ON THE PITMAN EFFICIENCY OF ONE-SIDED KOLMOGOROV AND SMIRNOV TESTS FOR NORMAL ALTERNATIVES[1]

By P. V. Ramachandramurty

University of California, Berkeley

1. Summary. Lower bounds for the Pitman efficiency of the one-sided Kolmogorov test for all normal alternatives and for the Smirnov test for normal shift alternatives are obtained. In the one-sample case this does not fall below 36 per cent at the usual levels of significance and power. In the two-sample case the lower bound depends on the ratio of the two sample sizes and does not fall below 36 per cent when the ratio is larger than 40. We assume that the results of [1] are known.

2. Definitions and notation. We take the following definition of Pitman efficiency given in [4]. Let $\{t_n\}$ and $\{t_n^*\}$ be two sequences of tests all of the same size α to test the hypothesis $\theta = \theta_0$. Let θ_i be a sequence of parametric alternatives which converge to θ_0. Let $\{n_i\}$ and $\{m_i\}$ be two sequences such that

$$\lim_{i\to\infty} P_i(\theta_i) = \lim_{i\to\infty} P_i^*(\theta_i) = \beta, \qquad 0 < \beta < 1, \tag{2.1}$$

where P_i is the power of $\{t_{n_i}\}$ at θ_i and P_i^* is the power of $\{t_{m_i}^*\}$ at θ_i. Then the relative efficiency of $\{t_n\}$ with respect to $\{t_n^*\}$ is defined to be $e[\{t_n\}, \{t_n^*\}] = \lim_{i\to\infty} m_i/n_i$ if this limit exists and is the same for all m_i, n_i satisfying (2.1). In the present paper n_i are replaced by larger values and the resulting lower bound for efficiency depends both on α and β. $\Phi(x)$ and $\varphi(x)$ stand for the c.d.f. and density of the standard normal distribution, respectively. K_α is the root of the equation in x, $\Phi(x) = 1 - \alpha$. All the distributions that figure in are assumed to be continuous.

3. Lower bound for efficiency in the one-sample case. We shall first prove a lemma useful for the main result. We use the familiar Slutsky-type arguments (see, for example, [2] p. 254) and only outline the proof.

Lemma 3.1. *Suppose* $\{X_n\}$ *and* $\{Y_n\}$ *are two sequences of random variables with continuous* c.d.f.'s *and* c_n *a sequence of constants with the following properties*

(i) $Y_n \to_P Y$ *with a continuous distribution* F

(ii) $\exists$ *a sequence of constants* $a_n \ni P\{|X_n - a_n| > \epsilon\} \to 0\ \forall \epsilon > 0$

(iii) $\lim_{n\to\infty} c_n = c$

(iv) $\lim_{n\to\infty} P\{Y > c - a_n\} = \beta$.

Then $\lim_{n\to\infty} P\{Y_n > c_n - X_n\}$ *exists and equals* β.

Outline of Proof. Let F_n be the c.d.f. of Y_n. Since F is uniformly continuous

Received 17 September 1963; revised 1 February 1966.

[1] This paper was prepared with the partial support of the Office of Naval Research, Contract NONR-222-43, and the partial support of Case Research Fund. This paper in whole or in part may be reproduced for any purpose of the United States Government.

on $(-\infty, \infty)$ given $\delta > 0 \ \exists \epsilon(\delta) \ni$

$$|F(c - a_n \pm \epsilon) - F(c - a_n)| < \delta/4 \text{ for all } a_n . \tag{3.1}$$

Now,

$$\begin{aligned} P\{Y_n > c_n - X_n\} &= P[\{Y_n > c_n - X_n\} \cap \{|X_n - a_n| > \epsilon\}] \\ &\quad + P[\{Y_n > (c_n - a_n) - (X_n - a_n)\} \cap \{|X_n - a_n| \leqq \epsilon\} \end{aligned} \tag{3.2}$$

Now choose $N_1\{\delta, \epsilon\}$ such that

$$n > N_1(\delta, \epsilon) \Rightarrow P\{|X_n - a_n| > \epsilon\} < \delta/4.$$

An examination of the right hand side of (3.2) shows that

$$\begin{aligned} n > N_1(\delta, \epsilon) \Rightarrow 1 - F_n(c_n - a_n + \epsilon) - \delta/4 &\leqq P\{Y_n > c_n - X_n\} \\ &\leqq 1 - F_n(c_n - a_n - \epsilon) + \delta/4. \end{aligned} \tag{3.3}$$

Since $F_n(x) \rightarrow F(x)$ uniformly in x, which can easily be seen from the hypothesis (i) $\exists N_2(\delta, \epsilon) \ni$

$$n > N_2(\delta, \epsilon) \Rightarrow |F_n(c_n - a_n \pm \epsilon) - F(c_n - a_n \pm \epsilon)| < \delta/4. \tag{3.4}$$

Now, making use of the fact that F is continuous and $c_n \rightarrow c$ and using (3.4) in (3.3) we get $|1 - F(c - a_n) - P\{Y_n > c_n - X_n\}| < \delta$ for sufficiently large values of n. Using now the hypothesis (iv) the conclusion follows.

Now let $F_n(x)$ be the empirical c.d.f. based on $X_1, X_2, \cdots, X_n$, a random sample from an unknown distribution F. We would like to test the hypothesis $H_0 : F(x) = \Phi(x)$ against $H_1 : F(x) = \Phi((x - \theta)/\sigma)$, $\theta > 0$, $\sigma > 0$. Kolmogorov's one-sided test is given by: reject $\Phi(x)$ if and only if

$$n^{\frac{1}{2}} \sup_x \{\Phi(x) - F_n(x)\} > \epsilon_{n,\alpha} \tag{3.5}$$

with $\lim_{n\rightarrow\infty} \epsilon_{n,\alpha} = (\frac{1}{2} \log \alpha^{-1})^{\frac{1}{2}}$. The t-test based on a sample of size m is given by: reject $\Phi(x)$ if and only if

$$m^{\frac{1}{2}}\bar{X}/s_m > t_{(m-1),\alpha} \tag{3.6}$$

with $s_m^2 = (1/(m-1)) \sum_{j=1}^{m} (X_j - \bar{X})^2$ and $S_{m-1}\{t_{(m-1),\alpha}\} = 1 - \alpha$ where $S_{m-1}(x)$ is the c.d.f. of the Student's t-distribution with $(m - 1)$ d.f.

THEOREM 3.1. *A lower bound for the Pitman efficiency of* (3.5) *compared to* (3.6) *is given by*

$$(2/\pi)[(K_\alpha + K_{1-\beta})/\{K_{1-\beta} + (2 \log \alpha^{-1})^{\frac{1}{2}}\}]^2 \tag{3.7}$$

PROOF. Let θ_i and σ_i be two sequences such that $\lim_{i\rightarrow\infty} \theta_i = 0$ and $\lim_{i\rightarrow\infty} \sigma_i = 1$. Let $\delta_i = \sup_x \{\Phi(x) - \Phi((x - \theta_i)/\sigma_i)\} = \Phi(a_i) - \Phi((a_i - \theta_i)/\sigma_i)$ where $0 < a_i < \theta_i$ and let $U_{0i} = \Phi(a_i) - \delta_i = \Phi((a_i - \theta_i)/\sigma_i)$. Let n_i be a sequence of sample sizes such that

$$\lim_{i\rightarrow\infty} \Phi[\{U_{0i}(1 - U_{0i})\}^{-\frac{1}{2}}\{n_i^{\frac{1}{2}}\delta_i - (\tfrac{1}{2} \log \alpha^{-1})^{\frac{1}{2}} - n_i^{-\frac{1}{2}}\}] = \beta \tag{3.8}$$

Now, $U_{0i}(1 - U_{0i}) = \Phi((a_i - \theta_i)/\sigma_i)\Phi((\theta_i - a_i)/\sigma_i) \rightarrow \frac{1}{4}$ as $i \rightarrow \infty$ and from the mean value theorem $n_i^{\frac{1}{2}}\delta_i = n_i^{\frac{1}{2}}\{a_i - ((a_i - \theta_i)/\sigma_i)\}\varphi(h_i)$ where $h_i \rightarrow 0$ as $i \rightarrow \infty$. Thus,

$$n_i^{\frac{1}{2}}\delta_i = n_i^{\frac{1}{2}}\{(a_i(\sigma_i - 1) + \theta_i)/\sigma_i\}\varphi(h_i) = n_i^{\frac{1}{2}}\theta_i\{(a_i/\theta_i)((\sigma_i - 1)/\sigma_i) + \sigma_i^{-1}\}\varphi(h_i)$$

since $\{(a_i/\theta_i)((\sigma_i - 1)/\sigma_i) + \sigma_i^{-1}\}\varphi(h_i) \rightarrow \varphi(0) = (2\pi)^{-\frac{1}{2}}$ as $i \rightarrow \infty$ it follows from (3.8) that $\lim_{i\rightarrow\infty} n_i^{\frac{1}{2}}\theta_i$ exists and is given by

$$\lim_{i\rightarrow\infty} n_i^{\frac{1}{2}}\theta_i = (\pi/2)^{\frac{1}{2}}\{K_{1-\beta} + (2 \log \alpha^{-1})^{\frac{1}{2}}\}. \tag{3.9}$$

It follows that $n_i \rightarrow \infty$ and hence from (4.2) of [1] the left-hand side of (3.8) is a lower bound for the power of θ_i for sufficiently large i. The power of the test (3.6) based on a sample size m_i at the alternative (θ_i, σ_i) is given by

$$P\{Y_i > t_{(m_1-1),\alpha} - m_i^{\frac{1}{2}}\theta_i/s_i\} \tag{3.10}$$

where $s_i = s_{m_i}$ and $Y_i = m_i^{\frac{1}{2}}(\bar{X} - \theta_i)/s_i$ is a Student's t with $(m_i - 1)$ degrees of freedom.

Now, we choose the sequence m_i in such a way that

$$\lim_{i\rightarrow\infty} [1 - \Phi\{K_\alpha - m_i^{\frac{1}{2}}\theta_i/\sigma_i\}] = \beta. \tag{3.11}$$

Since Y_i converges in law to the standard normal distribution and $P\{|m_i^{\frac{1}{2}}\theta_i/s_i - m_i^{\frac{1}{2}}\theta_i/\sigma_i| > \epsilon\} \rightarrow 0$, it follows from Lemma 3.1 that $\lim_{i\rightarrow\infty} P\{Y_i > t_{(m_i-1),\alpha} - m_i^{\frac{1}{2}}\theta_i/s_i\}$ exists and is equal to β, i.e. the choice of m_i is such that the power of (3.6) tends to β.

Now, from (3.11) it follows

$$\lim_{i\rightarrow\infty} m_i^{\frac{1}{2}}\theta_i = K_\alpha + K_{1-\beta}. \tag{3.12}$$

The result now follows from (3.9) and (3.12). The following gives numerical lower bounds for some values of α and β.

α	β		
	.90	.95	.99
.01	.45	.46	.48
.05	.39	.41	.44
.10	.36	.38	.42

It can be seen that (3.3) decreases with α and increases with β so that for α between .01 and .10 and β between .90 and .99 the efficiency is larger than .36. By making use of the approximation for the "tail" of the normal distribution (see [3], p. 166) it is easy to see that $K_\alpha \sim (-2 \log \alpha)^{\frac{1}{2}} = (2 \log \alpha^{-1})^{\frac{1}{2}}$ as $\alpha \rightarrow 0$ and we see that the limiting lower bound for efficiency is $2/\pi$ as $\alpha \rightarrow 0$. It is interesting to note that this is independent of β.

4. Bounds for the power in the two-sample case. Before deriving a lower bound for Pitman efficiency in the two-sample case we shall extend the results of [1] to two samples. Let $X_1, X_2, \cdots, X_m$ be a sample from an unknown distribution F and $Y_1, Y_2, \cdots Y_n$ from an unknown distribution G. Let H_0 be $F = G$ and $H_1 : F = F^{(1)}, G = G^{(1)}$ with $F^{(1)} > G^{(1)}$. Let $F_m(z)$ and $G_n(z)$ be the empirical c.d.f.'s of X's and Y's respectively. Then the one-sided test (see [5]) rejects H_0 if and only if

$$\text{(4.1)} \qquad \{mn/m + n\}^{\frac{1}{2}} \sup_z \{F_m(z) - G_n(z)\} > \epsilon_{m,n,\alpha}$$

with $\lim_{m,n\to\infty} \epsilon_{m,n,\alpha} = (\frac{1}{2} \log \alpha^{-1})^{\frac{1}{2}}$ where $m, n \to \infty$ in such a way that $m/n = \tau > 0$.

LEMMA 4.1. *A lower bound for the asymptotic power of* (4.1) *is given by*

$$(1 - \alpha)\Phi[\{U_0(1 - U_0)\}^{-\frac{1}{2}}\{n^{\frac{1}{2}}\delta - [((1 + (\tau + 1)^{\frac{1}{2}}(\tfrac{1}{2} \log \alpha^{-1})^{\frac{1}{2}})/\tau^{\frac{1}{2}}] - n^{-\frac{1}{2}}\}]$$

where $\delta = \sup_z \{F^{(1)}(z) - G^{(1)}(z)\} = F^{(1)}(z_0) - G^{(1)}(z_0)$ *say, and* $U_0 = G^{(1)}(z_0) = F^{(1)}(z_0) - \delta$.

PROOF. $F_m(z) - G_n(z) = \{F^{(1)}(z) - G_n(z)\} - \{F^{(1)}(z) - F_m(z)\}$ so that $\sup_z \{F_m(z) - G_n(z)\} \geqq \sup_z \{F^{(1)}(z) - G_n(z)\} - \sup_z \{F^{(1)}(z) - F_m(z)\}$ and since $\{X_i\}$ and $\{Y_i\}$ are independent,

$$P_{F^{(1)},G^{(1)}}[\{mn/(m + n)\}^{\frac{1}{2}} \sup_z \{F_m(z) - G_n(z)\} > \epsilon_{m,n,\alpha}]$$
$$\geqq P_{G^{(1)}}[n^{\frac{1}{2}} \sup_z \{F^{(1)}(z) - G_n(z)\} > (1 + \gamma)\{(m + n)/m\}^{\frac{1}{2}}\epsilon_{m,n,\alpha}]$$
$$\cdot P_{F^{(1)}}[m^{\frac{1}{2}} \sup_z \{F^{(1)}(z) - F_m(z)\} < \gamma\{(m + n)/n\}^{\frac{1}{2}}\epsilon_{m,n,\alpha}]$$

for any $0 < \gamma < 1$. By choosing $\gamma = \{n/(m + n)\}^{\frac{1}{2}}$ independent of m and n and proceeding to the limit we get asymptotic power

$$\geqq (1 - \alpha) \lim_{n\to\infty} P_{G^{(1)}}[n^{\frac{1}{2}} \sup_z \{F^{(1)}(z) - G_n(z)\}$$
$$> \{1 + (\tau + 1)^{\frac{1}{2}}\}(\tfrac{1}{2} \log \alpha^{-1})^{\frac{1}{2}}/\tau^{\frac{1}{2}}]$$

A lower bound for the last expression can be obtained from (4.1) of [1] and this completes the proof of the lemma.

Now, let the null hypothesis and the alternative be given by

$$\text{(4.2)} \qquad H_0 : F = G = \Phi(x/\sigma); \qquad H_1 : F = \Phi(x/\sigma), \qquad G = \Phi(x - \theta)/\sigma)$$

with $\theta > 0$ and $\sigma > 0$ unknown. The "best" parametric test is given by:

$$\text{(4.3)} \qquad (m^{-1} + n^{-1})^{-\frac{1}{2}}(\bar{Y} - \bar{X})/\omega_{m,n} > t_{m+n-2,\alpha}$$

where $\omega_{m,n}^2 = (m + n - 2)^{-1}\{\sum_{i=1}^m (X_i - \bar{X})^2 + \sum_{j=1}^n (Y_j - \bar{Y})^2\}$ and $t_{(m+n-2),\alpha}$ is defined as in (3.6).

THEOREM 4.1. *A lower bound for the Pitman efficiency of* (4.1) *compared to* (4.3) *for normal shift alternatives* (4.2) *is given by*

$$(2/\pi\tau)(\tau + 1)(K_\alpha + K_{1-\beta})^2\{K_{1-(\beta/(1-\alpha))} + [(1 + (1 + \tau)^{\frac{1}{2}})/\tau^{\frac{1}{2}}](2 \log \alpha^{-1})^{\frac{1}{2}}\}^{-2}$$

The proof is analogous to that of Theorem 3.1 and is therefore omitted. We now use Lemma 4.1 for the lower bound for the asymptotic power of the Smirnov test.

When α is small and τ is large we notice that this lower bound is approximately the same as in the one-sample case. The following are some numerical values for the lower bound.

α, β	τ				
	1	5	10	20	40
$\alpha = .01$ $\beta = .90$	.22	.27	.30	.33	.36
$\alpha = .01$ $\beta = .95$	.24	.29	.32	.34	.36

5. Acknowledgments. I am grateful to Professor E. L. Lehmann, University of California, Berkeley for suggesting that I investigate the Pitman efficiency of Kolmogorow-type tests and to the referee whose comments and suggestions led to the improvement of both the content and the presentation.

REFERENCES

[1] Birnbaum, Z. W. (1953). On the power of a one-sided test of fit for continuous probability functions. *Ann. Math. Statist.* **24** 484–489.

[2] Cramér, Harald (1951). *Mathematical Methods of Ststistics.* Princeton University Press, Princeton.

[3] Feller, William (1950). *An Introduction to Probability Theory and its Applications.* Vol. 1, 2nd ed. Wiley, New York.

[4] Fraser, D. A. S. (1957). *Nonparametric Methods in Statistics.* Wiley, New York.

[5] Smirnov, N. (1939). On the estimation of the discrepancy between empirical curves of distribution for two independent samples. *Bull. Math.*, Univ. Moscow, Serie Internationale. **2** 3–16.