SOME PROBLEMS IN THE THEORY OF OPTIMAL STOPPING RULES[1]

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1. Introduction and summary. Let $y_1, y_2, \cdots$ be a sequence of random variables with known joint distribution. We are allowed to observe the y's sequentially. We must terminate the observation process at some point, and if we stop at time n, we receive a reward which is a known function of $y_1, \cdots, y_n$. Our decision to stop at time n is allowed to depend on the observations we have previously made but may not depend on the future, which is still unknown. We are interested in finding stopping rules which maximize our expected terminal reward.

More formally, let $(x_n, F_n)_{1 \leqq n}$ be a stochastic sequence on a probability space (W, F, P), i.e., let (F_n) be an increasing sequence of sub-sigma-algebras of F and for each $n \geqq 1$ let x_n be a random variable (rv) measurable F_n. In terms of the intuitive background of the preceding paragraph, $F_n = B(y_1, \cdots, y_n)$; and although it is convenient to keep this interpretation in mind, our general results do not depend on it. A *stopping rule* or *stopping variable* (sv) is a rv t with values $1, 2, \cdots, +\infty$, such that $P(t < \infty) = 1$ and for each $n \geqq 1$ $(t = n) \,\varepsilon\, F_n$. x_t is (up to an equivalence) a rv, and if $v = \sup Ex_t$, where the supremum is taken over all sv's such that Ex_t exists, we are interested in answering the following questions:

(a) What is v?

(b) Is there an *optimal* sv, i.e., one for which Ex_t exists and equals v?

(c) If there exists an optimal sv, what is it?

The problem stated above is not sufficiently well formulated, as the class of sv's t such that Ex_t exists may be vacuous. To avoid this and other uninteresting complications we shall add the assumption that $E|x_n| < \infty, n \geqq 1$.

We recall that the essential supremum (e. sup) of a family of rv's $\{q_t, t \,\varepsilon\, T\}$ is a rv Q such that (1) $Q \geqq q_t$ a.s. $t \,\varepsilon\, T$, and (2) if Q' is any rv such that $Q' \geqq q_t$ a.s., $t \,\varepsilon\, T$, then $Q' \geqq Q$ a.s. It is known that the essential supremum of a family of rv's always exists and can be assumed to be the supremum of some countable subfamily (e.g., [12], p. 44).

Let C_n be the class of all sv's t such that $P(t \geqq n) = 1$ and $Ex_t^- < \infty$. Let $f_n = \text{e. sup}_{t \varepsilon C_n} E(x_t \mid F_n)$, $v_n = \sup_{C_n} Ex_t$. It is known (Theorem 2 of [3]) that if $v < \infty$ and an optimal rule exists, then $s = \text{first}\, n \geqq 1$ such that $x_n = f_n$ is an

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optimal rule. For this and various other reasons which will become apparent, e.g., Theorem 1 below, it is desirable to have a constructive method for computing the f_n. The technique of backward induction and taking limits, originating with [1] and described in Theorem 2 below, achieves the desired result under certain conditions (see Theorem 2 of [4] for a general statement of these conditions). The central theorem of Section 2 provides completely general methods for computing the f_n. Although it seems unlikely that one would ever find it desirable to carry out these computations, there are, nevertheless, several interesting applications of the results to the theory of optimal stopping rules, and it is these applications which concern us throughout the remainder of this paper. In the course of these investigations we find it convenient to introduce the notion of an *extended* sv, i.e., we drop the requirement that t be finite with probability one while defining x_∞ to be $\limsup x_n$. We show that $\bar{f}_n = f_n$, where, relative to the class of extended sv's, $\bar{f}_n$ is defined analogously to f_n. We utilize extended sv's as a technical device within the framework of the usual theory and give examples which illustrate the inherent value of these sv's.

In Section 3 we define the *Markov case.* We show that by paying proper attention to the Markovian structure of many stopping rule problems we are able to simplify somewhat the general theory and to give relatively simple descriptions of optimal rules when they exist. We also define *randomized* sv's and show that randomization does not increase v. We then apply this result to prove the monotonocity and continuity of $v = v(p)$ in the case where x_n is the proportion of heads in n independent tosses of a coin having probability p of heads on each toss.

2. A basic theorem and applications.

The following basic results [3] will be used frequently and are assumed to be known.

THEOREM 1. *For each* n

$$f_n = \max(x_n, E(f_{n+1} \mid F_n)), \quad \text{a.s.}, \tag{1}$$

$$v_n = Ef_n. \tag{2}$$

THEOREM 2. *Define for* $N = 1, 2, \cdots$

$$f_N^{\ N} = x_N, \qquad f_n^{\ N} = \max(x_n, E(f_{n+1}^N \mid F_n)), \qquad (1 \leqq n \leqq N - 1);$$

then if $E(\sup x_n^-) < \infty$, $\lim_N f_n^{\ N} = f_n$ a.s.

For the most part statements are assumed to hold up to an event of probability 0, and where no confusion can result we make no mention of this fact. Following [3] we shall denote by $A+$ the hypothesis that $E(\sup x_n^+) < \infty$.

LEMMA 1. *Under* $A+$, *let* $(g_n, F_n)_{1 \leqq n}$ *be a stochastic sequence satisfying for* $n = 1, 2, \cdots$.

(a) $f_n \leqq g_n \leqq E(u \mid F_n)$, *for some integrable* u,

(b) $g_n = \max(x_n, E(g_{n+1} \mid F_n))$,

(c) $\limsup g_n = \limsup x_n$;

then $g_n = f_n$ $(n = 1, 2, \cdots)$.

PROOF. It suffices to show $Eg_n = Ef_n$.

If $0 < \epsilon < Eg_n - Ef_n$, let t = first $k \geqq n$ such that $x_k \geqq g_k - \epsilon$.

$$Eg_n = \int_{t=n} g_n + \int_{t>n} g_{n+1} = \cdots = \int_{n \leqq t \leqq N} g_t + \int_{t>N} g_N .$$

From (a) we see that Fatou's lemma applies to $I_{(t \geqq N)} g_N$ and thus from (c)

$$-\infty < Eg_n \leqq \int_{t<\infty} g_t + \int_{t=\infty} (\limsup x_k). \tag{3}$$

Letting $x_\infty = \limsup x_k$, it follows that $P(t = \infty, x_\infty = -\infty) = 0$. But by (c) $P(t = \infty, x_\infty > -\infty) = 0$. Hence $t < \infty$ a.s. and it then follows from (3) that

$$Ef_n + \epsilon < Eg_n \leqq Eg_t \leqq Ex_t + \epsilon$$

contradicting Theorem 1.

For the following we shall relax somewhat the definition of a stopping rule by no longer requiring that our rule stop with probability one, and we agree that if we go on forever, our reward is the upper limit of the rewards we might have received by stopping at some finite time n. More formally, let $x_\infty = \limsup x_n$, and let $\bar{C}$ be the class of all rv's t assuming values in $\{1, \cdots, +\infty\}$ such that $(t = k) \varepsilon F_k$ and $Ex_t^- < \infty$. Let $\bar{f}_n = \text{e.}\sup_{\bar{C}_n} E(x_t \mid F_n)$, $1 \leqq n < \infty$, where $\bar{C}_n = \{\max(t, n) : t \varepsilon \bar{C}\}$, and let $f_\infty = x_\infty$.

We shall use the following notation throughout the rest of this section. Let $-\infty \leqq a < 0 < b \leqq +\infty$,

$$\begin{aligned} x_n(a, b) &= b \quad \text{if} \quad x_n > b \\ &= x_n \quad \text{if} \quad b \geqq x_n \geqq a \\ &= a \quad \text{if} \quad x_n < a; \\ x_n(b) &= x_n(-\infty, b); \\ x_n(a) &= x_n(a, +\infty). \end{aligned}$$

Let $f_n(a, b) (f_n^N(b)$, etc.) denote the f_n's (f_n^N's, etc.) associated with the sequence $(x_n(a, b), F_n)_{1 \leqq n} ((x_n(b), F_n)_{1 \leqq n \leqq N}$, etc.)

LEMMA 2. *Assume* $A+$, *and let* N *be a positive integer. Define* $\tilde{f}_N^N = E(\sup_{k \geqq N} x_k \mid F_N)$,

$$\tilde{f}_n^N = \max(x_n, E(\tilde{f}_{n+1}^N \mid F_n)), \qquad (1 \leqq n \leqq N - 1),$$

$$\tilde{f}_n = \lim_{N \to \infty} \tilde{f}_n^N.$$

Then (a) $\tilde{f}_n = \bar{f}_n = f_n$, (b) $\lim_{a \to -\infty} f_n(a) = f_n$.

PROOF. (a) It can be shown by induction that $\tilde{f}_n^n \geqq \tilde{f}_n^{n+1} \geqq \cdots$ and hence that the limit defining $\tilde{f}_n$ exists.

It is almost obvious that $\tilde{f}_n \geqq \bar{f}_n \geqq f_n$; for if $t \varepsilon \bar{C}_n$, $t' = \max(t, n + 1) \varepsilon \bar{C}_{n+1}$, and from

$$\begin{aligned} E(x_t \mid F_n) &= I_{(t=n)}x_n + I_{(t>n)}E(x_{t'} \mid F_n) \\ &= I_{(t=n)}x_n + I_{(t>n)}E(E(x_{t'} \mid F_{n+1}) \mid F_n) \\ &\leqq I_{(t=n)}x_n + I_{(t>n)}E(\bar{f}_{n+1} \mid F_n) \\ &\leqq \max(x_n, E(\bar{f}_{n+1} \mid F_n)) \end{aligned}$$

it follows that

$$\bar{f}_n \leqq \max(x_n, E(\bar{f}_{n+1} \mid F_n)). \tag{4}$$

Obviously $\bar{f}_n \leqq \tilde{f}_n^n$, and by backwards induction, using (4) and the relations defining $\tilde{f}_n^N$, we have $\bar{f}_n \leqq \tilde{f}_n^N$, $N \geqq n$, and hence $\bar{f}_n \leqq \tilde{f}_n$.

We also see by the monotone convergence theorem for conditional expectations that $\tilde{f}_n = \max(x_n, E(\tilde{f}_{n+1} \mid F_n))$; and since $\tilde{f}_n \leqq \tilde{f}_n^n \leqq E(\sup_{k \geqq m} x_k \mid F_n)$, $(m \leqq n)$, we have $\limsup_n \tilde{f}_n \leqq \sup_{k \geqq m} x_k \to \limsup_n x_n$ as $m \to \infty$ by a convergence theorem for conditional expectations. Hence $\limsup \tilde{f}_n = \limsup x_n$, and we complete the proof of (a) by appealing to Lemma 1.

(b) $f_n(a)$ is a decreasing function of a; let $f_n^* = \lim_{a \to -\infty} f_n(a)$. Then $f_n^* \geqq f_n$. Since $f_n(a) = \max(x_n(a), E(f_{n+1}(a) \mid F_n))$, we have by the monotone convergence theorem for conditional expectations $f_n^* = \max(x_n, E(f_{n+1}^* \mid F_n))$. By the proof of (a) we see that $\limsup x_n(a) = \limsup f_n(a)$, all a. Hence $\limsup f_n^* \leqq \limsup x_n(a) = \max(\limsup x_n, a) \downarrow \limsup x_n$ as $a \downarrow -\infty$. Lemma 1 completes the proof.

The above lemma says, among other things, that if we relax the definition of a stopping rule in the prescribed manner, then the f_n are not increased provided $A+$ is satisfied. That this result holds without such a restriction is the content of the following:

Lemma 3. *$\lim_{b \to \infty} \bar{f}_n(b) = \bar{f}_n$, and consequently $\lim_b f_n(b) = f_n$.*

Proof. Clearly $\bar{f}_n(b)$ is increasing as a function of b; let $f_n^* = \lim \bar{f}_n(b)$. Then $f_n^* \leqq \bar{f}_n$. If $t \in \bar{C}_n$, $x_t^-(b) = x_t^-$ and $Ex_t^- < \infty$ imply that $E(x_t(b) \mid F_n) \leqq \bar{f}_n(b) \leqq f_n^*$. But by the monotone convergence theorem for conditional expectations,

$$E(x_t(b) \mid F_n) \uparrow E(x_t \mid F_n).$$

Hence

$$E(x_t \mid F_n) \leqq f_n^*$$

and the lemma follows, as t is arbitrary.

Theorem 3. (a) $f_n = \bar{f}_n = \lim_{b \to \infty} \lim_{N \to \infty} \tilde{f}_n^N(b)$,

(b) $f_n = \lim_{b \to \infty} \lim_{a \to -\infty} \lim_{N \to \infty} f_n^N(a, b)$.

Proof. (a) Lemmas 2(a) and 3;

(b) Theorem 2, Lemmas 2(b) and 3.

Corollary. (a) $v_n = \lim_b \lim_N E\tilde{f}_n^N(b)$,

(b) $v_n = \lim_b \lim_a \lim_N Ef_n^N(a, b)$.

A different proof of part (b) of the above theorem was given by the author in [3].

We shall spend the remainder of this section and the next exploring applications of the above methods and results to various problems in the theory of optimal stopping rules. We shall defer what seems to be the most important single application, i.e., the Markov case, until Section 3.

Since the computations involved in the above theorem appear prohibitive, it it is of some interest to know under what conditions various limits involved can be interchanged, thus simplifying the problem. With the possible exception of the limits on a and b in part (b) it is easily seen that in general we cannot interchange limits. That this is also true in the case of a and b is seen by the following

EXAMPLE 1. Let $a_1, a_2, \cdots, b_1, b_2, \cdots$ be increasing sequences of nonnegative real numbers and W the space of sequences

$$w_j = (a_1, \cdots, a_j, -b_{j+1}, -b_{j+2}, \cdots).$$

Let F be all subsets of W and $P(w_j) = 1/j - 1/j + 1$. For each $n \geqq 2$ let $x_n(w_j)$ be the nth coordinate of w_j and $F_n = B(x_2, \cdots, x_n)$. It is easily seen that

$$P(x_n = a_n) = 1/n = 1 - P(x_n = -b_n),$$

$$E(x_{n+1} \mid x_n = a_n) = a_{n+1}n/(n+1) - b_{n+1}(1 - n/(n+1)),$$

$$Ex_n = a_n/n - b_n(1 - 1/n).$$

Let $t =$ first $k \geqq 2$ such that $x_k = -b_k$ and $t_n = \min(t, n)$. It is easily seen that the rules t_n are the only ones which need be considered in searching for optimal rules or in computing the value of the sequence $(x_n, F_n)_{2 \leqq n}$. Elementary calculations give

$$Ex_{t_n}^+ = a_n/n, \qquad Ex_{t_n}^- = \textstyle\sum_{j=2}^n b_j/j(j-1).$$

Now put $a_j = j(j-1) = b_j$; then $Ex_{t_n}^+ = n - 1 \to +\infty$. Thus $v(a) = \infty$, all $a > -\infty$. But $Ex_{t_n} = n - 1 - \sum_{j=2}^n 1 = 0$, $v = 0$, and it is not true that $\lim v(a) = v$.

Various special cases of the above parameterized stochastic sequence are quite useful in constructing counterexamples, and we shall have reason to refer to it again on several occasions. I am indebted to H. E. Robbins for introducing me to the above class of examples via his original construction of the particular case which constitutes Example 3 later in this section.

DEFINITION. Let

$$s = \text{first } n \geqq 1, \quad \text{such that} \quad x_n = f_n$$

$$(= \infty \quad \text{if} \quad x_n < f_n \quad \text{for all} \quad n).$$

LEMMA 4. *Under* $A+$ *if* t *is any extended* sv *such that* $t \leqq s$, *then* $E(f_t \mid F_n) \geqq f_n$ *on* $(t \geqq n)$ (*a deeper analysis shows that equality holds, but we shall use only the above inequality*).

PROOF. First observe that because of the hypothesis $A+$ there is no question about the existence of Ef_t. In fact

$$Ef_t^+ \leqq \textstyle\sum_{n=1}^\infty \int_{t=n} E(\sup x_j^+ \mid F_n) + \int_{t=\infty} \sup x_j^+ = E(\sup x_j^+) < \infty.$$

Then for $A \varepsilon F_n$

$$\int_{A(t\geqq n)} f_n = \int_{A(t=n)} f_n + \int_{A(t>n)} f_{n+1} = \cdots = \int_{A(n\leqq t\leqq N)} f_t + \int_{A(t>N)} f_N .$$

By the proof of Lemma 2 we see that $\lim\sup f_n = \lim\sup x_n = f_\infty$. Thus letting $N \to \infty$, we have by Fatou's lemma

$$\int_{A(t\geqq n)} f_n \leqq \int_{A(n\leqq t<\infty)} f_t + \int_{A(t=\infty)} f_\infty = \int_{A(t\geqq n)} f_t .$$

THEOREM 4. *Under* $A+$ *s is optimal in* $\bar{C}$.

PROOF. Since $x_s = f_s$, Lemma 4 implies that $Ex_s = v$. Since Theorem 3(a) shows that $Ex_t \leqq v$, $t \varepsilon \bar{C}$, the theorem follows.

COROLLARY. *If* $A+$ *and* $\lim x_n = -\infty$, *then s is optimal in C.*

PROOF. By the theorem s is optimal in $\bar{C}$. Clearly then s cannot assume the value $+\infty$ (and hence collect a reward of $-\infty$) with positive probability.

The existence of an optimal rule under the conditions of the corollary was first demonstrated by Snell [13], who extended an earlier result of Arrow, Blackwell, and Girshick [1]. Snell's approach to the optimal stopping problem uses the "minimal regular generalized supermartingale" where the f_n sequence is used above. The introduction of the f_n sequence and its identification with Snell's process is due to Haggstrom [10] and Chow and Robbins [3].

Note that by making a straightforward generalization of Lemma 1 of [5] we might prove the more general result that if Ex_s exists and for $n \geqq 1$, $E(x_s \mid F_n) > x_n$ on $(s > n)$, then $Ex_s = v$. We could then show much in the fashion of Theorem 4 that under $A+$ s has the required property.

Contrasted to the above discussion in which the class $\bar{C}$ plays the role of a technical device for use in the usual optimal stopping theory, there do exist quite natural examples of problems where one is concerned with deciding whether or not he should continue sampling indefinitely. The following example resembles a problem of Bellman [2], for which a similar result holds.

EXAMPLE 2. Suppose that conditional on p, y_1, y_2, $\cdots$, are iid such that $P(y_1 = 1) = p = 1 - P(y_1 = -1)$, that $x_n = \sum_{i=1}^n d^{i-1} y_i$, for some $0 < d < 1$, and that an *a priori* distribution of p is known. The above theorem then states that s is an optimal rule. Suppose for simplicity that the *a priori* distribution is a member of the beta family with equal parameters r. Then using well known properties of the beta family relating *a priori* and *a posteriori* distributions in the above problem, denoting by $v(r, q)$ the value of the reward sequence as a function of the parameters r, q, it is plausible as a result of heuristic invariance arguments made precise by Theorem 6 that if $S_n = y_1^+ + \cdots + y_n^+$, then $s =$ first $n \geqq 1$ such that $v(r + S_n, r + n - S_n) \leqq 0$. Now $E_p x_1 = 2p - 1$ and hence $v(r, q) \geqq (r - q)/(r + q)$, where the inequality follows from considering the rule $t = 1$ and the fact that the expectations of a $B(r, q)$ rv is $r/(r + q)$. Thus for s to be infinite, it suffices that $S_n > n/2$, $n \geqq 1$, which occurs with positive probability when $p > \frac{1}{2}$.

The following examples show that the condition $A+$ cannot be removed from Theorem 4 or its Corollary 1.

EXAMPLE 3. With the model of Example 1 put $a_n = n + 1$, $b_n = 0$; we have

$$E(x_{n+1} \mid x_n = a_n) > a_n$$

and x_n is eventually 0. Hence s = first $n \geqq 2$ such that $x_n = 0$. s belongs to C_2 and $P(x_s = 0) = 1$. Clearly s is not optimal.

EXAMPLE 4. This time let $a_n = n^2 - 1$, $b_n = n(n - 1)$. Then $x_n \to -\infty$ and in fact $Ex_n \to -\infty$, but $Ex_{t_n} = 1 - 1/n$ and no optimal rule exists.

For our next example we introduce some additional notation. The optimal stopping problem associated with $(-x_n, F_n)$ has an obvious interpretation as a problem in which x_n represents the "loss" we incur if we stop at time n, and we are trying to choose a sv t so as to minimize our expected loss. Hence we define C_n^*, $v_n^* = \inf_{C_n^*} Ex_t$, $f_n^* = \text{e. }\inf_{t \varepsilon C_n^*} E(x_t \mid F_n)$, etc. analogously to the usual C_n, v_n, f_n. Suppose now that $(x_n, F_n)_{1 \leqq n}$ is a sub-martingale such that $\sup E|x_n| < \infty$. It is known [8] that $x_n \to x_\infty$, $E|x_\infty| < \infty$, and $E|x_t| < \infty$ for every sv t. It seems reasonable that we should also have $v_n = Ex_\infty$, $v_n^* = Ex_n$. A precise statement, motivated by a result of Chow's [7] (note that our methods also suffice to prove Chow's result), is

EXAMPLE 5. Let $(x_n, F_n)_{1 \leqq n}$ be a sub-martingale such that $\sup E|x_n| < \infty$. The following are equivalent:

(i) (x_n^+) is uniformly integrable.

(ii) $v_n^* = Ex_n$.

(iii) $v_n = Ex_\infty$.

PROOF.

(i) $\Rightarrow$ (ii): See Doob [8], pg. 302.

(ii) $\Rightarrow$ (iii): (ii) $\Rightarrow f_n^* = x_n \Rightarrow$ (from Theorem 3) $x_n \leqq E(x_\infty \mid F_n) \Rightarrow Ex_t \leqq Ex_\infty$, all $t \Rightarrow v_n = Ex_\infty$.

(iii) $\Rightarrow$ (i): From Theorem 3 $f_n \geqq E(x_\infty \mid F_n)$. Thus $x_n \leqq E(x_\infty \mid F_n)$ and it follows that (x_n^+) is uniformly integrable.

3. Markov case. In a large class of optimal stopping rule problems we can in our search for optimal and near optimal rules confine our attention to a subclass of C and in so doing we can frequently give a relatively simple description of optimal rules when they exist. In particular we are interested in finding conditions under which the rule s, which is certainly a candidate for optimality, has a relatively simple structure. Consequently we shall in this section define the Markov case and show that as suggested by intuition, which says that the future behavior of a Markov process depends on the past only through the present, we can in the Markov case restrict consideration to rules "without memory."

DEFINITION. In an optimal stopping rule problem as defined in Section 1, if x_n can be expressed as a Baire function, say u_n, of z_n, where z_n is an F_n-measurable rv taking values in Z (assumed for convenience to be a complete separable metric space) such that for any Borel set B

$$P(z_{n+1} \varepsilon B \mid F_n) = P(z_{n+1} \varepsilon B \mid z_n),$$

then we say that we are in the Markov case. If in addition the conditional probability on the right is invariant under changes in the subscript n, we say that we are in the stationary Markov case. We say we are in the independent case if the z_n are independent.

LEMMA 5. *In the Markov case f_n is z_n-measurable* (i.e., *is measurable with respect to the σ-algebra generated by z_n*).

(See [3], Corollary 2 to Theorem 9.)

PROOF. By Theorem 3

$$f_n = \lim_b \lim_a \lim_N f_n^N(a, b).$$

It is easy to see using the Markov property and backwards induction that $f_n^N(a, b)$ is z_n-measurable and the lemma follows.

DEFINITION. In the Markov case we shall designate by D_n that subset of C_n having the property that if $t \varepsilon D_n$, $k \geqq n$, there exists a subset $B(k, t)$ of Z such that $(t = k) = (t \geqq k, z_k \varepsilon B)$.

THEOREM 5. *In the Markov case*

$$f_n = \text{e. sup}_{t \varepsilon D_n} E(x_t \mid z_n), \qquad v_n = \sup_{D_n} Ex_t .$$

PROOF. Suppose initially that the x_n are truncated above at $b > 0$. We are still in the Markov case. If $\epsilon > 0$, let t_b = first $k \geqq n$ such that $x_k(b) \geqq f_k(b) - \epsilon$. By the argument of Lemmas 1 and 2, $t_b \varepsilon C_n$ and $E(x_{t_b}(b) \mid F_n) \geqq f_n(b) - \epsilon$. Lemma 5, moreover, shows that t_b is in fact in D_n and thus $E(x_{t_b} \mid z_n) \geqq f_n(b) - \epsilon$. Thus letting $b \uparrow \infty$ and applying Lemma 3

$$\text{e. sup}_{t \varepsilon D_n} E(x_t \mid z_n) \geqq \sup_b E(x_{t_b} \mid z_n) \geqq \lim_b f_n(b) - \epsilon = f_n - \epsilon,$$

$$\sup_{D_n} E(x_t) \geqq \sup_b E(x_{t_b}) \geqq \lim_b Ef_n(b) - \epsilon \geqq v_n - \epsilon.$$

As ϵ is arbitrary, the theorem follows.

Observe that in the case of independence, the fundamental recursion relation (1) becomes $f_n = \max(x_n, v_{n+1})$, or taking expectations $v_n = E(\max(x_n, v_{n+1}))$. Hence in this case s = first $n \geqq 1$ such that $x_n \geqq v_{n+1}$.

For the following theorem we are interested in the stationary case. By the above result there is no loss in generality in assuming that the basic "observed" sequence is in fact the z_n, $n \geqq 1$. We can and do assume that there exists a transition probability which together with an initial distribution determines the probability structure of the process. Conditional expectations are to be regarded as integrals with respect to appropriate transition probabilities and to avoid confusion we shall distinguish carefully between relations holding everywhere and those holding almost everywhere.

To be specific then, we may assume (by means of an obvious transformation if necessary) that $W = Z \times Z \times \cdots$, $F_n = B(z_1, \cdots, z_n)$, $F = B(z_1, z_2, \cdots)$, where the z_n's are the coordinate variables. Denote by P_z the measure on F governing the behavior of the Markov sequence $z_1, z_2, \cdots$ starting from z, and for $n = 0, 1, 2, \cdots$ let $V_n(z)$ be the value of the stochastic sequence $(u_{n+k}(z_k), F_k)_{1 \leqq k}$ on (W, F, P_z).

THEOREM 6. *In the stationary Markov case, there is a version of (f_n) such that $V_n(z) = E(f_{n+1} \mid z_n = z)$, $z \varepsilon Z$, $n \geqq 1$.*

PROOF. The version of (f_n) in which we are interested is the triple limit of Theorem 3. We shall write $f_n(z)$ $(f_n^N(z)$, etc.) to denote the value at z of that function on Z which equals $f_n(w)$ $(f_n^N(w)$, etc.) when $z_n(w) = z$. Then

$$f_n(z) = \lim_b \lim_a \lim_N f_n^N(a, b)(z), \qquad z \, \varepsilon \, Z, \tag{5}$$

where for each $z \, \varepsilon \, Z$

$$\begin{aligned} f_N^N(z) &= u_N(z), \\ f_n^N(z) &= \max\,(u_n(z), E(f_{n+1}^N(z_{n+1}) \mid z_n = z)), \qquad n = N-1, \cdots, 1, \end{aligned} \tag{6}$$

and as usual $f_n^N(a, b)(\cdot)$ is defined analogously with respect to the u_n's truncated above at b and below at a. Then if $V_n^N(z)$ is defined analogously to $V_n(z)$ relative to the class of rules which take at most $N - n$ observations, it suffices by Theorem 3 and its corollary to show for $n = 1, 2, \cdots$

$$V_n^N(z) = E(f_{n+1}^N(z_{n+1}) \mid z_n = z), \tag{7}$$

$N = n + 1, n + 2, \cdots$. It is easy to see, however, by backward induction using (6), that for $N = 2, 3, \cdots$, (7) holds for $n = N - 1, \cdots, 1$.

COROLLARY 1. *Under the assumptions of Theorem* 6, *suppose* $u_n = u - n$ *and* $B = \{z : u(z) \geqq V_0(z)\}$. *Then*

$$(s = n) = (z_1 \not\varepsilon B, \cdots, z_{n-1} \not\varepsilon B, z_n \, \varepsilon \, B).$$

PROOF. Obviously $V_n(z) = V_0(z) - n$.

COROLLARY 2. *A Bayes solution to the problem of testing a simple hypothesis against a simple alternative with constant cost and independent, identically distributed observations is a Wald sequential probability ratio test.*

PROOF. The reduction of the problem of this corollary to a stopping rule problem having properties described in the assumptions of Corollary 1 is known (see, e.g., [4]). Corollary 1 of Theorem 4 assures us that s is optimal and Corollary 1 above tells us that s = first $n \geqq 0$ such that $z_n \, \varepsilon \, B$, where z_n is the *a posteriori* probability of H_0, say. The well known convexity of $V_0(\cdot)$ (concavity when the problem in stated in terms of "loss") then gives us the explicit representation of B as $[0, z_1^*] \cup [z_2^*, 1]$ for numbers $z_1^*, z_2^* \, \varepsilon \, [0, 1]$ (in this regard see [11]).

We shall now apply the above theorem to extend some known results (see [4] or [5] and the references given there).

COROLLARY 3. *Let* $y, y_1, y_2, \cdots$ *be* iid *such that* $E|y| < \infty$, *and for some* $r \geqq 1$ *let* $F_n = B(y_1, \cdots, y_n)$, $x_n = \max\,(y_1, \cdots, y_n) - n^r$, $n = 1, 2, \cdots$. *If* $v < \infty$, *then* s *is defined by*

$$s = \textit{first } n \quad \textit{such that} \quad \max\,(y_1, \cdots, y_n) \geqq c_n,$$

where c_n *is the unique solution of*

$$E(y - c_n)^+ = (n + 1)^r - n^r, \tag{8}$$

and s *is optimal.*

PROOF. For convenience take $r = 1$; let $z_n = \max\,(y_1, \cdots, y_n)$, $n = 1, 2, \cdots$.

We are in the stationary Markov case and $V_0(z) = V_n(z) + n$, $(n \geqq 1)$. Obviously $V_0(\cdot)$ is convex. Moreover,

$$V_0(z) - z \geqq E(y_1 - z)^+ - 1 \to \infty \quad \text{as} \quad z \to -\infty;$$

and if $z \geqq V_0(z)$, then for every $t \varepsilon C$ and $z' \geqq z$

$$0 \geqq E[\max(0, y_1 - z, \cdots, y_t - z) - t]$$
$$\geqq E[\max(0, y_1 - z', \cdots, y_t - z') - t],$$

i.e., $z' \geqq V_0(z')$. It follows that there exists a constant $c \leqq \infty$ such that $\{z : z \geqq V_0(z)\} = [c, \infty)$, and from Corollary 1

$$s = \text{first } n \quad \text{such that} \quad z_n \geqq c$$
$$= \infty \quad \text{if no such} \quad n \quad \text{exists.}$$

If $P\{y \geqq c\} > 0$, it is easy to see that $P\{s < \infty\} = 1$, $Ez_s^+ < \infty$ and

$$v = Ef_1 = \int_{s \leqq n} x_s + \int_{s>n} V_n(z_n)$$
$$\leqq \int_{s \leqq n} x_s + V_0^+(c)P\{s > n\} \to Ex_s \quad \text{as} \quad n \to \infty.$$

Suppose then that $P\{y < c\} = 1$. From Lemma 8 of [3] and Theorem 6

$$P\{z_n \geqq V_0(z_n) - 1, \text{ i.o.}\} = 1$$

and arguing as above there exists a constant b such that $P\{y \geqq b\} > 0$ and $\{z : z \geqq V_0(z) - 1\} = [b, \infty)$. Since $P\{s = \infty\} = 1$

$$v = Ef_{n+1} = \int_{\{z_n < b\}} V_0(z_n) + \int_{\{V_0(z_n) < z_n + 1\}} V_0(z_n) - n$$
$$\leqq V_0^+(b) + Ez_n^+ + 1 - n \to -\infty \quad \text{as} \quad n \to \infty,$$

since as is easily verified $Ez_n^+ = o(n)$. Hence $v = -\infty$, a contradiction. Thus s is optimal and it is now easy to see that c satisifies (8).

The following example gives rigorous foundation to a result of Elfving [9].

EXAMPLE 6. Suppose that $y_1, y_2, \cdots$ are iid non-negative rv's with finite expectation, and that $N = N(\sigma)$ is a Poisson process independent of the y_n with the time between events denoted by

$$\tau_i - \tau_{i-1}, \qquad i = 1, 2, \cdots, (\tau_0 \equiv 0).$$

Let $r(\cdot)$ be a real-valued non-increasing function defined on the non-negative real numbers such that $r(0) = 1$, $r(\sigma) \geqq 0$, $\sigma > 0$; and let

$$F_\sigma = B(y_1, \cdots, y_{N(\sigma)}, \tau_1, \cdots, \tau_{N(\sigma)}), \qquad \sigma \geqq 0.$$

We are interested in finding an optimal extended sv for $(x_n, F_{\tau_n})_{1 \leqq n}$ where $x_n = y_n r(\tau_n)$, $n = 1, 2, \cdots, x_\infty = 0$.

The above problem is in an obvious formal sense a problem with a discrete time parameter; the usual theory applies, and as usual the backward induction and passage to the limit are difficult to carry out explicitly. Under the general assumption

$$\int_0^\infty r(\sigma)\, d\sigma < \infty,$$

Elfving [9], by making use of the continuous time aspects of the problem, has been able to derive a differential equation for the boundary of an optimal stopping region. Using completely analytic techniques, he then proves an appropriate existence and uniqueness theorem and calculates exact solutions in a number of particular cases. Elfving's derivation relies, however, on several additional assumptions which it seems desirable to remove. To be precise he assumes:

(a) There is an optimal rule in the class of rules t defined by piece-wise continuous functions $y(\cdot)$, where a sv t is said to be defined by a function $y(\cdot)$ if

$$t = \text{first } n \geqq 1 \quad \text{such that} \quad y_n \geqq y(\tau_n).$$

(b) If t^* denotes the rule assumed to exist in (a), then $E(x_{t^*} \mid F_\sigma) = y(\sigma)r(\sigma)$ on $\{\tau_{t^*} > \sigma\}$.

Retaining the assumption $\int_0^\infty r(\sigma)\, d\sigma < \infty$ we shall prove as an application of our main theorems that s is optimal in $\bar{C}$ and is defined by a piece-wise continuous function $y(\cdot)$. A proof of (b) will be included for completeness.

For ease of exposition we assume that $r(\sigma) > 0$ all $\sigma > 0$. If this is not the case the same remarks apply to the problem restricted to the interval $[0, T)$, where $T = \inf\{\sigma: r(\sigma) = 0\}$.

We first observe that $E(\sup x_n) < \infty$ and $x_n \to 0$ (and hence our convention about x_∞ agrees with Elfving's). In fact an easy calculation shows that

$$E(\textstyle\sum_1^\infty y_n r(\tau_n)) = Ey_1 \int_0^\infty r(\sigma)\, d\sigma,$$

which we have assumed to be finite. Theorem 4 implies that s is optimal in $\bar{C}$. Putting $z_n = (y_n, \tau_n)$ we see that we are in the stationary Markov case. Moreover, $V_n(z) = V_n((y, \sigma)) = \sup_t E(y_t r(\sigma + \tau_t))$ is a function of σ only, say $U(\sigma)$, and thus by Theorem 6,

$$s = \text{first } n \geqq 1 \quad \text{such that} \quad y_n r(\tau_n) \geqq U(\tau_n).$$

Let $y(\sigma) = U(\sigma)/r(\sigma)$, $(\sigma \geqq 0)$.

To show that $y(\cdot)$ is piecewise continuous, it suffices to show that $U(\cdot)$ is continuous. The continuity of $U(\cdot)$ follows from a generalization of Theorem 3(b).

THEOREM 3*. *Let $(x_n(p), F_n)_{1 \leqq n}$ be a family of stochastic sequences depending on an extended real parameter p assuming values in some closed (perhaps infinite) interval $[a, b]$. Let $p_0 \varepsilon [a, b]$ and assume that $x_n(p) \uparrow\downarrow x_n(p_0)$ as $p \uparrow\downarrow p_0$, $n = 1, 2, \cdots$.*

(a) *If there exists a $p^* > p_0$ such that $E(\sup x_n^+(p^*)) < \infty$ and if*

$$\limsup x_n(p) \downarrow \limsup x_n(p_0) \quad \textit{as} \quad p \downarrow p_0,$$

then $f_n(p) \downarrow f_n(p_0)$ as $p \downarrow p_0$, $n = 1, 2, \cdots$.

(b) *If there exists a* $p_* < p_0$ *such that*

$$f_n(p) = \text{e. sup}_{t \varepsilon C_n(p^*)} E(x_t \mid F_n), \qquad p_* \leqq p \leqq p_0,$$

then $f_n(p) \uparrow f_n(p_0)$ *as* $p \uparrow p_0$.

Proof. The proof may be inferred from that of Theorem 3(b). Note that an analogous statement holds if $x_n(p)$ is monotonically *decreasing* in p. Similarly if $p_0 = a$ or $p_0 = b$, the statement of the theorem must be modified accordingly.

In the problem under consideration, the parametric family of stochastic sequences is

$$(x_n(\sigma), F_{\tau_n})_{1 \leqq n} = (y_n r(\sigma + \tau_n), F_{\tau_n})_{1 \leqq n}, \qquad \sigma \geqq 0,$$

where $x_n(\sigma) \uparrow\downarrow x_n(\sigma_0)$ as $\sigma \uparrow\downarrow \sigma_0$ for any fixed $\sigma_0 > 0$, since $P\{\sigma_0 + \tau_n \varepsilon$ set of discontinuities of $r(\cdot)\} = 0$. The conditions of the above theorem are easily checked, and it follows that $U(\sigma) = Ef_1(\sigma)$ is continuous.

It remains to show that

$$E(y_s r(\tau_s) \mid F_\sigma) = U(\sigma) \quad \text{on} \quad \{\tau_s > \sigma\}.$$

From well-known properties of the exponential distribution it is easy to see that the conditional joint distribution given F_σ of $(y_{N(\sigma)+1}, \tau_{N(\sigma)+1}), (y_{N(\sigma)+2}, \tau_{N(\sigma)+2}), \cdots$ is the same as the unconditional joint distribution of $(y_1, \sigma + \tau_1), (y_2, \sigma + \tau_2), \cdots$. Letting

$$s(\sigma) = \text{first } n \geqq 1 \quad \text{such that} \quad y_n r(\sigma + \tau_n) \geqq U(\sigma + \tau_n) \quad (s(0) = s),$$

and observing that on $\{\tau_s > \sigma\}$,

$$s = \text{first } k \geqq 1 \quad \text{such that} \quad y_{N+k} r(\tau_{N+k}) \geqq U(\tau_{N+k}),$$

we see that on $\{\tau_s > \sigma\}$ the conditional distribution given F_σ of $y_s r(\tau_s)$ is the same as the unconditional distribution of $y_{s(\sigma)} r(\sigma + \tau_{s(\sigma)})$. Remark (b) now follows from the fact that $E(y_{s(\sigma)} r(\sigma + \tau_{s(\sigma)})) = U(\sigma)$.

Comments on Randomization. Theorem 5 while intuitively trivial, is proved in the present development by relying on the rather complicated Theorem 3. The most direct and presumably most obvious approach to Theorem 5 is via randomized rules. For any possibly randomized rule which does not stop before time n and which depends on $z_1, z_2, \cdots$, we may define an "equivalent" randomized rule depending only on $z_n, z_{n+1}, \cdots$ by constructing a quasi $z_1, \cdots, z_{n-1}$, say $z_1^*, \cdots, z_{n-1}^*$, using the known conditional joint distribution of $z_1, \cdots, z_{n-1}$ given z_n, and then apply the original rule to $z_1^*, \cdots, z_{n-1}^*, z_n, z_{n+1}, \cdots$. To complete the discussion along these lines it is desirable to "recover" the original non-randomized rules, which means proving that randomization does not increase the value of the f_n or what is sufficient that any randomized rule of the above form is equivalent to a randomization on the space of non-randomized stopping rules prior to experimentation.

The fact that randomization does not increase v can also be inferred as a corollary to Theorem 3. For our discussion of this result we restrict ourselves to the case where there exists a sequence of random variables $y_1, y_2, \cdots$ with known joint distribution and $F_n = B(y_1, \cdots, y_n)$. We observe that although we always work with a fixed underlying probability space (W, F, P), the particular space is irrelevant provided that it is consistent with the specified joint distribution of $y_1, y_2, \cdots$. If $G_1, G_2, \cdots$ is an increasing sequence of sub-sigma-algebras of F such that

(a) $F_n \subset G_n$ and

(b) $P(A \mid G_n) = P(A \mid F_n), A \varepsilon B(\bigcup_{i=1}^{\infty} F_i)$,

then we call any sv relative to the sequence (G_n) a randomized rule relative to (F_n); and the class of randomized sv's for the original stochastic sequence is precisely the class of sv's which can be obtained by considering such sequences (G_n), choosing when necessary a different underlying probability space (W, F, P) to accommodate this additional structure. Observe that the intuitive method of randomization whereby at stage n one performs an auxiliary random experiment depending (measurably) on $y_1, \cdots, y_n$ in order to decide whether to stop fits easily into the above scheme. In fact in this case we would have $G_n = B(y_1, \cdots, y_n$ and all auxiliary experiments preceding (and including) time n).

COROLLARY (to Theorem 3). *If t is any randomized rule such that Ex_t exists, then $Ex_t \leqq v$.*

PROOF. Let (G_n) be an increasing sequence of sigma-algebras satisfying (a) and (b) such that t is a sv relative to (G_n). We denote by an asterisk (*) the f_n's, v_n's, etc., associated with $(x_n, G_n)_{1 \leq n}$. It suffices to show $v^* = v$; and hence by Theorem 3 to show $f_n^N(a, b) = f_n^{N*}(a, b)$. But this result follows easily from backward induction and condition (b).

Another reason for considering the introduction of randomized rules into the theory involves the value ∞. If $v = \sup_i Ex_{t_i} = \infty$, where we can assume $Ex_{t_i} \geqq 2^i$, then by using rule t_i with probability 2^{-i}, $i = 1, 2, \cdots$, we presumably have a randomized rule the expected return of which is infinite. In other words we would like to prove the following theorem: If $v = \infty$, there exists a possibly randomized optimal rule. That such a theorem is not true in general is shown by the following

EXAMPLE 7. In the set up of Example 1, let

$$a_n = n^2(n + 2), \qquad b_n = 2n^2(n - 1)$$

Then

$$Ex_{t_n} = n + 2 \uparrow +\infty.$$

But $1 \geqq (n + 2)/(n^2 + n - 2) = Ex_{t_n}/Ex_{t_n}^- \rightarrow 0$.

Hence any mixture of the t_n which formally gives $\sum_{n=2}^{\infty} p_n Ex_{t_n} = \infty$ must in fact define a rule the expected return of which does not exist.

The following is an application of the results and methods of the preceding sections.

EXAMPLE 8. Let $y_1, y_2, \cdots$ be iid, $P(y_1 = 1) = p = 1 - P(y_1 = 0)$, $x_n = (y_1 + \cdots + y_n)/n$. We shall show that $v = v(p)$ is increasing and continuous in

p. The essential step is to rephrase the problem so that instead of a fixed sequence of rv's on a measurable space on which is defined a family of measures, we have a family of sequences of rv's on a fixed probability space. Let $Y_1, Y_2, \cdots$ be independent and uniform on $(0, 1)$, $G_n = B(Y_1, \cdots, Y_n)$, $x_n(p) =$ proportion of terms among $Y_1, \cdots, Y_n$ which are $\leqq p$.

It is easily seen that the increasing sequence $G_1, G_2, \cdots$ of σ-algebras has properties (a) and (b) relative to the sequence generated by the events $(Y_i \leqq p)$, $i = 1, \cdots, n$, which is the sequence involved in the problem as it was originally formulated. It follows from the above corollary that $v(p)$ is unchanged. Since $x_n(\cdot)$ is increasing in p, it follows that $v(\cdot)$ is increasing. The continuity of $v(\cdot)$ then follows from Theorem 3*.

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