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A SIMPLER PROOF OF SMITH'S ROULETTE THEOREM[1]

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A roulette-table is governed by two parameters w and r with $0 < w < r < 1$, where: w is the probability that a player who stakes a unit amount of money on a single hole on a particular spin of the wheel will, on that particular spin, win; and $1/r$ is the number of units that the house then returns to him if he wins that bet, that is, $(1/r) - 1$ is the amount that he gains from that bet. (In many real-world casinos, w is 1/38 and r is 1/36.)

How should someone with an infinitely divisible fortune play so as to maximize the probability of ultimately attaining a specified larger fortune? A step toward answering this question was made in [2], Chap. 6, where it was shown that bold play is optimal if a positive stake may be placed on only one hole on each spin. The second and final step was taken by Smith in [3] where he showed that (if w and r are reciprocals of integers) there is no advantage in placing positive stakes on more than one hole. (Theorem 1 below.)

The purpose of this note is to give a shorter and simpler proof of Smith's result. Though valid for all real w and r, $0 < w < r < 1$, the proof given here is in large measure simply a reorganization of Smith's. This simplification (and slight generalization) is achieved by establishing and exploiting (7), and (7) is an immediate consequence of this inequality:

PROPOSITION 1. *For every subfair casino function* U,

$$U(f/(1-f)) \geqq U(f)/(1-U(f)) \quad \textit{for} \quad 0 \leqq f \leqq \tfrac{1}{2}, \tag{1}$$

and, more generally, for each integer $n \geqq 1$,

$$U(f/(1-nf)) \geqq U(f)/(1-nU(f)) \quad \textit{for} \quad 0 \leqq f \leqq 1/(n+1). \tag{2}$$

PROOF. As was shown for primitive casino functions in [2], Chapter 6, and for all subfair casino functions in [1],

$$U(f+g) \geqq U(f) + U(g) \quad \text{for} \quad 0 \leqq f+g \leqq 1. \tag{3}$$

Moreover,

$$U(fg) \geqq U(f)U(g), \tag{4}$$

as was pointed out in [2], Chapter 4.

Hence, for $0 \leqq f \leqq \frac{1}{2}$,

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$$U(f/(1-f)) = U(f + f(f/(1-f)))$$
$$(5) \qquad \geqq U(f) + U(f(f/(1-f)))$$
$$\geqq U(f) + U(f)U(f/(1-f)),$$

which proves (1).

Suppose now that (2) holds for some n and that $0 \leqq (n+2)f \leqq 1$. Let $f^* = f/(1-nf)$, and calculate thus.

$$U(f/(1-(n+1)f)) = U(f^*/(1-f^*))$$
$$\geqq U(f^*)/(1-U(f^*))$$
$$(6) \qquad = U(f/(1-nf))[1 - U(f/(1-nf))]^{-1}$$
$$\geqq [U(f)/(1-nU(f))]$$
$$\cdot [1 - (U(f)/(1-nU(f)))]^{-1}$$
$$= U(f)[1-(n+1)U(f)]^{-1}.$$

This completes the proof.

Let $U_{w,r}$ be the U of the primitive casino $\Gamma_{w,r}$.

COROLLARY 1. *For all positive integers n with* $(n+1)r \leqq 1$,

$$(7) \qquad U_{w,r}(r/(1-nr)) \geqq w/(1-nw).$$

PROOF. Since bold play at r is plainly available in Γ, $U(r) \geqq w$. Therefore, the special case of (2) in which f equals r, implies (7).

Let $\Gamma' = \Gamma'_{w,r}$ be the roulette-table corresponding to w, r. (It is intended that Γ' be a casino in the technical sense of [2].) Every γ available in Γ' which stakes strictly positive amounts on precisely n holes is of *order n*. For n to be the order of a γ available in $\Gamma'_{w,r}$, nw cannot exceed 1.

Here is the main lemma, which is due to Smith [3].

LEMMA 1. *Let* $0 \leqq f \leqq 1$ *and let* $\gamma \varepsilon \Gamma'_{w,r}(f)$ *be of order* $n+1$. *Then there is a* $\gamma^* \varepsilon \Gamma'_{w,r}(f)$ *of order n such that* $\gamma U_{w,r} \leqq \gamma^* U_{w,r}$.

PROOF. If γ stakes positive amounts on $n+1$ distinct holes where $(n+1)r \geqq 1$, then the required γ^* is easily obtained by reducing each of these $n+1$ stakes by their minimum.

Suppose therefore that $(n+1)r < 1$, and let s_1 be the minimum of the $n+1$ positive stakes $s_1, \cdots, s_{n+1}$. Let t_i, $i = 1, \cdots, n+1$, be the gambler's fortune after the play if the ball falls in the hole on which he staked s_i, and let it be t_0 otherwise. There is a γ^* available at f such that

$$(8) \qquad \gamma^*\{t_i\} = \gamma\{t_i\} \quad \text{for} \quad 2 \leqq i \leqq n+1,$$

and which stakes positive amounts on only n holes. Namely, let $\alpha = r/(1-nr)$ and define γ^* thus. For each j, $2 \leqq j \leqq n+1$, γ^* stakes $s_j - \alpha s_1$ on that hole on which γ staked s_j, and γ^* stakes nothing on all other holes. If the ball fails to fall in one of the n holes on which positive stakes were placed, then the gambler's fortune decreases to $\alpha t_1 + (1 - \alpha)t_0$, an event which occurs with probability $1 - nw$, as is easily checked.

This is the required γ^*. Why? The only nontrivial point to verify is that $\gamma^* U \geqq \gamma U$. Introduce the momentary abbreviation τ for $\alpha t_1 + (1 - \alpha)t_0$ and verify that $\gamma^* U \geqq \gamma U$ if, and only if,

$$(9) \qquad (1 - nw)U(\tau) \geqq wU(t_1) + (1 - (n + 1)w)U(t_0).$$

Dividing both sides of (9) by $(1 - nw)$ and letting $\beta = w/(1 - nw)$, (9) becomes

$$(10) \qquad U(\alpha t_1 + (1 - \alpha)t_0) \geqq \beta U(t_1) + (1 - \beta)U(t_0).$$

Since U is a casino function, the left side of (10) is at least as large as $U(\alpha)U(t_1) + (1 - U(\alpha))U(t_0)$, as the casino inequality of [2], Chapter 4, states. Therefore, for (10) to hold, it certainly suffices that $U(\alpha) \geqq \beta$. But this is the content of (7).

LEMMA 2. *$U_{w,r}$ is excessive for $\Gamma'_{w,r}$, that is, $\gamma U_{w,r} \leqq U_{w,r}(f)$ for all f and all $\gamma \in \Gamma'_{w,r}(f)$.*

PROOF. Let $\gamma \in \Gamma'(f)$. As Lemma 1 implies, there is a $\gamma' \in \Gamma'(f)$ of order 1 such that $\gamma U \leqq \gamma' U$. But for such γ', $\gamma' \in \Gamma(f)$. Since U is the U of Γ, it is excessive for Γ, as is easily seen, for example, with the aid of Theorem 2.14.1 in [2]. So, $\gamma' U \leqq U(f)$. Consequently, $\gamma U \leqq U(f)$, so U is excessive for Γ'.

THEOREM 1. (*Smith*). *The U of the primitive casino $\Gamma_{w,r}$ is the U of the roulette-table $\Gamma'_{w,r}$.*

PROOF. Apply Lemma 2 together with the basic Theorem 2.12.1 in [2].

COROLLARY 2. (*Smith*). *Every strategy that is optimal for the primitive casino $\Gamma_{w,r}$ at f is optimal for the roulette-table $\Gamma'_{w,r}$ at f.*

Incidentally, the fact that $U_{w,r}$ is the primitive casino function $S_{w,r}$ of [2], or, equivalently, that bold play is optimal for subfair primitive casino functions, has not been used in this derivation of Theorem 1 and its Corollary.

REMARK. For inequalities (1) and (2) to hold, U may be any bounded solution to the special casino inequalities of Chapter 4 in [2], since any such U is superadditive, as shown in [1]. Moreover, for any such U, not only does (2) hold, but the dual inequality also holds. Namely, (2) is an instance of an inequality of the form

$$(11) \qquad U(\varphi(f)) \geqq \varphi(U(f)).$$

When such an inequality holds for a monotone increasing function φ it also holds for φ^* where

$$(12) \qquad \varphi^*(x) = 1 - \varphi(1 - x).$$

Similar phenomena were reported in [2] and in [1].

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