non-orthogonal partitions of $n \times n$ Latin squares into $(n + 1)$ or $(n - 1)$ groups may be useful when further treatments are to be added to experimental designs in Latin squares. The different methods of constructing these partitions for $n = 4$, 5 or 6 are considered here. It is found that a partition into $(n + 1)$ groups is always possible when the Latin square has a directrix and a partition into $(n - 1)$ groups is possible whether or not there is a directrix. A complete enumeration of all possible partitions is given for the partition of 4×4 squares into 3 or 5 groups and that of 5×5 squares into 6 groups. Examples only are given for the partition of the 5×5 squares into 4 groups and the partition of the 6×6 squares into either 5 or 7 groups.

2. Introduction. When a Latin square design has been used for an experiment, it is sometimes necessary to add further treatments [6]. If the new set of treatments is not likely to interact with the old set, the simplest way of doing this is by means of a Graeco-Latin square, if the numbers of treatments in the two sets are the same. Sometimes, however, a Graeco-Latin square is unavailable, or inappropriate for the particular circumstances. Finney [1], [2], [3] has shown how more general orthogonal solutions can be obtained for squares of side 4, 5 or 6, but the new treatments will then have unequal replication. Freeman [6] has shown how the replication can be more nearly equalized, though the design will not be orthogonal, and has given methods of analysis and examples of designs for the addition of $(n - 1)$ and $(n + 1)$ treatments to an $n \times n$ Latin square. In the present paper the various ways in which this can be done are studied for 4×4, 5×5 and 6×6 squares.

If the n^2 cells of an $n \times n$ Latin square can be divided into r groups $S_1, S_2, \cdots, S_r$, where S_p has nk_p members and $k_1 + k_2 + \cdots + k_r = n$, such that S_p has k_p cells on each row, column and letter, then the subdivision is said to constitute a $(k_1, k_2, \cdots, k_r)$ orthogonal partition of the square [1]. In particular, a part for which $k_p = 1$ is called a *directrix* of the square. If two directrices have no cells in common, they are said to be *parallel*; if they have one cell in common, they are *orthogonal*. These properties are illustrated in the 4×4 square shown below, in which the letters form a Latin square and the numbers a (1, 1, 2) partition. The 1's and 2's form two parallel directrices and the cells underlined lie on a directrix orthogonal to each of these two.

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$\underline{A\ 1}$	$B\ 2$	$C\ 3$	$D\ 3$
$B\ 3$	$A\ 3$	$D\ 1$	$\underline{C\ 2}$
$C\ 3$	$\underline{D\ 3}$	$A\ 2$	$B\ 1$
$D\ 2$	$C\ 1$	$\underline{B\ 3}$	$A\ 3$

When the n^2 cells are divided into groups $S_1, S_2, \cdots, S_r$, where S_p has m_p members and, in general, m_p is not an integral multiple of n, the resulting partition is non-orthogonal. Two non-orthogonal partitions will be considered here, one containing $(n + 1)$ groups and the other $(n - 1)$ groups. It is convenient to change the notation so that the $(n + 1)$ groups are numbered from 0 to n and the $(n - 1)$ groups from 0 to $(n - 2)$. Then, with $(n + 1)$ groups S_0 will have n members, one on each row, column and letter, and so will be a directrix, while $S_p(p \neq 0)$ will have $(n - 1)$ members, one on each of $(n - 1)$ rows, columns and letters, but none on the nth. With $(n - 1)$ groups S_0 will have $(n + 2)$ members, two on two rows, columns and letters, and one on each of the other $(n - 2)$, and $S_p(p \neq 0)$ will have $(n + 1)$ members, two on one row, column and letter, and one on each of the other $(n - 1)$; for all p, n of the members may constitute a directrix, if one exists, but need not do so. Of the two squares shown below, that on the left is partitioned into $(n + 1)$ groups and that on the right into $(n - 1)$ groups. There is one directrix in the first partition, given by the 0's, and three in the second, given by the underlined 0's, 1's and 2's.

$A\ 0$	$B\ 1$	$C\ 2$	$D\ 3$
$B\ 2$	$A\ 3$	$D\ 4$	$C\ 0$
$C\ 3$	$D\ 0$	$A\ 1$	$B\ 4$
$D\ 1$	$C\ 4$	$B\ 0$	$A\ 2$

$\underline{A\ 0}$	$B\ 0$	$\underline{C\ 1}$	$\underline{D\ 2}$
$\underline{B\ 1}$	$\underline{A\ 2}$	$\underline{D\ 0}$	$C\ 0$
$\underline{C\ 2}$	$\underline{D\ 1}$	$A\ 1$	$\underline{B\ 0}$
$D\ 2$	$\underline{C\ 0}$	$\underline{B\ 2}$	$\underline{A\ 1}$

Latin squares are enumerated in *transformation sets* [4]. A transformation set contains all squares generated from one of its members by permutation of rows, columns and letters. By permuting these categories whole sets can be changed into other sets. When everything possible has been permuted there remain two sets each of 4×4 and 5×5 squares and 12 basic or *adjugate* sets of 6×6 squares [4], [5]. Similarly, permutations are possible after a non-orthogonal partition. With these conditions and another, simplifying one given below, every non-orthogonal partition of the 4×4 squares into 3 or 5 groups and the 5×5 squares into 6 groups is given here, examples are given of the non-orthogonal partitions of the 5×5 squares into 4 groups and the 6×6 squares into 7 groups and their number given, while examples only are given for the partition of the 6×6 squares into 5 groups.

3. Methods of constructing non-orthogonal partitions. Where there is an $n \times n$ Graeco-Latin square it is always possible to partition the square into $(n - 1)$ groups as described, and where there is another orthogonal classification the partition into $(n + 1)$ groups is always possible. The partition into $(n - 1)$ groups is achieved by taking $(n - 1)$ parallel directrices and replacing the nth directrix parallel to all these by two members of one group and one member of each of the remaining $(n - 2)$. This is even possible for the trivial case of the

3×3 square, as is immediately obvious. The partition into $(n + 1)$ groups is achieved by taking n parallel directrices and their intersections with another, orthogonal, directrix. The partitions of 4×4 squares into 5 or 3 groups shown in Section 2 are examples of these types of partition. 4×4 squares of Set II and 5×5 squares of Set II, which both have complete orthogonal solutions, can *only* be partitioned into $(n + 1)$ groups by the method of parallel directrices. No partition into $(n + 1)$ groups is possible, however, where there are no directrices, as with the first transformation set of the 4×4 squares and Sets VII, VIII, XI, XIII, XIV and XVII of the 6×6 squares. A non-orthogonal partition into $(n + 1)$ groups is possible for all 4×4, 5×5 and 6×6 squares containing a directrix, and any directrix may be taken as S_0. It is also possible to partition all squares of these sizes, with or without directrices, into $(n - 1)$ groups. It may be conjectured that such partitions are also possible for larger squares, but this seems difficult to prove.

The simplification referred to above is as follows: Two partitions are regarded as the same if, for some non-zero value of p, one can be obtained from the other by interchanging all the members of S_p with all but one of S_0. Thus, the following two partitions of the standard 4×4 square of the second set into 5 groups are the same.

A 0	B 1	C 2	D 3
B 2	A 3	$\underline{D\ 0}$	C 4
C 1	D 2	A 4	B 0
D 4	C 0	B 3	A 1

A 1	B 0	C 2	D 3
B 2	A 3	$\underline{D\ 0}$	C 4
C 0	D 2	A 4	B 1
D 4	C 1	B 3	A 0

Similarly, the following two partitions of the standard 4×4 square of the first set into 3 groups are the same.

$\underline{A\ 0}$	B 0	C 1	D 2
B 1	A 2	D 0	C 0
C 0	D 1	B 2	A 1
D 2	C 2	A 0	B 1

$\underline{A\ 0}$	B 1	C 0	D 2
B 0	A 2	D 1	C 1
C 1	D 0	B 2	A 0
D 2	C 2	A 1	B 0

One of the partitions into 5 groups can be obtained from the other by interchanging all the 0's and 1's except the 0 on D in the second row: one of the partitions into 3 groups can be obtained from the other by interchanging all the 0's and 1's except the 0 on A in the first row. This is always possible for the partition into $(n + 1)$ groups by the method of parallel directrices, as used here, and must work whenever the $(n - 1)$ members of S_p form a directrix with one member of S_0. It is more valuable, however, for the partitition into $(n - 1)$ groups, and happens if and only if there is a member common to one of the rows, one of the columns and one of the letters on which 0 occurs twice. This technique may be called *changeability*, and partitions which can be altered in this way will be called *changeable*, those which cannot be altered being called *unchangeable*. With a partition into $(n - 1)$ groups, no unchangeable partition can have n 0's forming a directrix, because the two extra 0's would occur in such positions as to make the partition changeable.

4. 4×4 **Latin squares.** The first transformation set has no directrices, so no partition into 5 groups is possible. The only partition of the second transformation set into 5 groups is by the method of parallel directrices, as shown in the last section.

The only partition of the first transformation set into 3 groups is as shown in the last section, this being a changeable partition. Two different partitions of the second set into three groups are possible, one by the method of parallel directrices and another, quite different, unchangeable one. The two partitions are as follows:

A 0	B w	C 1	D 2
B 1	A 2	D 0	C x
C 2	D 1	A y	B 0
D z	C 0	B 2	A 1

A 0	B 0	C 1	D 2
B 1	A 2	D 0	C 0
C 2	D 1	A 1	B 0
D 0	C 2	B 2	A 1

In the first of these partitions w, x, y, z represent 0, 0, 1, 2 in any order, but any of these 12 partitions can be derived from any other by permutation of rows, columns and letters. In the second partition two parallel directrices each contain four 1's and four 2's; of the other directrices parallel to these, one has three 0's and a 1 and the other three 0's and a 2.

5. 5×5 **Latin squares—partition into six groups.** These partitions have been fully investigated by means of an electronic computer. Taking the first standard square of Fisher and Yates [5] for both Set I and Set II all possible orthogonal partitions having the standard first row $A0\ B1\ C2\ D3\ E4$ have been determined; there are 60 of them for Set I and 36 for Set II.

There are three different partitions for Set I, as follows. It will be seen that they can all be derived from each other by permutations of the first column.

A 0	B 1	C 2	D 3	E 4
B 3	A 2	E 0	C 1	D 5
C 4	D 0	A 1	E 5	B 2
D 2	E 3	B 5	A 4	C 0
E 1	C 5	D 4	B 0	A 3

A 0	B 1	C 2	D 3	E 4
B 4	A 2	E 0	C 1	D 5
C 3	D 0	A 1	E 5	B 2
D 2	E 3	B 5	A 4	C 0
E 1	C 5	D 4	B 0	A 3

A 0	B 1	C 2	D 3	E 4
B 3	A 2	E 0	C 1	D 5
C 4	D 0	A 1	E 5	B 2
D 1	E 3	B 5	A 4	C 0
E 2	C 5	D 4	B 0	A 3

There are only three directrices of the 5×5 square of Set I [2], and they all go through the cell in the first row and column of the first standard square of Fisher and Yates. Thus, S_0 has to have this cell as a member, and the three partitions given can be changed into another three by interchanging all the 0's and 5's except this 0. Of the 60 standard partitions, 36 are derivable from the first given here after various permutations and changes, 18 from the second and 6 from the third.

For the second set of 5 × 5 squares the 36 standard partitions can all be derived from each other, being obtained by the method of parallel directrices. A typical partition is as follows:

A 0	B 1	C 2	D 3	E 4
B 2	C 0	E 3	A 1	D 5
C 3	E 5	D 0	B 4	A 2
D 4	A 3	B 5	E 0	C 1
E 1	D 2	A 4	C 5	B 0

Here, the partitions $S_p (p \neq 0)$ are on parallel directrices cut across by S_0, a directrix orthogonal to the others.

6. 5 × 5 Latin squares—partition into four groups. These partitions can be enumerated in two ways, by whether they are changeable or unchangeable and by the number of directrices. Table I gives the full number of partitions of each type for both sets of squares, the number in parentheses referring to the examples given below.

TABLE I

Numbers of partitions into four groups of each type for the 5 × 5 squares

	Directrices	Changeable	Unchangeable
Set I	0	10 (6a)	30 (6b)
	1	5 (6c)	15 (6d)
Set II	1	—	2 (6e)
	2	1 (6f)	4 (6g)
	3	—	6 (6h)
	4	1 (6i)	—

Examples of partitions (6a)–(6d) are as follows. It will be seen that (6a) and (6b) are the same apart from the numbers on C and A in the last column: similarly (6c) and (6d) are the same apart from the numbers on E and A in the fourth row. In (6c) and (6d) there are five 1's on a directrix.

(6a)

A 0	B 0	C 1	D 2	E 3
B 1	A 2	E 2	C 0	D 3
C 3	D 1	A 3	E 0	B 2
D 2	E 3	B 0	A 1	C 0
E 1	C 2	D 0	B 3	A 1

(6b)

A 0	B 0	C 1	D 2	E 3
B 1	A 2	E 2	C 0	D 3
C 3	D 1	A 3	E 0	B 2
D 2	E 3	B 0	A 1	C 1
E 1	C 2	D 0	B 3	A 0

(6c)

A 1	B 0	C 0	D 2	E 3
B 0	A 2	E 1	C 3	D 2
C 2	D 1	A 3	E 0	B 3
D 3	E 0	B 2	A 1	C 1
E 2	C 3	D 0	B 1	A 0

(6d)

A 1	B 0	C 0	D 2	E 3
B 0	A 2	E 1	C 3	D 2
C 2	D 1	A 3	E 0	B 3
D 3	E 1	B 2	A 0	C 1
E 2	C 3	D 0	B 1	A 0

An example of partition (6e) is as follows; it has five 2's on a directrix.

(6e)

A 0	B 0	C 1	D 2	E 3
B 3	C 2	E 1	A 3	D 0
C 2	E 0	D 3	B 1	A 2
D 1	A 1	B 2	E 0	C 3
E 2	D 3	A 0	C 0	B 1

The only partition (6f) and one example of partition (6g) are as follows; these being the same apart from the numbers on A and C in the last column; there are five 2's and five 3's on directrices in each example.

(6f)

A 0	B 0	C 1	D 2	E 3
B 0	C 3	E 2	A 1	D 1
C 2	E 1	D 0	B 3	A 0
D 3	A 2	B 1	E 0	C 2
E 1	D 3	A 3	C 0	B 2

(6g)

A 0	B 0	C 1	D 2	E 3
B 0	C 3	E 2	A 1	D 1
C 2	E 1	D 0	B 3	A 2
D 3	A 2	B 1	E 0	C 0
E 1	D 3	A 3	C 0	B 2

The six possible partitions (6h) can all be represented by the following example, in which x, y and z represent 1, 2 and 3 in some order; there are five 1's, five 2's and five 3's on directrices.

(6h)

A 0	B 0	C 1	D 2	E 3
B 1	C 3	E 2	A 0	D x
C 2	E 0	D 0	B 3	A 1
D 3	A 2	B y	E 1	C 0
E z	D 1	A 3	C 0	B 2

The partition (6i) is that obtained by the method of parallel directrices, and is as follows, where v, w, x, y, z represent 0, 0, 1, 2, 3 in some order.

(6i)

A v	B 0	C 1	D 2	E 3
B 2	C w	E 0	A 3	D 1
C 3	E 2	D x	B 1	A 0
D 0	A 1	B 3	E y	C 2
E 1	D 3	A 2	C 0	B z

7. 6×6 **Latin squares—directrix properties.** There are far too many partitions of 6×6 Latin squares for a complete enumeration to be attempted, and so examples only are given, these being related to properties of the directrices. The relevant directrix properties are given in this section, all being either stated in Finney's paper [3] or derived directly from it. There are no directrices for squares of Sets VII, VIII, XI, XIII, XIV or XVII, so only Sets I, III, IV, V, X and XV are considered.

There are 8 directrices of Set I, these being numbered by Finney as follows:

1.	*B*	*F*	*E*	*D*	*C*	*A*
2.	*C*	*D*	*F*	*B*	*E*	*A*
3.	*D*	*C*	*B*	*F*	*E*	*A*
4.	*E*	*C*	*B*	*D*	*F*	*A*
5.	*F*	*B*	*E*	*A*	*C*	*D*
6.	*F*	*B*	*A*	*E*	*D*	*C*
7.	*F*	*A*	*C*	*E*	*D*	*B*
8.	*F*	*D*	*C*	*B*	*A*	*E*

The square is assumed to be the standard one of Fisher and Yates, and the numbering is by rows, i.e., the rth element in the directrix is that in the rth row of the square. The directrices may be divided into two groups of four, directrices 1, 2, 5 and 8 forming the first group and 3, 4, 6 and 7 the second. The members of the first group each have three parallel directrices and the members of the second group each have four. It is not possible to find three mutually parallel directrices but there are pairs of parallel directrices in three ways, within the first group, such as 1 and 8, within the second group, such as 3 and 6, and with one member from each group, such as 1 and 6.

For Set III there are again 8 directrices. These can be grouped into four parallel pairs, there being no groups of three mutually parallel directrices.

The 8 directrices of Set IV may be divided into two groups of four mutually parallel directrices. Further, each directrix is also parallel to one directrix from the other group.

The 8 directrices of Set V may also be divided into two groups of four mutually parallel directrices, but here no two directrices from different groups are parallel.

Set X has 32 directrices which give rise to many groups of two, three or four mutually parallel directrices. They are numbered by Finney as follows; as with the directrices of Set I the standard square of Fisher and Yates is assumed and the numbering is by rows.

1.	*A*	*C*	*E*	*F*	*B*	*D*	17.	*D*	*B*	*F*	*E*	*C*	*A*
2.	*A*	*C*	*D*	*E*	*F*	*B*	18.	*D*	*C*	*B*	*F*	*E*	*A*
3.	*A*	*F*	*E*	*C*	*D*	*B*	19.	*D*	*C*	*A*	*E*	*B*	*F*
4.	*A*	*F*	*D*	*B*	*C*	*E*	20.	*D*	*E*	*B*	*A*	*C*	*F*
5.	*A*	*D*	*F*	*E*	*B*	*C*	21.	*E*	*C*	*B*	*D*	*F*	*A*
6.	*A*	*D*	*B*	*C*	*F*	*E*	22.	*E*	*C*	*D*	*B*	*A*	*F*
7.	*A*	*E*	*F*	*B*	*C*	*D*	23.	*E*	*F*	*C*	*B*	*D*	*A*
8.	*A*	*E*	*B*	*F*	*D*	*C*	24.	*E*	*A*	*B*	*C*	*D*	*F*
9.	*B*	*F*	*E*	*D*	*C*	*A*	25.	*F*	*B*	*E*	*A*	*C*	*D*
10.	*B*	*A*	*D*	*E*	*C*	*F*	26.	*F*	*B*	*A*	*E*	*D*	*C*
11.	*B*	*D*	*C*	*E*	*F*	*A*	27.	*F*	*C*	*E*	*D*	*A*	*B*
12.	*B*	*D*	*E*	*C*	*A*	*F*	28.	*F*	*C*	*A*	*B*	*E*	*D*
13.	*C*	*B*	*E*	*F*	*D*	*A*	29.	*F*	*A*	*C*	*E*	*D*	*B*
14.	*C*	*D*	*F*	*B*	*E*	*A*	30.	*F*	*A*	*B*	*D*	*C*	*E*
15.	*C*	*D*	*E*	*A*	*B*	*F*	31.	*F*	*D*	*C*	*B*	*A*	*E*
16.	*C*	*E*	*A*	*B*	*D*	*F*	32.	*F*	*D*	*B*	*A*	*E*	*C*

All the directrices have three members on one or the other of the two leading diagonals of the standard square and three members amongst the other 24 cells of the square. The directrices are divided by Finney into two sets, Set I containing 8 directrices and Set II 24; the distinction between the sets is that the three members of a directrix of Set I on the leading diagonals are all on the same diagonal while the three members of Set II on the leading diagonals are two on one diagonal, one on the other. The parallelism properties of the directrices can best be illustrated by displaying them as in Table II, using Finney's numbering.

TABLE II

Relationship between the directrices of Set X

32	19	13	7	4	27	24	11
17	8	28	15	12	23	2	30
16	25	5	18	21	10	31	3
1	14	20	26	29	6	9	22

The 8 directrices of Set I are in the middle two columns of Table II and the 24 of Set II in the outer six columns. There are 16 groups of four mutually parallel directrices amongst the 8 of Set I, formed by taking either directrix from each of the four rows; the two directrices of Set I in the same row have the same elements on the leading diagonal of the Latin square. There are 24 groups of four mutually parallel directrices amongst the 24 of Set II, 6 formed by taking the columns of Table II and the rest obtained by taking two directrices from one column and two from one of the columns on the opposite side in such a way that one directrix is taken from each row; for example, 1, 10, 23, 32 and 1, 16, 11, 30 and 1, 17, 24, 31 are all groups of four mutually parallel directrices. There are 16 groups of four mutually parallel directrices with three from Set II and one from Set I obtained by taking the three directrices of Set II in one row at one side and either directrix of Set I in the same row, e.g., 1, 14, 20, 26 or 1, 14, 20, 29. There are also 72 groups of three mutually parallel directrices of Set II that do not give rise to four parallel directrices; two of the directrices are in the same row and the same side and the third in a different row on the other side, e.g., 1, 14, 10, or 1, 14, 24. The final directrix property that is useful is that in either half of Table II the same elements occur in the four directrices of a row and column with a common directrix, e.g., 1, 16, 17, 32 and 1, 14, 20, 26 and 7, 15, 18, 26 all contain the same elements.

Set XV has 24 directrices giving rise to 30 groups of four mutually parallel directrices with each directrix occurring in 5 of these groups. With respect to any one directrix there are 10 parallel directrices, 5 of which occur in two different groups of four mutually parallel directrices and 5 in only one group. There are also 40 groups of three mutually parallel directrices not giving rise to a group of four parallel directrices, any one directrix occurring in 5 of these groups.

8. 6 × 6 Latin squares—partition into seven groups. These partitions have been fully investigated by means of an electronic computer. The partition is always possible where the Latin square has a directrix and further, for Sets I and X, where it is important which directrix is used, any directrix may be taken as S_0. Table III gives the full number of partitions, changeable and unchangeable, the numbers in parentheses referring to the examples below. Defining *pseudo-directrix* x as a group containing 5 members of directrix x, it is convenient to make a further subdivision of the partitions using a given directrix as S_0 by means of pseudo-directrices. Even so, for reasons of space examples cannot be given for some squares using each pseudo-directrix.

TABLE III

Numbers of partitions into seven groups of each type for the 6 × 6 squares

Set	Directrix as S_0	Changeable	Unchangeable
I	1	—	30 (8a, 8b, 8c)
I	3	—	24 (8d, 8e)
III	Any	12 (8f)	27 (8g, 8h)
IV	Any	—	9 (8i, 8j)
V	Any	—	24 (8k, 8l, 8m)
X	4	—	2 (8n, 8o)
X	1	—	1 (8p)
XV	Any	2 (8q, 8r)	—

For Set I using directrix 1 as S_0 there are 10 partitions with no pseudo-directrices 8 with pseudo-directrix no. 6, 8 with no. 7, 2 with no. 8 and 2 with nos. 6 and 8, Three examples are given: in (8a) there are no pseudo-directrices, in (8b) pseudo-directrix no. 7 is underlined and in (8c) nos. 6 and 8 are underlined.

(8a)

A 1	B 0	C 2	D 3	E 4	F 5
B 2	C 5	F 0	A 4	D 1	E 6
C 3	F 1	B 4	E 0	A 6	D 2
D 0	E 2	A 5	B 6	F 3	C 4
E 5	A 3	D 6	F 2	C 0	B 1
F 6	D 4	E 3	C 1	B 5	A 0

(8b)

A 1	B 0	C 2	D 3	E 4	$\underline{F\ 5}$
B 3	C 4	F 0	A 6	D 1	E 2
$\underline{C\ 5}$	F 1	B 6	E 0	A 2	D 4
D 0	$\underline{E\ 5}$	A 4	B 2	F 3	C 6
E 6	A 3	$\underline{D\ 5}$	F 4	C 0	B 1
F 2	D 6	E 3	C 1	$\underline{B\ 5}$	A 0

(8c)

A 1	B 0	C 2	D 3	E 4	$\underline{F\ 5}$
$\underline{B\ 5}$	C 3	F 0	A 2	$\underline{D\ 6}$	E 1
$\underline{C\ 6}$	F 1	B 4	E 0	$\underline{A\ 5}$	D 2
D 0	$\underline{E\ 5}$	A 3	$\underline{B\ 6}$	F 2	C 4
E 2	$\underline{A\ 6}$	D 1	F 4	C 0	B 3
F 3	D 4	$\underline{E\ 6}$	$\underline{C\ 5}$	B 1	A 0

For Set I using directrix 3 as S_0 6 partitions have pseudo-directrix no. 6, 2 have no. 7, 8 have no. 8, 4 have nos. 5 and 8, and 4 have nos. 6 and 8. In the examples the pseudo-directrices are underlined, no. 7 in (8d) and nos. 6 and 8 in (8e).

(8d)

A 1	B 2	C 3	D 0	E 4	$\underline{F\ 5}$
B 4	C 0	F 1	$\underline{A\ 5}$	D 6	E 3
$\underline{C\ 5}$	F 3	B 0	E 6	A 2	D 4
D 3	$\underline{E\ 5}$	A 6	B 1	F 0	C 2
E 0	A 4	$\underline{D\ 5}$	F 2	C 1	B 6
F 6	D 1	E 2	C 4	B 3	A 0

(8e)

A 1	B 2	C 3	D 0	E 4	$\underline{F\ 5}$
$\underline{B\ 5}$	C 0	F 1	A 2	$\underline{D\ 6}$	E 3
$\underline{C\ 6}$	F 4	B 0	E 1	$\underline{A\ 5}$	D 2
D 3	$\underline{E\ 5}$	A 4	$\underline{B\ 6}$	F 0	C 1
E 0	$\underline{A\ 6}$	$\underline{D\ 5}$	F 3	C 2	B 4
F 2	D 1	$\underline{E\ 6}$	C 4	B 3	A 0

For Set III the 12 changeable partitions are in three groups. Using directrix 1 as S_0, 4 have pseudo-directrix no. 2, 4 have no. 6 and 4 have no. 7; directrices 2, 6 and 7 are all orthogonal to directrix 1. Of the unchangeable partitions 21 have no pseudo-directrices and 6 have no. 4, directrix 4 being parallel to directrix 1. In example (8g) there are no pseudo-directrices, while in (8f) and (8h) nos. 2 and 4 respectively are underlined.

(8f)

A 0	B 1	C 2	D 3	E 4	F 5
B 2	A 4	F 0	E 1	C 3	$\underline{D\ 6}$
C 1	$\underline{F\ 6}$	B 3	A 5	D 0	E 2
D 4	C 5	$\underline{E\ 6}$	B 0	F 1	A 3
E 5	D 2	A 1	F 4	$\underline{B\ 6}$	C 0
F 3	E 0	D 5	$\underline{C\ 6}$	A 2	B 4

(8g)

A 0	B 1	C 2	D 3	E 4	F 5
B 2	A 3	F 0	E 5	C 6	D 1
C 3	F 4	B 6	A 1	D 0	E 2
D 4	C 5	E 3	B 0	F 1	A 6
E 1	D 6	A 4	F 2	B 5	C 0
F 6	E 0	D 5	C 4	A 2	B 3

(8h)

A 0	$\underline{B\ 1}$	C 2	D 3	E 4	F 5
B 4	A 2	F 0	E 6	C 3	$\underline{D\ 1}$
C 6	F 4	B 3	A 5	D 0	E 2
D 2	C 5	$\underline{E\ 1}$	B 0	F 6	A 3
E 5	D 6	A 4	$\underline{F\ 1}$	B 2	C 0
F 3	E 0	D 5	C 4	$\underline{A\ 1}$	B 6

For Set IV, 6 of the partitions have no pseudo-directrices. Using directrix 1 as S_0, one partition has pseudo-directrices nos. 4 and 5, one has nos. 4 and 8 and one has nos. 5 and 8. Directrices 1, 4, 5 and 8 are mutually parallel. In the examples there are no pseudo-directrices in (8i), and in (8j) nos. 4 and 5 are underlined.

(8i)

A 1	B 2	C 0	D 3	E 4	F 5
B 3	A 4	E 2	F 1	C 6	D 0
C 5	F 0	B 4	A 6	D 2	E 1
D 4	E 6	A 5	B 0	F 3	C 2
E 0	D 5	F 6	C 4	B 1	A 3
F 2	C 3	D 1	E 5	A 0	B 6

(8j)

A 1	B 2	C 0	$\underline{D\ 3}$	E 4	F 5
B 4	A 5	E 2	$\underline{F\ 6}$	$\underline{C\ 3}$	D 0
$\underline{C\ 6}$	F 0	$\underline{B\ 3}$	A 4	D 2	E 1
D 5	$\underline{E\ 3}$	$\underline{A\ 6}$	B 0	F 1	C 2
E 0	$\underline{D\ 6}$	F 4	C 1	B 5	$\underline{A\ 3}$
F 2	C 4	D 1	E 5	A 0	$\underline{B\ 6}$

For Set V, 12 of the partitions have no pseudo-directrices. Using directrix 1 as S_0 there is one partition with pseudo-directrix no. 3, one with no. 6, 6 with no. 8, 2 with nos. 3 and 8, and 2 with nos. 6 and 8. Directrices 1, 3, 6 and 8 are mutually parallel. In the examples there are no pseudo-directrices in (8k), no. 6 is underlined in (8l) and nos. 3 and 8 in (8m).

(8k)

A 1	B 2	C 0	D 3	E 4	F 5
B 3	A 4	E 1	C 2	F 6	D 0
C 6	F 0	B 4	A 5	D 2	E 3
D 4	E 5	F 3	B 0	C 1	A 6
E 0	D 6	A 2	F 1	B 5	C 4
F 2	C 3	D 5	E 6	A 0	B 1

(8l)

A 1	B 2	C 0	D 3	E 4	F 5
B 3	A 4	E 1	$\underline{C\ 6}$	F 2	D 0
C 4	F 0	$\underline{B\ 6}$	A 5	D 1	E 3
D 2	E 5	F 4	B 0	C 3	$\underline{A\ 6}$
E 0	$\underline{D\ 6}$	A 3	F 1	B 5	C 2
$\underline{F\ 6}$	C 1	D 5	E 2	A 0	B 4

(8m)

A 1	B 2	C 0	$\underline{D\ 3}$	E 4	$\underline{F\ 5}$
B 4	A 6	$\underline{E\ 5}$	C 2	F 1	D 0
$\underline{C\ 3}$	F 0	B 1	A 4	D 2	E 6
$\underline{D\ 5}$	$\underline{E\ 3}$	F 4	B 0	C 6	A 2
E 0	D 1	$\underline{A\ 3}$	F 6	$\underline{B\ 5}$	C 4
F 2	$\underline{C\ 5}$	D 6	E 1	A 0	$\underline{B\ 3}$

All the partitions of Set X have three pseudo-directrices parallel to the directrix forming S_0, this directrix and the pseudo-directrices being all of the same set. Of the two distinct partitions using a directrix of Set I, one has all its pseudo-directrices in the other column of the scheme in Table II, while the other has one pseudo-directrix in the same column and two in the other column. In the examples directrix 4 is taken as S_0; in (8n) pseudo-directrices nos. 15, 18 and 26 are underlined and in (8o) nos. 15, 21 and 26 are underlined.

(8n)

A 0	B 1	C 2	$\underline{D\ 3}$	E 4	$\underline{F\ 5}$
B 4	$\underline{C\ 3}$	F 0	A 1	$\underline{D\ 6}$	E 2
C 1	F 4	$\underline{B\ 3}$	$\underline{E\ 6}$	$\underline{A\ 5}$	D 0
D 2	$\underline{A\ 6}$	$\underline{E\ 5}$	B 0	F 1	C 4
$\underline{E\ 3}$	$\underline{D\ 5}$	A 4	F 2	C 0	$\underline{B\ 6}$
$\underline{F\ 6}$	E 0	D 1	$\underline{C\ 5}$	B 2	$\underline{A\ 3}$

(8o)

A 0	B 1	$\underline{C\ 2}$	D 3	$\underline{E\ 4}$	$\underline{F\ 5}$
B 3	$\underline{C\ 4}$	F 0	A 1	$\underline{D\ 2}$	E 6
C 6	F 3	$\underline{B\ 4}$	$\underline{E\ 2}$	$\underline{A\ 5}$	D 0
$\underline{D\ 4}$	A 6	$\underline{E\ 5}$	B 0	F 1	C 3
E 1	$\underline{D\ 5}$	A 3	F 6	C 0	$\underline{B\ 2}$
$\underline{F\ 2}$	E 0	D 1	$\underline{C\ 5}$	B 6	$\underline{A\ 4}$

In the partition using a directrix of Set II as S_0 the directrix and the parallel pseudo-directrices are two from one column of Table II and two from a column on the opposite side. In example (8p) directrix 1 is taken as S_0 and pseudo-directrices nos. 10, 23 and 32 are underlined.

(8p)

A 0	$\underline{B\ 1}$	C 2	D 3	$\underline{E\ 4}$	$\underline{F\ 5}$
B 2	C 0	F 3	$\underline{A\ 1}$	$\underline{D\ 5}$	E 6
$\underline{C\ 4}$	F 6	$\underline{B\ 5}$	E 0	A 3	D 2
D 6	A 2	$\underline{E\ 1}$	$\underline{B\ 4}$	F 0	C 3
$\underline{E\ 5}$	$\underline{D\ 4}$	A 6	F 2	$\underline{C\ 1}$	B 0
$\underline{F\ 1}$	E 3	D 0	$\underline{C\ 5}$	B 6	$\underline{A\ 4}$

For Set XV both the partitions have six pseudo-directrices, the maximum possible. In one, 5 of the pseudo-directrices are those occurring in two different groups of four mutually parallel directrices with the directrix forming S_0, while in the other, 5 of the pseudo-directrices are those occurring in only one group of mutually parallel directrices with S_0. In both partitions the sixth pseudo-directrix is orthogonal to S_0, thus permitting changeability. In the examples

(8q) has the pseudo-directrices from two parallel groups, (8r) those from one only.

(8q)

A 0	B 1	C 2	D 3	E 4	F 5
B 3	A 4	F 0	E 2	D 6	C 1
C 4	D 2	A 5	B 0	F 3	E 6
D 5	F 6	E 3	A 1	C 0	B 4
E 1	C 5	B 6	F 4	A 2	D 0
F 2	E 0	D 1	C 6	B 5	A 3

(8r)

A 0	B 1	C 2	D 3	E 4	F 5
B 5	A 2	F 0	E 6	D 1	C 3
C 1	D 4	A 3	B 0	F 6	E 2
D 2	F 3	E 5	A 4	C 0	B 6
E 3	C 6	B 4	F 1	A 5	D 0
F 4	E 0	D 6	C 5	B 2	A 1

9. 6 × 6 **Latin squares—partition into five groups.** These partitions have not been enumerated completely but are very numerous. In Table IV the existence of partitions only is shown, not the full number, the numbers in parentheses referring to the examples below; a plus sign shows that there are some partitions, a minus sign that there are none. The partitions are classified by whether or not they are changeable, and by the number of directrices. For Set X with four directrices it matters which directrices are used, but in all other cases it does not. Where there are both changeable and unchangeable partitions, the examples given are of unchangeable partitions from which changeable ones can be derived by the interchange of the two elements underlined. If there are n directrices the example has 61's, $\cdots$, 6 n's on directrices.

TABLE IV

Existence of partitions into five groups of each type for the 6 × 6 *squares*

Set	Directrices	Changeable	Unchangeable
I	0, 1, 2	+	+ (9a, 9b, 9c)
III	0, 1, 2	+	+ (9d, 9e, 9f)
IV	0, 1, 2, 3	+	+ (9g, 9h, 9i, 9j)
IV	4	+ (9k)	−
V	0, 1, 2, 3	+	+ (9l, 9m, 9n, 9o)
V	4	+ (9p)	−
VII	0	+	+ (9q)
VIII	0	+	+ (9r)
X	0, 1, 2, 3	+	+ (9s, 9t, 9u, 9v)
X	4: directrices 7, 15, 18, 29 or 1, 14, 20, 29	+	+ (9w)
X	4: directrices 4, 12, 18, 26 or 1, 16, 30, 11	+ (9x)	−
X	4: directrices 7, 15, 18, 26 or 1, 16, 17, 32 or 1, 14, 20, 26	−	−
XI	0	+	+ (9y)
XIII	0	+	+ (9z)
XIV	0	+	+ (9A)
XV	0, 1, 2, 3	+	+ (9B, 9C, 9D, 9E)
XV	4	+ (9F)	−
XVII	0	+	+ (9G)

In the examples of Set I, (9a) has no directrices, (9b) has directrix 1 from the first group and (9c) has directrices 3 and 7 from the second group.

(9a)

A 0	B 0	C 1	D 2	E 3	F 4
B $\underline{0}$	C 1	F 2	A $\underline{3}$	D 4	E 1
C 3	F 2	B 4	E 0	A 1	D 0
D 4	E 4	A 3	B 1	F 0	C 2
E 2	A 4	D 1	F 3	C 0	B 3
F 1	D 3	E 0	C 4	B 2	A 2

(9b)

A 2	B 1	C $\underline{1}$	D $\underline{0}$	E 3	F 4
B 2	C 2	F 1	A 3	D 0	E 4
C 4	F 3	B 3	E 1	A 0	D 2
D 1	E 0	A 0	B 4	F 2	C 3
E 3	A 4	D 4	F 2	C 1	B 0
F 0	D 3	E 2	C 0	B 4	A 1

(9c)

A 0	B 0	C 3	D 1	E 4	F 2
B $\underline{0}$	C 1	F 1	A 2	D $\underline{4}$	E 3
C 2	F 4	B 1	E 2	A 3	D 0
D 4	E 2	A 4	B 3	F 1	C 0
E 1	A 3	D 2	F 0	C 0	B 4
F 3	D 3	E 0	C 4	B 2	A 1

In the examples of Set III, (9d) has no directrices, (9e) has one directrix and (9f) has two directrices.

(9d)

A 0	B 0	C 1	D 2	E 3	F 4
B 1	A 0	F $\underline{1}$	E 4	C 2	D 3
C 4	F 2	B 3	A 3	D 0	E 1
D 2	C 3	E 0	B 1	F $\underline{0}$	A 4
E 2	D 1	A 2	F 3	B 4	C 0
F 3	E 4	D 4	C 0	A 1	B 2

(9e)

A 1	B 0	C 0	D 2	E 3	F 4
B 0	A 4	F 1	E 2	C 3	D $\underline{0}$
C 2	F 2	B 3	A 4	D 1	E 0
D $\underline{1}$	C 4	E 0	B 1	F 2	A 3
E 4	D 3	A 2	F 0	B 4	C 1
F 3	E 1	D 4	C 3	A 0	B 2

(9f)

A 1	B 2	C 0	D 0	E 3	F 4
B $\underline{0}$	A 0	F 1	E 4	C 3	D 2
C 2	F 1	B 3	A 4	D 1	E 0
D 2	C 4	E 2	B 1	F 0	A 3
E $\underline{4}$	D 3	A 0	F 2	B 4	C 1
F 3	E 1	D 4	C 3	A 2	B 0

In the examples of Set IV, (9g) has no directrices, (9h) has one directrix, (9i) has two directrices, (9j) has three and (9k), which is essentially changeable, has four.

(9g)

A $\underline{1}$	B 2	C 3	D 3	E 4	F $\underline{0}$
B 0	A 3	E 4	F 2	C 2	D 1
C 2	F 4	B 1	A 1	D 0	E 3
D 4	E 2	A 4	B 3	F 1	C 0
E 1	D 0	F 0	C 4	B 3	A 2
F 3	C 1	D 2	E 0	A 0	B 4

(9h)

A 0	B 0	C 1	D 2	E 3	F 4
B 1	A 2	E 3	F 0	C 4	D 1
C 2	F 1	B 4	A 3	D $\underline{0}$	E 0
D 4	E 2	A 0	B 1	F $\underline{2}$	C 3
E 1	D 3	F 2	C 0	B 3	A 4
F 3	C 4	D 0	E 4	A 1	B 2

(9i)

A 3	B 3	C 1	D 0	E 2	F 4
B 3	A 0	E $\underline{1}$	F 2	C 4	D 1
C 0	F 1	B 2	A 4	D 3	E 0
D 2	E 4	A $\underline{0}$	B 1	F 0	C 3
E 1	D 2	F 3	C 0	B 4	A 2
F 4	C 2	D 4	E 3	A 1	B 0

(9j)

A 0	B 0	C 1	D 2	E 3	F 4
B 3	A 4	E 0	F 3	C 2	D 1
C 3	F 1	B 2	A 0	D 0	E 4
D 4	E 2	A 3	B 1	F 2	C 0
E 1	D 3	F 0	C 4	B 4	A 2
F 2	C $\underline{0}$	D 4	E $\underline{1}$	A 1	B 3

(9k)

A 0	B 0	C 1	D 2	E 3	F 4
B 0	A 0	E 4	F 3	C 2	D 1
C 3	F 1	B 2	A 4	D 1	E 0
D 4	E 2	A 3	B 1	F 0	C 2
E 1	D 3	F 3	C 0	B 4	A 2
F 2	C 4	D 0	E 4	A 1	B 3

In the examples of Set V, (9l) has no directrices, (9m) has one directrix, (9n) has two directrices, (9o) has three and (9p), which is essentially changeable, has four.

(9l)

A $\underline{1}$	B 0	C 2	D 3	E 1	F 4
B 2	A 3	E 0	C 1	F 3	D 4
C $\underline{0}$	F 2	B 3	A 1	D 4	E 0
D 2	E 3	F 1	B 4	C 0	A 2
E 4	D 1	A 4	F 0	B 2	C 3
F 3	C 4	D 0	E 2	A 0	B 1

(9m)

A $\underline{2}$	B 2	C 1	D 3	E 4	F 0
B 1	A 3	E 2	C 0	F 4	D 1
C 4	F 1	B 3	A 4	D 2	E 0
D $\underline{0}$	E 0	F 4	B 1	C 3	A 2
E 1	D 4	A 0	F 2	B 0	C 3
F 3	C 2	D 0	E 3	A 1	B 4

(9n)

A 3	B 3	C 1	D 2	E 4	F 0
B 0	A 0	E 4	C 3	F 2	D 1
C 2	F 1	B 4	A 0	D 3	E 0
D 4	E 2	F 3	B 1	C 0	A 4
E 1	D $\underline{2}$	A 2	F 4	B $\underline{0}$	C 3
F 1	C 4	D 0	E 3	A 1	B 2

(9o)

A 0	B 0	C 1	D 2	E 3	F 4
B 4	A 4	E 0	C 3	F 2	D 1
C 2	F 1	B 3	A 1	D 4	E 0
D 0	E 2	F 4	B 1	C 0	A 3
E 1	D 3	A 2	F 0	B 2	C 4
F 3	C $\underline{0}$	D $\underline{3}$	E 4	A 1	B 2

(9p)

A 0	B 0	C 1	D 2	E 3	F 4
B 0	A 0	E 4	C 3	F 2	D 1
C 2	F 1	B 3	A 4	D 1	E 0
D 4	E 2	F 2	B 1	C 0	A 3
E 1	D 3	A 2	F 0	B 4	C 3
F 3	C 4	D 0	E 4	A 1	B 2

Sets VII, VIII, XI, XIII, XIV and XVII have no directrices; examples of the partitions are (9q), (9r), (9y), (9z), (9A) and (9G) respectively.

Set VII (9q)

A 1	B 0	C $\underline{2}$	D 3	E 2	F 4
B 1	C 1	D 3	E 4	F 2	A 0
C 2	E 3	A 4	F 1	B 0	D 0
D 4	F 0	B 3	A 2	C 4	E 1
E 0	D 2	F 1	B 4	A 3	C 3
F 3	A 4	E $\underline{0}$	C 0	D 1	B 2

Set VIII (9r)

A 0	B $\underline{0}$	C 1	D 2	E 3	F 4
B 1	A 0	E 2	F 2	C 4	D 3
C 3	F 1	A 4	E 0	D 1	B 2
D 4	C 2	B 3	A 3	F 0	E 1
E 2	D 3	F 0	C 4	B $\underline{4}$	A 1
F 3	E 4	D 0	B 1	A 2	C 0

Set XI (9y)

A 0	B 0	C 1	D 2	E 3	F 4
B 0	C 1	A 2	F 3	D 4	E 2
C 2	A 4	B 3	E 1	F 0	D 0
D 3	F 2	E 0	B 4	A 1	C 3
E 4	D 1	F 4	C 0	B 2	A 3
F 1	E 3	D $\underline{0}$	A 2	C 4	B 1

Set XIII (9z)

A 0	B $\underline{1}$	C 2	D 3	E 1	F 4
B 0	C 2	A 3	F 0	D 1	E 4
C 1	A $\underline{0}$	B 4	E 3	F 2	D 0
D 2	F 3	E 0	B 1	A 4	C 3
E 3	D 4	F 1	A 2	C 0	B 2
F 4	E 2	D 0	C 4	B 3	A 1

Set XIV (9A)

A 0	B $\underline{1}$	C 2	D 3	E 4	F 1
B 0	C 2	A 3	E 0	F 4	D 1
C 1	A $\underline{0}$	B 4	F 3	D 0	E 2
D 4	F 3	E 1	B 2	A 2	C 0
E 3	D 2	F 0	C 4	B 1	A 4
F 2	E 4	D 0	A 1	C 3	B 3

Set XVII (9G)

A $\underline{0}$	B 0	C 1	D 2	E 3	F 4
B 0	C 1	A 2	F 3	D 1	E 4
C 2	A 4	B 3	E 1	F 0	D 0
D 4	E 2	F 0	A 1	B 2	C 3
E 3	F 2	D 4	C 0	A $\underline{3}$	B 1
F 1	D 3	E 0	B 4	C 4	A 2

In the examples of Set X (9s) has no directrices, (9t) has directrix 4 from Set I, (9u) has directrix 1 from Set II and directrix 26 from Set I, and (9v) has directrices 1, 10 and 31 from Set II; directrices 1, 10 and 31 do not have a fourth mutually parallel directrix. (9w) has directrices 1 and 14 from Set II and directrices 20 and 29 from Set I, and (9x), which is essentially changeable, has directrices 4, 12, 18 and 26 from Set I.

(9s)

A 0	B 1	C 1	D 2	E 3	F 4
B $\underline{3}$	C 0	F 2	A 1	D 4	E 3
C 4	F 3	B $\underline{0}$	E 2	A 0	D 1
D 3	A 2	E 4	B 4	F 1	C 0
E 1	D 4	A 3	F 0	C 2	B 2
F 2	E 0	D 0	C 3	B 1	A 4

(9t)

A 1	B 0	C 0	D 2	E 3	F 4
B $\underline{0}$	C $\underline{1}$	F 1	A 3	D 4	E 2
C 4	F 3	B 2	E 0	A 2	D 1
D 3	A 4	E 4	B 1	F 0	C 2
E 0	D 2	A 3	F 4	C 1	B 3
F 2	E 1	D 0	C 3	B 4	A 0

(9u)

A 1	B 0	C 0	D 3	E 4	F 2
B 2	C 1	F 0	A 0	D 4	E 3
C 4	F 3	B 4	E 1	A 2	D 0
D $\underline{0}$	A 4	E 2	B 3	F 1	C $\underline{1}$
E 0	D 2	A 3	F 4	C 3	B 1
F 3	E 2	D 1	C 2	B 0	A 4

(9v)

A 1	B 2	C 0	D 0	E 4	F 3
B $\underline{0}$	C 1	F $\underline{1}$	A 2	D 3	E 4
C 3	F 0	B 4	E 1	A 0	D 2
D 4	A 4	E 2	B 3	F 1	C 0
E 0	D 2	A 3	F 4	C 2	B 1
F 2	~~E 3~~	~~D 1~~	~~C 4~~	B 0	A 3

(9w)

A 1	B 1	C 2	D 3	E 0	F 4
B 0	C 1	F 0	A 4	D 2	E 3
C 4	F 2	B 3	E 1	A 2	D 0
D 3	A 3	E 4	B 2	F 1	C 0
E 2	D 4	A 0	F $\underline{4}$	C 3	B 1
F 3	E 0	D 1	C $\underline{0}$	B 4	A 2

(9x)

A 1	B 2	C 0	D 3	E 0	F 4
B 4	C 3	F 1	A 1	D 2	E 0
C 0	F 2	B 3	E 2	A 4	D 1
D 3	A 0	E 4	B 1	F 3	C 2
E 3	D 4	A 2	F 0	C 1	B 4
F 2	E 1	D 0	C 4	B 0	A 3

In the examples of Set XV, (9B) has no directrices, (9C) has one directrix, (9D) has two directrices, (9E) has three and (9F), which is essentially changeable, has four.

(9B)

A 1	B 2	C 3	D 0	E 4	F 1
B 0	A 0	F 2	E 3	D 1	C 4
C 3	D 2	A 0	B 1	F 4	E 0
D 4	F 3	E 1	A 4	C 0	B 2
E 2	C 1	B 4	F 0	A 2	D 3
F $\underline{0}$	E 4	D $\underline{1}$	C 2	B 3	A 3

(9C)

A 1	B 0	C $\underline{2}$	D 3	E 4	F 2
B 0	A 3	F 1	E 1	D 4	C 2
C 3	D 2	A 4	B 1	F 3	E 0
D 0	F 0	E 3	A 2	C 1	B 4
E 2	C 4	B 3	F 4	A 0	D 1
F 4	E 1	D $\underline{0}$	C 0	B 2	A 3

(9D)

A 1	*B* 2	*C* 0	*D* $\underline{2}$	*E* 3	*F* 4
B 0	*A* 4	*F* 1	*E* 3	*D* 2	*C* 1
C 2	*D* 0	*A* 3	*B* 1	*F* 4	*E* 0
D 4	*F* 0	*E* 2	*A* $\underline{0}$	*C* 1	*B* 3
E 4	*C* 3	*B* 4	*F* 2	*A* 0	*D* 1
F 3	*E* 1	*D* 3	*C* 4	*B* 0	*A* 2

(9E)

A 1	*B* 2	*C* 0	*D* 0	*E* 4	*F* 3
B 0	*A* 0	*F* 1	*E* 3	*D* 4	*C* 2
C 4	*D* 1	*A* 3	*B* 1	*F* 2	*E* 0
D 3	*F* 4	*E* $\underline{0}$	*A* 2	*C* 1	*B* $\underline{2}$
E 2	*C* 3	*B* 4	*F* 3	*A* 0	*D* 1
F 0	*E* 1	*D* 2	*C* 4	*B* 3	*A* 4

(9F)

A 1	*B* 2	*C* 0	*D* 0	*E* 3	*F* 4
B 1	*A* 0	*F* 1	*E* 4	*D* 2	*C* 3
C 2	*D* 3	*A* 4	*B* 1	*F* 0	*E* 2
D 4	*F* 3	*E* 2	*A* 3	*C* 1	*B* 0
E 0	*C* 4	*B* 3	*F* 2	*A* 4	*D* 1
F 3	*E* 1	*D* 0	*C* 0	*B* 4	*A* 2

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