FUNCTIONS OF FINITE MARKOV CHAINS[1]

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1. Statement of the Problem. This paper is a presentation of some results in the study of random processes which arise as functions of finite, continuous parameter Markov chains. For reference purposes throughout this paper the definition of such processes is contained in:

HYPOTHESIS H: The processes $X(t)$ and $Y(t)$ will be said to satisfy hypothesis H whenever $X(t)$ is a basic Markov chain (Definition 2.1) with state space $\mathfrak{N} = \{1, 2, \cdots, N\}$, there is a function f mapping $\mathfrak{N}$ onto $\mathfrak{M} = \{1, 2, \cdots, M\}$, where $M \leqq N$, and the process $Y(t)$ is equal in joint distribution to the process $f[X(t)]$.

The process $Y(t)$ is termed a function of a finite Markov chain. The process $Y(t)$ need not be Markov and in fact the question motivating this research is that of finding necessary and sufficient conditions for $Y(t)$ to be Markov. Such conditions are given in Theorem 2.5. The conditions given are in terms of exponential type processes (Definition 2.2) which arise as functions of basic Markov chains (Theorem 2.3), and their order (Definition 2.4).

Aspects of this problem have been considered previously by C. J. Burke and M. Rosenblatt [1], M. Rosenblatt [11], J. Hachigian and M. Rosenblatt [8] and J. Hachigian [7]. Burke and Rosenblatt [1] gave necessary and sufficient conditions for $Y(t)$ to be Markov when $X(t)$ was a discrete parameter reversible Markov chain. They also gave necessary and sufficient conditions for $Y(t)$ to be Markov whatever the initial probabilities in the case when $X(t)$ was a continuous time parameter Markov chain. J. Hachigian and M. Rosenblatt [8] extended the results of [1] to reversible, continuous time parameter Markov processes with arbitrary state space. Related questions concerning functions of discrete time finite Markov chains have been considered by Gilbert [6], Dharmadhikari [3], [4], [5] and most recently by Heller [9] who completed the problem of characterizing processes which arise as functions of Markov chains.

This paper is organized in four main sections. Section 2 is introductory and contains a statement of the main theorem. The theorem of Section 3 has interest of its own, in that it identifies regeneration states (Definition 3.1) for certain exponential type processes. The results in Section 4 are special ones needed for the proof of Theorem 2.5 which makes up Section 5.

2. Basic Markov chains and exponential type processes. Let $X(t)$ be a standard [2, p. 123] Markov chain with a stationary transition matrix $P(t) = (P_{ij}(t))$, and initial column vector $\mathbf{P} = (P_i)$ whose state space is

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$\mathfrak{N} = \{1, 2, \cdots, N\}$. One may express $P(t)$ as $P(t) = \exp(\Lambda t)$ where $\Lambda = (\lambda_{ij})$ is a matrix whose elements satisfy the conditions $\lambda_{ij} \geqq 0$ for $i \neq j$, $\sum_{j=1}^{N} \lambda_{ij} = 0$ for each i, $1 \leqq i \leqq N$.

DEFINITION 2.1. A finite standard Markov chain will be called a basic Markov chain whenever the eigenvalues of Λ, $\nu_1, \nu_2, \cdots, \nu_N$, are real and distinct and the initial probabilities of $X(t)$, P_i, are non-zero.

For a basic Markov chain since Λ has distinct eigenvalues there exists a non-singular matrix $C = (c_{ij})$ such that $C^{-1}\Lambda C = D = (\nu_i \delta_{ij})$ where $1 \leqq i, j \leqq N$, and δ_{ij} is the Kronecker delta. Thus one can write:

$$P(t) = Ce^{Dt}C^{-1}. \tag{2.1}$$

For an arbitrary random process $Z(t)$ with parameter set $[0, \infty)$ if $z = \{z_1, z_2, \cdots, z_n\}$ is a sequence of n states for $Z(t)$ and $s = \{s_1, s_1 + s_2, \cdots, s_1 + s_2 + \cdots + s_n\}$ is a corresponding sequence of n parameters, (z, s) is termed a sequence pair of length n for $Z(t)$. Define the joint probability function for $Z(t)$ by

$$p_Z(z, s) = \text{Prob}\,[Z(\tau_i) = z_i, i = 1, 2, \cdots, n],$$

where $\tau_i = \sum_{k=1}^{i} s_k$.

Suppose $X(t)$ and $Y(t)$ satisfy hypothesis H. Let (y, s) be a sequence pair of length n for the process $Y(t)$, where $y = \{y_1, y_2, \cdots, y_n\}$ and s is as above. Then $p_Y(y, s) = \sum p_X(x, s)$ where the sum is extended over all sequences of states of $X(t)$, $x = (x_1, \cdots, x_n)$ for which $f[x_i] = y_i$. In terms of $N \times N$ matrices and N-vectors

$$p_Y(y, s) = \mathbf{P}'P(s_1)A(y_1) \cdots P(s_{n-1})A(y_{n-1})P(s_n)\mathbf{A}(y_n) \tag{2.2}$$

where $\mathbf{P}'$ is the transpose of $\mathbf{P}$. Define the matrices $A(m)$ for each m in $\mathfrak{M}$ by $A(m) = (a_{ij}(m)) = (a_i(m)\delta_{ij})$ where $a_i(m) = 1$ if $f(i) = m$ and 0 otherwise. Define $\mathbf{A}(m) = (a_i(m))$.

Whenever $m \varepsilon \mathfrak{M}$ define the matrix $B(m) = (b_{ij}(m)) = C^{-1}A(m)C$ where C is as in (2.1). The first column of C is taken as the eigenvector corresponding to $\nu_1 = 0$ and is normalized to be a column of ones. The remaining columns of C can be normalized to make the vector $\mathbf{B}' = \mathbf{P}'C$ an N-vector of terms, the first of which is one and the remaining of which are either zero or one. Finally note $C^{-1}\mathbf{A}(y_n)$ is the first column of the matrix $B(y_n)$, so this vector will be written as $\mathbf{B}(y_n)$. With these conventions and using (2.1) and (2.2), $p_Y(y, s)$ takes the form

$$p_Y(y, s) = \mathbf{B}'e^{Ds_1}B(y_1)e^{Ds_2} \cdots e^{Ds_n}\mathbf{B}(y_n). \tag{2.3}$$

The discussion above motivates the definition of an exponential type chain.

DEFINITION 2.2. A finite random chain $Y(t)$ with state space $\mathfrak{R} = \{1, 2, \cdots R,\}$ is an exponential type process if for some finite positive integer K there exist K distinct non-positive real numbers $\nu_1 = 0, \nu_2, \cdots, \nu_K$ and a set of $K \times K$ matrices $B(r)$, $1 \leqq r \leqq R$ such that for any sequence pair (y, s) for $Y(t)$, $p_Y(y, s)$ can be expressed as

$$(2.4) \qquad p_Y(y, s) = \mathbf{B}'e^{Ds_1}B(y_1) \cdots e^{Ds_n}\mathbf{B}(y_n)$$

where the K-vector $\mathbf{B} = (b_i)$ is of the form $b_1 = 1, b_j = 0$ or $1, 2 \leqq j \leqq K$, the K-vector $\mathbf{B}(y_i)$ is the first column of $B(y_i)$, and $D = (\delta_{ij}\nu_j)1 \leqq i, j \leqq K$.

The discussion thus far forms the proof of Theorem 2.3.

THEOREM 2.3. *Let $X(t)$ and $Y(t)$ satisfy hypothesis H. Then $Y(t)$ is an exponential type chain.*

Clearly, a basic Markov chain is an exponential type process. Note that (2.3) can be written in terms of the elements of the vectors and matrices as:

$$(2.5) \qquad p_Y(y, s) = \sum_{\alpha_1=1}^{K} \cdots \sum_{\alpha_n=1}^{K} [b_{\alpha_1} \prod_{m=1}^{n} b_{\alpha_m,\alpha_{m+1}}(y_m)]\Gamma(\alpha, s)$$

where

$$(2.6) \qquad \Gamma(\alpha, s) = [\exp(\sum_{m=1}^{n} \nu_{\alpha_m} s_m)], \quad \text{and} \quad \alpha_{n+1} \equiv 1.$$

DEFINITION 2.4. The K of Definition 2.2 will be termed the order of $Y(t)$ if there exists no $K' < K$ for which the representation (2.4) is possible for all sequence pairs of finite length for $Y(t)$.

The main theorem can now be stated.

THEOREM 2.5. *Let $X(t)$ and $Y(t)$ satisfy hypothesis H and let $Y(t)$ have order K. Then $Y(t)$ is Markov* if *and only* if *$K = M$.*

3. Regeneration states.

DEFINITION 3.1. Let $Z(t)$ be a random chain. If there exists a state m for the process $Z(t)$ with the property that

$$P[Z(t) = k \mid Z(t_0) = m] = P[Z(t) = k \mid Z(t_0) = m, z(\tau) \quad \text{for all} \quad \tau < t_0]$$

for all states k of the state space for $Z(t)$, the state m is termed a regeneration state for the process $Z(t)$.

This definition is related to that of Smith [12, p. 13] for a regeneration point. The emphasis here is on the states rather than the time epochs at which a process exhibits a regenerative behavior. A set of $K \times K$ idempotent matrices $H(r), 1 \leqq r \leqq R$, each of which has only zero and unit (equal to one) eigenvalues and for which $\sum_{r=1}^{R} H(r) = I$ where I is the $K \times K$ identity matrix is termed a set of factor matrices. If $K \geqq R$, there exists a non-singular matrix C such that for each r, $1 \leqq r \leqq R$, $CH(r)C^{-1}$ is a diagonal matrix. This is a consequence of the definition for factor matrices and Theorem 7 of [10], p. 134.

THEOREM 3.2. *Let $Y(t)$ be an exponential type process with order K. Let the set of matrices $\{B(r)\}$ of Definition 2.2 be a set of factor matrices. Let k be a state for $Y(t)$ such that $B(k)$ has exactly one unit eigenvalue. Then k is a regeneration state for $Y(t)$.*

PROOF. Fix $\tau > 0$. Let (y, s) be any sequence pair of length n for $Y(t)$ with $y_n = k$. Define $B_{(y,s)} = B = \{Y(s_1 + \cdots + s_i) = y_i \text{ for } 1 \leqq i \leqq n\}$, and for each state ξ in $\mathfrak{M}$ and each time $t \geqq \sum_{i=1}^{n} s_i = \tau$ define $A_{(\xi,t)} = A = \{Y(t) = \xi\}$. The conditional probability $P(A \mid B)$ must satisfy the equation $P(AB) =$

$P(A \mid B)P(B)$ whenever $P(AB) \neq 0$. It must also be shown that $P(A \mid B)$ is a function only of $Y(\tau)$, and $Y(t)$. Note that $P(AB) = p_Y(y\xi, st) = \mathbf{R}'C^{-1}A(k)CH(\xi, t - \tau)$, where $\mathbf{R}' = (r_i)'$ is the row vector of elements preceding $B(k)$ in the representation (2.4), $B(k) = C^{-1}A(k)C$, and $\mathbf{H}(\xi, t - \tau) = (h_i(\xi, t - \tau)) = e^{D(t-\tau)}\mathbf{B}(\xi)$. The matrix $\mathrm{A}(k)$ is diagonal with exactly one unit term, say $a_{jj}(k) = 1$. Thus

$$p_Y(y\xi, st) = \sum_{i=1}^{K} r_i c^{ij} \cdot \eta = P(B)\eta, \quad \text{where} \quad \eta = \sum_{m=1}^{K} c_{jm} h_m(\xi, t - \tau)$$

η satisfies the equation $P(AB) = \eta P(B)$, and $\eta = P(A \mid B)$. Note finally that η depends only on $Y(\tau)$ and $Y(t)$ as was to be shown.

4. A new representation for $p_Y(y, s)$. Throughout this section let $X(t)$ and $Y(t)$ satisfy hypothesis H. Clearly $K \leqq N$ since the ν_i of Definition 2.2 are the eigenvalues of the matrix Λ associated with $X(t)$. It is of interest to examine the case $K < N$. Define the set of integers $\mathcal{K} = \{i \mid$ for some sequence pair (y, s), ν_i appears in a term $\Gamma(\alpha, s)$ of (2.5) having a non-zero coefficient$\}$.

LEMMA 4.1. *Let $X(t)$ and $Y(t)$ satisfy hypothesis H. Then* $1 \in \mathcal{K}$.

PROOF. It must be shown that for some sequence pair (y, s), $p_Y(y, s)$ has a constant term in its representation (2.5). The set of joint probability functions corresponding to the set of all sequence pairs of length 1 must form a distribution. Should none of these functions have a constant term, their sum would tend to zero as the parameter s increases.

If u and v are elements of $\mathfrak{N}$ then u is said to be linked to v if there exist finite sequences $\{\xi_0, \xi_1, \cdots, \xi_{n-1}\}$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ of elements of $\mathfrak{M}$ and $\mathfrak{N}$ respectively such that

$$b_{u,\alpha_1}(\xi_0) b_{\alpha_1,\alpha_2}(\xi_1) \cdots b_{\alpha_{n-1},v}(\xi_{n-1}) \tag{4.1}$$

is non-zero where $b_{ij}(m)$ is defined in (2.5). If u is linked to v, then v is said to be linked from u. Any non-zero term of the form (4.1) is a linking term.

It is immediate that the sets $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ form a partition of $\mathfrak{N} - \mathcal{K}$, the set theoretic complement of $\mathcal{K}$ relative to $\mathfrak{N}$ where by definition

$$\mathcal{U} = \{v \mid v \in \mathfrak{N} - \mathcal{K} \text{ and } v \text{ is linked from some but not to any element in } \mathcal{K}\}$$

$$\mathcal{V} = \{v \mid v \in \mathfrak{N} - \mathcal{K} \text{ and } v \text{ is linked to some but not from any element in } \mathcal{K}\}$$

$$\mathcal{W} = \{v \mid v \in \mathfrak{N} - \mathcal{K} \text{ and } v \text{ is neither linked to nor from any element in } \mathcal{K}\}.$$

Without loss of generality the eigenvalues, ν_i of the matrix Λ can be indexed so that $\nu_1 = 0$ and $\mathcal{K} = \{i \mid 1 \leqq i \leqq K\}$, $\mathcal{U} = \{i \mid K + 1 \leqq i \leqq K + U\}$, $\mathcal{V} = \{i \mid K + U + 1 \leqq i \leqq K + U + V\}$, and $\mathcal{W} = \{i \mid K + U + V + 1 \leqq i \leqq N\}$. Henceforth the diagonal matrix $D = (\delta_{ij}\nu_j)$ and the non-singular matrix C of (2.1) will be taken to conform to the above indexing. That is, the ith column of C will be the eigenvector corresponding to the eigenvalue ν_i. Within the individual partition sets however, no further order is specified for the indices. Using this indexing convention and the definitions of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ one has that for each element k of $\mathfrak{M}$ the matrix $B(k)$ of (2.3) can be written in the block form

$$B(k) = \begin{bmatrix} B^*(k) & * & 0 & 0 \\ 0 & B_U(k) & 0 & 0 \\ * & * & B_V(k) & * \\ 0 & * & 0 & B_W(k) \end{bmatrix} \tag{4.2}$$

where $B^*(k)$ is $K \times K$, $B_U(k)$ is $U \times U$, $B_V(k)$ is $V \times V$, $B_W(k)$ is $W \times W$ and the blocks indicated by * may contain non-zero elements.

Observe that the product of two matrices each of which has zero submatrices where the matrix (4.2) does will again have zero submatrices in the same positions. From (4.2) the vector $\mathbf{B}(k) = (b_i(k))$ has zero elements for $K + 1 \leqq i \leqq K + U$ and $K + U + 1 \leqq i \leqq N$ for each k of $\mathfrak{M}$.

Note finally that for $v \in \mathfrak{V}$, $b_v = 0$. By definition of $\mathfrak{V}$ there exists a linking term $L_1(v, k)$ linking v to some element k of $\mathfrak{K}$. Since in (2.6) $\alpha_{n+1} = 1$, every element of $\mathfrak{K}$ is linked to 1. Thus there exists a linking term $L_2(k, 1)$ linking k to 1. If b_v were non-zero, $b_v L_1(v, k) L_2(k, 1)$ would be a non-zero coefficient for a term of the form in (2.6) contradicting the assumption that $v \notin \mathfrak{K}$. These observations when applied to (2.3) yield Theorem 4.2.

THEOREM 4.2. *Let $X(t)$ and $Y(t)$ satisfy hypothesis H and let $Y(t)$ have order K. Then for any sequence pair (y, s) of length n,*

$$p_Y(y, s) = \mathbf{B}^{*\prime} e^{D^* s_1} B^*(y_1) e^{D^* s_2} B^*(y_2) \cdots e^{D^* s_n} \mathbf{B}^*(y_n), \tag{4.3}$$

where $D^* = (\delta_{ij} \nu_j)$, $1 \leqq i, j \leqq K$, $\mathbf{B}^* = (b_i)$, $1 \leqq i \leqq K$ and $B^*(y_n) = (b_i(y_n))\cdot$ $1 \leqq i \leqq K$.

LEMMA 4.3. *Let $X(t)$ and $Y(t)$ satisfy hypothesis H and let $Y(t)$ have order K Then for each element k in $\mathfrak{M}$, $\mathbf{B}^*(k) \neq 0$.*

PROOF. Suppose for some k in $\mathfrak{M}$ $\mathbf{B}^*(k) = 0$. Consider the sequence pair of length one, (k, τ). From Theorem 4.2 and from (2.3) one concludes that in the context of the basic Markov chain $X(t)$

$$p_Y(k, \tau) = \sum_{f(j)=k} \sum_{i=1}^{N} PiP_{ij}(\tau) = 0. \tag{4.4}$$

Each P_i is positive. Thus (4.4) implies that $P_{ij}(\tau) = 0$ for each i and every j with $f(j) = k$, and that the matrix Λ for $X(t)$ has more than one zero eigenvalue, contrary to the assumption that $X(t)$ is a basic Markov chain.

THEOREM 4.4. *Let $X(t)$ and $Y(t)$ satisfy hypothesis H and let $Y(t)$ have order K. The set $\{B^*(m)\}$ of $K \times K$ matrices of representation* (4.3) *is a set of factor matrices.*

PROOF. From (4.2) the characteristic equation for $B(k)$ is

$$|B(k) - \mu I_N| = |B^*(k) - \mu I_K| \cdot |B_U(k) - \mu I_U| \cdot |B_V(k) - \mu I_V| \cdot |B_W(k) - \mu I_W| = 0$$

where the vertical bars denote determinant and I_j is the $j \times j$ identity matrix for $j = N, K, U, V, W$. Thus the set of eigenvalues of $B^*(k)$ is contained in the set of zero and unit eigenvalues of $B(k)$. That the remaining properties hold follows readily from (4.2) and the observation that the set of matrices $\{B(k)\}$ of (2.3) is a set of factor matrices.

5. The proof of Theorem 2.5. Let $Y(t)$ be Markov. For any sequence pair (y, s), $p_Y(y, s)$ will be a linear combination of exponential terms involving the set $\{\mu_j\}$, $1 \leqq j \leqq M$ of eigenvalues of the Λ matrix for $Y(t)$. Since these exponential terms are linearly independent functions, the representations for $p_Y(y, s)$ are unique. Hence $\{\mu_j\} \subseteq \{\nu_j\}$ and the elements of $\{\mu_j\}$ are distinct. The initial probabilities of $Y(t)$ can easily be seen to be positive. Thus $Y(t)$ is a basic Markov chain, and $K \leqq M$.

For each element k in $\mathfrak{M}$ let $B^*(k) = (b_{ij}^*(k))$ be defined as for Theorem 4.2. Let $m(k)$ be the number of unit eigenvalues of $B^*(k)$.

From Theorem 4.2 observe $\sum_{k \varepsilon \mathfrak{M}} b_{ii}^*(k) = 1$ for each $i \varepsilon \mathcal{K}$. For each k in $\mathfrak{M}$ the characteristic equation of the matrix $B^*(k)$ is

$$(5.1) \quad |B^*(k) - \mu I_K| = [\mu^K - [\textstyle\sum_{i \varepsilon \mathcal{K}} b_{ii}^*(k)]\mu^{K-1} + \cdots \pm |B^*(k)|](-1)^K$$

$$(5.2) \quad = [\mu K^{-m(k)}(\mu - 1)^{m(k)}](-1)^K = 0.$$

Equating the coefficients in (5.1) and (5.2) for μ^{K-1} yields

$$(5.3) \quad \textstyle\sum_{i \varepsilon \mathcal{K}} b_{ii}^*(k) = m(k).$$

If expression (5.3) is summed over all k in $\mathfrak{M}$, one observes there are K unit eigenvalues among the M matrices $B^*(k)$. If $K < M$, then at least $M - K$ of these matrices would have no unit eigenvalues and hence would be zero matrices. This contradicts Lemma 4.3. Hence $K = M$.

Conversely suppose $K = M$. There exist exactly K matrices $B^*(k)$, each $K \times K$, in the representation (4.3) for $p_Y(y, s)$. By Theorem 4.4 the set of matrices $\{B^*(k)\}$ is a set of factor matrices. Since none can be a zero matrix by Lemma 4.3, each has exactly one unit eigenvalue. By Theorem 3.2 each state of $Y(t)$ is a regeneration state and $Y(t)$ is a Markov chain.

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