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ON A THEOREM BY DOBRUSHIN

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1. Introduction. In 1956 Dobrushin obtained in [2] an interesting theorem on the asymptotic convergence to the Poisson process of a randomly translated one-dimensional infinite particle system. Within the context of his Theorem 1 a necessary and sufficient condition for convergence was obtained.

It was pointed out by J. Goldman that Dobrushin's sufficiency proof was wrong, a serious error occuring in equation (17) of [2]. It turns out also that equation (22) used in giving a trivial proof of the necessary condition is wrong, and the necessary condition is incorrect as stated in the theorem.

In this paper we will correct these two errors of Dobrushin and also generalize the results to d-dimensions. Also we will consider analogous results when the sets may possess boundaries having positive measure.

Finally, we will give examples where the conditions are satisfied. These examples which have been treated in special cases by other authors, involve renewal theory, random walks, and processes with independent increments, and processes with random constant velocities.

In addition to the cited work of Dobrushin, related work has been done by Doob [3], Marayuma [6], Watanabe [10], Lamperti [5], Breiman [1], Thedeen [8], Goldman [4], and Warnshuis [9].

2. Definitions and statements of results. Let X denote a d-dimensional closed subgroup of R^d. With no loss of generality we can assume that X is of the form

$$X = \{x = (x^1, \cdots, x^d) \mid x^k \text{ are integers for } d_1 < k \leqq d\}.$$

Set $Z^d = \{x \mid x^k \text{ are integers for } 1 \leqq k \leqq d\}$. If $d_1 = d$, then $X = R^d$; and if $d_1 = 0$, then $X = Z^d$.

Set $\Delta_m = \{x \varepsilon X \mid 0 \leqq x^k < m \text{ for } 1 \leqq k \leqq d\}$ and set $\Delta = \Delta_1$. Set $Z_m{}^d = Z^d \cap \Delta_m$. Finally, set

$$U = \{u \varepsilon Z \mid u^k = 1 \quad \text{for some } k, \text{ and } u^j = 0 \quad \text{for } j \neq k\}.$$

Then U consists of d "unit vectors." For $0 \leqq a < \infty$ and $x \varepsilon X$, set $a \odot x = (ax^1, \cdots, ax^{d_1}, x^{d_1+1}, \cdots, x^d)$.

Let $| \; |$ denote Haar measure on X, defined as the product of Lebesgue measure on the first d_1 coordinates of X and counting measure on the last $d - d_1$ coordinates. Let $\mathcal{B}$ denote the collection of all relatively compact Borel subsets of X. All subsets of X considered later on will be members of $\mathcal{B}$. Let $\mathcal{A}$ denote the subcollection of sets $A \subset \mathcal{B}$ such that $|\partial A| = 0$.

Let the time parameter set T be given either as $T = \{0, 1, 2, \cdots\}$ or

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$T = [0, \infty)$. Consider random counting measures $N_A(t)$, $t \varepsilon T$ and $A \varepsilon \mathcal{B}$, where $N_A(t)$ denotes the "number of particles" in A at time t. Set $N_A = N_A(0)$. Let ν_t, $t \varepsilon T$, be measures defined by $\nu_t(A) = EN_A(t)$, and set $\nu = \nu_0$. The particles in X at time zero are assumed to be translated independently according to stochastic processes isomorphic to a fixed stochastic process $Y(t)$, $t \varepsilon T$. Let μ_t denote the distribution of Y_t. Then

$$\nu_t = \nu * \mu_t, \qquad t \varepsilon T. \tag{2.1}$$

Our first result depends only on the relation (2.1).

THEOREM 1. *In order that for some fixed* λ, $0 \leqq \lambda < \infty$, *and all measures* ν *such that*

$$\lim_{m\to\infty} \nu(x - \Delta_m)m^{-d} = \lambda \tag{2.2}$$

uniformly for $x \varepsilon X$, *it should follow that for each* $A \varepsilon \mathcal{A}$

$$\lim_{t\to\infty} (\nu * \mu_t)(x + A) = \lambda|A| \tag{2.3}$$

uniformly for $x \varepsilon X$, *it is necessary and sufficient that for each* $u \varepsilon U$ *and* $a > 0$,

$$\lim_{t\to\infty} \sum_{n \varepsilon Z^d} |\mu_t(a \odot (n + u + \Delta)) - \mu_t(a \odot (n + \Delta))| = 0. \tag{2.4}$$

Suppose for each $A \varepsilon \mathcal{B}$ *such that* $|A| = 1$, *and each* $a > 0$ *and compact set* C *of* X, *that*

$$\lim_{t\to\infty} \sum_{n \varepsilon Z^d} \sup_{x \varepsilon C} \cdot |\mu_t(a \odot (n + x + A)) - \mu_t(a \odot (n + \Delta))| = 0. \tag{2.5}$$

Then for any measure ν *satisfying* (2.2) *its follows that* (2.3) *holds for all* $A \varepsilon \mathcal{B}$.

Note that if $d = 1$ and $X = R$, then (2.4) reduces to

$$\lim_{t\to\infty} \sum_{n=-\infty}^{\infty} |\mu_t([an, a(n + 1))) - \mu_t([a(n - 1), an))| = 0;$$

if $d = 1$ and $X = Z$, then (2.4) reduces to

$$\lim_{t\to\infty} \sum_{n=-\infty}^{\infty} |\mu_t(\{n\}) - \mu_t(\{n - 1\})| = 0.$$

An important role in the proof of this theorem will be played by the following lemma, which minimizes the apparent difference between equations (2.4) and (2.5). The main purpose of the lemma is to correct the error of Dobrushin in his equation (17) of the proof of Theorem 1 of [2]. The proof of the necessity part of Theorem 1 corrects an error in equation (22) of the proof of Theorem 1 of [2]. Though our Theorem 1 is closely related to his, the real extension of Dobrushin's result comes in Theorem 2 below.

LEMMA 1. *Suppose* μ_t, $t \varepsilon T$, *is such that* (2.4) *holds. Then for each* $a > 0$ *and compact subset* C *of* X

$$\lim_{t\to\infty} \sum_{n \varepsilon Z^d} \sup_{x \varepsilon C} \cdot |\mu_t(a \odot (x + n + \Delta)) - \mu_t(a \odot (n + \Delta))| = 0.$$

If N_A, $A \varepsilon \mathcal{A}$, defines a Poisson process with parameter λ, then according to a theorem of Doob [3], this property is preserved for all time. We say that $N_A(t)$, $A \varepsilon \mathcal{A}$ ($A \varepsilon \mathcal{B}$), is asymptotically distributed as a Poisson process with parameter

λ if for all $n > 1$ and $A_1, \cdots, A_n \varepsilon \mathfrak{A}$ $(A_1, \cdots, A_n \varepsilon \mathfrak{B})$, the asymptotic joint distribution as $t \to \infty$ of $N_{A_1}(t), \cdots, N_{A_n}(t)$ is that of a Poisson process with parameter λ. The next theorem is an extension and correction of Theorem 1 of [2].

THEOREM 2. *In order that for some* λ, $0 \leqq \lambda < \infty$, *and all initial point processes* N_A *such that*

$$\lim_{m\to\infty} E\,|N_{x-\Delta_m} m^{-d} - \lambda| = 0 \tag{2.7}$$

uniformly for $x \varepsilon X$, *it should follow that* $N_A(t)$, $A \varepsilon \mathfrak{A}$, *is asymptotically distributed as a Poisson process with parameter* λ, *it is necessary and sufficient that* (2.4) *hold.*

If (2.5) *and* (2.7) *hold, then* $N_A(t)$, $A \varepsilon \mathfrak{B}$, *is asymptotically distributed as a Poisson process with parameter* λ.

The remaining results provide examples in which the conditions of the above theorems are satisfied.

EXAMPLE 1. (*Renewal process*). Let $d = 1$. Then either $X = R = (-\infty, \infty)$ or $X = Z = \{0, \pm 1, \pm 2, \cdots\}$. Let S_n, $-\infty < n < \infty$, be X-valued random variables such that $S_0 = 0$ and $S_n - S_{n-1}$, $-\infty < n < \infty$ are independent identically distributed random variables having finite positive mean $1/\lambda$. Let N_A denote the number of values of n such that $S_n \varepsilon A$. We say that N_A comes from a renewal process with finite positive mean drift $1/\lambda$.

THEOREM 3. *Let* N_A *come from a renewal process with finite positive mean drift* $1/\lambda$. *Then* (2.2) *and* (2.7) *hold.*

EXAMPLE 2. (*Random walks and processes with independent increments*). Suppose that $\mu_{s+t} = \mu_s * \mu_t$ for $s, t \varepsilon T$. Then $Y(t)$ is called a random walk or a process with independent increments according as $T = \{0, 1, 2, \cdots\}$ or $T = [0, \infty)$. We say that $Y(t)$ is non-degenerate if for no $t > 0$ is μ_t supported by a translate of a proper closed subgroup of X.

THEOREM 4. *Let* $Y(t)$ *define a non-degenerate random walk or process with independent increments. Then* (2.4) *holds. If some* μ_t *is non-singular with respect to* $|\;|$, *then* (2.5) *holds.*

EXAMPLE 3. (*Random constant velocity*). Let $X = R^d$. We say that $Y(t)$ has random constant velocity V if $Y(t) = Vt$, $t \varepsilon T$, where the random variable V is independent of T.

THEOREM 5. *Let* $Y(t)$ *have random constant velocity* V *such that the distribution of* V *is absolutely continuous with respect to Lebesgue measure on* R^d. *Then* (2.5) *holds.*

Maruyama [6] (completed by Watanabe [10]) considered the case when N_A comes from Example 1 and $Y(t)$ comes from Example 2. He gave a rather special proof of (2.3). He also stated that convergence to the Poisson process holds, as stated in the first part of our Theorem 2. In his proof of this result, however, he incorrectly assumed that the S_n's are independent (this error was also pointed out by J. Goldman).

Special cases of the results on Example 3 were obtained by Breiman [1], Thedeen [8], Goldman [4], and Warnshius [9]. Special cases of the results on Example 2 were obtained by Warnshius [9] as well as Marayuma [6] and Watanabe [10]. None of these papers, other than Dobrushin's consider the type of assumption on N_A by (2.7).

3. Proofs. We first prove Lemma 1 starting out with

LEMMA 2. *Let μ denote a probability measure on X. Then for $m \geqq 1$*

$$(3.1) \quad \sum_{n \varepsilon Z^d} \max_{k \varepsilon Z_m{}^d} \mu(mn + k + \Delta) \leqq m^{-d} + \sum_{n \varepsilon Z^d} \max_{u \varepsilon U} |\mu(n + u + \Delta) - \mu(n + \Delta)|$$

PROOF OF LEMMA 2. For $n \varepsilon Z^d$ choose $b_n \varepsilon Z_m{}^d$ such that

$$(3.2) \qquad \mu(mn + b_n + \Delta) = \max_{k \varepsilon Z_m{}^d} \mu(mn + k + \Delta).$$

Then for $k \varepsilon Z_m{}^d$

$$(3.3) \quad \mu(mn + b_n + \Delta) - \mu(mn + k + \Delta) \leqq \sum_{z \varepsilon Z_m{}^d} \max_{u \varepsilon U} |\mu(mn + z + u + \Delta) - \mu(mn + z + \Delta)|.$$

Consequently,

$$(3.4) \quad \sum_{n \varepsilon Z^d} \mu(mn + b_n + \Delta) \leqq \sum_{n \varepsilon Z^d} \mu(mn + k + \Delta) + \sum_{n \varepsilon Z^d} \max_{u \varepsilon U} |\mu(n + u + \Delta) - \mu(n + \Delta)|.$$

Summing over $k \varepsilon Z_m{}^d$, we get that

$$(3.5) \quad m^d \sum_{n \varepsilon Z^d} \mu(mn + b_n + \Delta) \leqq 1 + m^d \sum_{n \varepsilon Z^d} \max_{u \varepsilon U} |\mu(n + u + \Delta) - \mu(n + \mu)|.$$

Equation (3.1) follows from (3.5) by dividing both sides by m^d.

PROOF OF LEMMA 1. In proving (2.6) it suffices to consider the case $C = \Delta$. Consequently, with no loss of generality, we can assume that $d_1 = d$. Also we need only prove (2.6) for $a = 1$. Under these specializations (2.6) becomes

$$(3.6) \qquad \lim_{t \to \infty} \sum_{n \varepsilon Z^d} \sup_{x \varepsilon \Delta} |\mu_t(n + x + \Delta) - \mu_t(n + \Delta)| = 0.$$

Let m denote a positive integer. Then

$$\begin{aligned} &\limsup_{t \to \infty} \sum_{n \varepsilon Z^d} \max_{k \varepsilon Z_m{}^d} |\mu_t(n + km^{-1} + \Delta) - \mu_t(n + \Delta)| \\ &\quad = \limsup_{t \to \infty} \sum_{n \varepsilon Z^d} \max_{k \varepsilon Z_m{}^d} |\mu_t(m^{-1}(mn + k + m\Delta)) \\ &\qquad - \mu_t(m^{-1}(mn + m\Delta))| \\ &\quad \leqq \limsup_{t \to \infty} \sum_{n \varepsilon Z^d} \sum_{k \varepsilon Z_m{}^d} \max_{u \varepsilon U} |\mu_t(m^{-1}(mn + k + u + m\Delta)) \\ &\qquad - \mu_t(m^{-1}(mn + k + m\Delta))| \\ &\quad = \limsup_{t \to \infty} \sum_{n \varepsilon Z^d} \max_{u \varepsilon U} |\mu_t(m^{-1}(n + u + m\Delta)) \\ &\qquad - \mu_t(m^{-1}(n + m\Delta))| = 0 \end{aligned}$$

by (2.4) since U is a finite set. In other words,

$$(3.7) \quad \lim_{t \to \infty} \sum_{n \varepsilon Z^d} \max_{k \varepsilon Z_m{}^d} |\mu_t(n + km^{-1} + \Delta) - \mu_t(n + \Delta)| = 0.$$

By Lemma 2

$$\sum_{n\varepsilon Z^d} \max_{k\varepsilon Z_m^d} \mu_t(n + km^{-1} + m^{-1}\Delta) = \sum_{n\varepsilon Z^d} \max_{k\varepsilon Z_m^d} \mu_t(m^{-1}(mn + k + \Delta))$$
$$\leqq m^{-d} + \sum_{n\varepsilon Z^d} \max_{u\varepsilon U} |\mu_t(m^{-1}(n + u + \Delta)) - \mu_t(m^{-1}(n + \Delta))| \to m^{-d}$$

as $t \to \infty$. In other words,

$$\limsup_{t\to\infty} \sum_{n\varepsilon Z^d} \max_{k\varepsilon Z_m^d} \mu_t(n + km^{-1} + m^{-1}\Delta) \leqq m^{-d}. \tag{3.8}$$

Let V denote the set of 2^d points in Z^d whose coordinates are all zero or one. Choose $x \varepsilon \Delta$. Choose $k = (k^1, \cdots, k^d) \varepsilon Z_m^d$ such that $(k^i)/m \leqq x < (k^i + 1)/m$ for $i = 1, 2, \cdots, d$. Then

$$|\mu_t(n + x + \Delta) - \mu_t(n + km^{-1} + \Delta)|$$
$$\leqq 2^d(m + 1)^{d-1} \sum_{v\varepsilon V} \max_{k\varepsilon Z_m^d} \mu_t(n + v + km^{-1} + m^{-1}\Delta).$$

Thus by (3.8)

$$\limsup_{t\to\infty} \sum_{n\varepsilon Z^d} \sup_{x\varepsilon X} \min_{k\varepsilon Z_m^d} \cdot|\mu_t(n + x + \Delta) - \mu_t(n + km^{-1} + \Delta)| \leqq 4^d(m + 1)^{d-1}m^{-d}. \tag{3.9}$$

Choose $\epsilon > 0$. Choose m such that

$$4^d(m + 1)^{d-1}m^{-d} < \epsilon. \tag{3.10}$$

By (3.7), (3.9), and (3.10), there is a t_0 such that for $t \geqq t_0$

$$\sum_{n\varepsilon Z^d} \max_{k\varepsilon Z_m^d} |\mu_t(n + km^{-1} + \Delta) - \mu_t(n + \Delta)| \leqq \epsilon/2, \tag{3.11}$$

and

$$\sum_{n\varepsilon Z^d} \sup_{x\varepsilon\Delta} \min_{k\varepsilon Z_m^d} |\mu_t(n + x + \Delta) - \mu_t(n + km^{-1} + \Delta)| \leqq \epsilon/2. \tag{3.12}$$

Therefore for $t \geqq t_0$

$$\sum_{n\varepsilon Z^d} \sup_{x\varepsilon\Delta} |\mu_t(n + x + \Delta) - \mu_t(n + \Delta)| \leqq \epsilon. \tag{3.13}$$

Since ϵ can be made arbitrarily small, this completes the proof of (3.6) and hence also of Lemma 1.

PROOF OF SUFFICIENCY PART OF THEOREM 1. We first prove (2.3), assuming that (2.2) and (2.4) hold. It suffices to prove (2.3) for sets of the form

$$A = \{x \varepsilon X \mid 0 \leqq x^k < r \quad \text{for} \quad 1 \leqq k \leqq d,$$
$$\text{and} \qquad x^k = 0 \quad \text{for} \quad d_1 < k \leqq d\}.$$

By a change of scale, it suffices to prove (2.3) for $A = \Delta$.

By Lemma 1 we have that

$$\lim_{t\to\infty} \sum_{n\varepsilon Z^d} \sup_{y\varepsilon\Delta_m} |\mu_t(n + y + \Delta) - \mu_t(n + \Delta)| = 0. \tag{3.14}$$

Observe next that for $k \varepsilon Z_m^d$

$$\lim_{t\to\infty} |(\nu * \mu_t)(x + \Delta) - \sum_{n\varepsilon Z^d} \nu(x - mn - \Delta_m)\mu_t(mn + k + \Delta)|$$

$$= \lim_{t\to\infty} |\textstyle\sum_{n\varepsilon Z^d} \int_{mn+\Delta_m} \nu(x - dy)[\mu_t(y + \Delta) - \mu_n(mn + k + \Delta)]|$$

$$\leqq 2 \lim_{t\to\infty} \sup_{n\varepsilon Z^d} \nu(x - mn - \Delta_m)$$

$$\cdot \textstyle\sum_{n\varepsilon Z^d} \sup_{y\varepsilon\Delta_m} |\mu_t(mn + y + \Delta) - \mu_t(mn + \Delta)| = 0$$

uniformly for $x \varepsilon X$ by (3.14) and (2.2). Summing on $k \varepsilon Z_m^{\ d}$, we get that

$$(3.15) \quad \lim_{t\to\infty} [(\nu * \mu_t)(x + \Delta) - m^{-d} \textstyle\sum_{n\varepsilon Z^d} \nu(x - mn - \Delta_m)\mu_t(mn + \Delta_m)] = 0$$

uniformly for $x \varepsilon X$.

Choose $\epsilon > 0$. There is an $m > 0$ such that

$$(3.16) \quad \lambda - \epsilon \leqq \nu(x - mn - \Delta_m)/m^{-d} \leqq \lambda + \epsilon, \qquad x \varepsilon X \quad \text{and} \quad n \varepsilon Z^d.$$

Then

$$(3.17) \quad \lambda - \epsilon \leqq m^{-d} \textstyle\sum_n \nu(x - mn - \Delta_m)\mu_t(mn + \Delta_m) \leqq \lambda + \epsilon, \qquad x \varepsilon X.$$

Equation (2.3) follows from (3.15) and (3.17) since ϵ can be made arbitrarily small.

In proving that (2.3) holds for all $A \varepsilon \mathfrak{B}$ if (2.4) holds, we can assume that $|A| = 1$. The proof is then almost exactly, the same as that given above with Δ replaced by A in (3.14) and (3.15).

Proof of Sufficiency Part of Theorem 2. By arguments of Dobrushin [2] it suffices to prove that for $A \varepsilon \mathfrak{A}$ (or $A \varepsilon \mathfrak{B}$ in the second statement of Theorem 2)

$$(3.18) \qquad \lim_{t\to\infty} E| \textstyle\int N_{x-dy}\, \mu_t(y + A) - \lambda |A|| = 0$$

uniformly for $x \varepsilon X$ (the uniformity in X is not actually needed). By reductions similar to those used in the proof of Theorem 1, we need only consider A such that $|A| = 1$ and for each $m \geqq 1$

$$(3.19) \quad \lim_{t\to\infty} \textstyle\sum_{n\varepsilon Z^d} \sup_{y\varepsilon\Delta_m} |\mu_t(n + y + A) - \mu_t(n + \Delta)| = 0.$$

Observe that for $k \varepsilon Z_m^{\ d}$

$$\limsup_{t\to\infty} E| \textstyle\int N_{x-dy}\, \mu_t(y + A) - \sum_n N_{x-mn-\Delta_m}\, \mu_t(mn + k + \Delta)|$$

$$= \limsup_{t\to\infty} E\, |\textstyle\sum_{n\varepsilon Z^d} \int_{mn+\Delta_m} N_{x-dy}\, [\mu_t(y + A) - \mu_t(mn + k + \Delta)]|$$

$$\leqq 2 \sup_{n\varepsilon Z^d} EN(x - mn - \Delta_m)$$

$$\cdot \limsup_{t\to\infty} \textstyle\sum_{n\varepsilon Z^d} \sup_{y\varepsilon\Delta_m} |\mu_t(mn + y + \Delta) - \mu_t(mn + \Delta)| = 0$$

uniformly for $x \varepsilon X$. Summing over $k \varepsilon Z_m^{\ d}$, we get

$$(3.20) \quad \lim_{t\to\infty} E\, |\textstyle\int N_{x-dy}\, \mu_t(y + A) - m^{-d} \sum_{n\varepsilon Z^d} N_{x-mn-\Delta_m}\, \mu_t(mn + \Delta_m)| = 0$$

uniformly for $x \varepsilon X$.

Choose $\epsilon > 0$. By (2.7) there is an $m > 0$ such that

$$(3.21) \qquad E\, |\, N_{x-mn-\Delta_m}/m^{-d} - \lambda| \leqq \epsilon, \qquad x \varepsilon X \quad \text{and} \quad n \varepsilon Z^d.$$

Then for this m

$$(3.22)\quad E\,|m^{-d}\textstyle\sum_{n\varepsilon Z^d} N_{x-mn-\Delta_m}\,\mu_t(mn+\Delta_m)-\lambda| = E\,|\textstyle\sum_{n\varepsilon Z^d}(N_{x-mn-\Delta_m}/m^{-d}-\lambda)\mu_t(mn+\Delta_m)| \leqq \epsilon$$

uniformly for $x\,\varepsilon\,X$. Equation (3.18) now follows from (3.20), (3.22), and the fact that $|A| = 1$.

PROOF OF NECESSITY PART OF THEOREM 1. We are going to show that if

$$(3.23)\qquad \lim_{t\to\infty}(\nu*\mu_t)(a\odot\Delta) = \lambda\,|a\odot\Delta|$$

for all measures ν satisfying (2.2) and concentrated on $a\odot Z^d$, then (2.4) holds. We can assume that $a = 1$. The general case now reduces to the lattice case $d_1 = 0$ and $X = Z^d$.

Assume now that $d_1 = 0$, so that $X = Z^d$. For $n\,\varepsilon\,Z^d$, set $\nu(n) = \nu(\{n\})$ and $\mu_t(n) = \mu_t(\{n\})$, $t\,\varepsilon\,T$. We need to show that if

$$(3.24)\qquad \lim_{t\to\infty}\textstyle\sum_{n\varepsilon Z^d}\nu(-n)\mu_t(n+k) = \lambda,\qquad k\,\varepsilon\,Z^d,$$

for all ν satisfying (2.2), then

$$(3.25)\qquad \lim_{t\to\infty}\textstyle\sum_{n\varepsilon Z^d}|\mu_t(n+u)-\mu_t(n)| = 0,\qquad u\,\varepsilon\,U.$$

Note that if (3.24) holds for all ν satisfying (2.2), then

$$(3.26)\qquad \lim_{t\to\infty}\mu_t(n) = 0,\qquad n\,\varepsilon\,Z.$$

It also follows from (3.24) that for all $u\,\varepsilon\,U$

$$(3.27)\qquad \lim_{t\to\infty}\textstyle\sum_{n\varepsilon Z^d}\nu(-n)(\mu_t(n+u)-\mu_n(n)) = 0.$$

In order to obtain (3.25), we will suppose that (3.26) holds but for some $u\,\varepsilon\,U$, (3.25) doesn't hold. We will then show that (3.27) doesn't hold for this value of u.

Thus we suppose that for some fixed $u\,\varepsilon\,U$

$$(3.28)\qquad \limsup_{t\to\infty}\textstyle\sum_{n\varepsilon Z^d}|\mu_t(n+u)-\mu_t(n)| = 4c > 0.$$

Let T_1 denote the set of $t\,\varepsilon\,T$ such that

$$(3.29)\qquad \textstyle\sum_{n\varepsilon Z^d}|\mu_t(n+u)-\mu_t(n)| \geqq 2c.$$

Then T_1 is unbounded from above.

For $t\,\varepsilon\,T$, let P_t denote the collection of $n\,\varepsilon\,Z^d$ such that

$$\mu_t(n+u)-\mu_t(n)\geqq 0.$$

Let T_2 denote the subset of T_1 on which

$$(3.30)\qquad \textstyle\sum_{n\varepsilon S_t}(\mu_t(n+u)-\mu_t(n))\geqq c.$$

We can assume that T_2 is unbounded.

Let m be a positive integer to be chosen later. For $t\,\varepsilon\,T$ and $n\,\varepsilon\,Z^d$ let $x_n = x_n(t)$

be in $mn + Z_m^d$ and such that

(3.31) $\mu_t(x_n + u) - \mu_t(x_n) = \max_{x \varepsilon mn + Z_m^d} \mu_t(x + u) - \mu_t(x).$

Let ν^t denote the measure supported on $-S_t = \{-x_n(t) \mid n \varepsilon Z^d\}$ and defined by

$$\nu^t(\{-x_n\}) = m^d. \tag{3.32}$$

Then ν^t satisfies (2.1) with $\lambda = 1$. Set $\nu^t(n) = \nu^t(\{n\})$.

Let $P_t' = P^t \cap S_t$. Then for $t \varepsilon T_2$

$$\textstyle\sum_{n \varepsilon P_t'} \nu^t(-n)(\mu_t(n + u) - \mu_t(n)) \geqq c. \tag{3.33}$$

Let $Q_t' = (Z^d - P_t) \cap S_t$. Then by (3.31), the definition of U, and the fact that $\mu_t(x) \geqq 0$ for $x \varepsilon Z^d$, it follows that for $n \varepsilon Z^d$ and $x_n \varepsilon Q_t'$

$$\begin{aligned} \mu_t(mn + \Delta_m) \\ &\geqq \tfrac{1}{2}m^{d-1}(m - 1)(m - 2)[-(\mu_t(x_n + u) - \mu_t(x_n))]. \end{aligned} \tag{3.34}$$

Consequently

$$\begin{aligned} &\textstyle\sum_{n \varepsilon Q_t'} \nu^t(-n)(\mu_t(n + u) - \mu_t(n)) \\ &\qquad \geqq -[2m\,[(m - 1)(m - 2)]^{-1}]\textstyle\sum_n \mu_t(mn + \Delta_m) \\ &\qquad = -2m/[(m - 1)(m - 2)]^{-1}. \end{aligned} \tag{3.35}$$

Choose m such that

$$2m/[(m - 1)(m - 2)]^{-1} \leqq c/2.$$

Then from (3.33) and (3.35) we have that for $t \varepsilon T_2$

$$\begin{aligned} &\textstyle\sum_{n \varepsilon Z^d} \nu^t(-n)(\mu_t(n + u) - \mu_t(n)) \\ &\qquad \geqq c - 2m/[(m - 1)(m - 2)]^{-1} \geqq c/2. \end{aligned} \tag{3.36}$$

For $n \varepsilon Z^d$ set $|n| = ((n^1)^2 + \cdots + (n^d)^2)^{\frac{1}{2}}$. We next define an increasing sequence of points $t_1, t_2, \cdots$ in T_2. Choose $t_1 \varepsilon T_2$ and set $a_0 = 0$. Once a_{i-1} and t_i are chosen, choose $a_i \geqq a_{i-1} + i$ such that

$$\textstyle\sum_{|n| \geqq a_i} m^d|\mu_{t_i}(n + u) - \mu_{t_i}(n)| \leqq c/16. \tag{3.37}$$

Once t_i and a_i are chosen, choose $t_{i+1} \varepsilon T_2$ such that $t_{i+1} \geqq t_i + 1$ and

$$\textstyle\sum_{|n| < a_i} m^d|\mu_{t_{i+1}}(n + u) - \mu_{t_{i+1}}(n)| \leqq c/16. \tag{3.38}$$

(This is possible by (3.26).) Clearly,

$$\lim_{i \to \infty} t_i = \infty. \tag{3.39}$$

Define ν by

$$\nu(\{n\}) = \nu^{t_i}(\{n\}), \qquad a_{i-1} \leqq |n| < a_i \tag{3.40}$$

and set $\nu(n \mid = \nu(\{n\})$. Then ν satisfies (2.2) and for $i \geqq 1$

$$\begin{aligned}&\textstyle\sum_{n\varepsilon Z^d} \nu(-n)(\mu_{t_i}(n+u) - \mu_{t_i}(n))\\ &\quad \geqq \textstyle\sum_{a_{i-1}\leqq|n|<a_i} \nu^{t_i}(-n)(\mu_{t_i}(n+u) - \mu_{t_i}(n))\\ &\qquad - \textstyle\sum_{|n|<a_{i-1}} m^d|\mu_{t_i}(n+u) - \mu_{t_i}(n)| - \sum_{|n|\geqq a_i} m^d|\mu_{t_i}(n+u) - \mu_{t_i}(n)|\\ &\quad \geqq \textstyle\sum_{n\varepsilon Z^d} \nu^{t_i}(-n)(\mu_{t_i}(n+u) - \mu_{t_i}(n)) - 2m^d\sum_{|n|<a_i} |\mu_{t_i}(n+\mu) - \mu_{t_i}(n)|\\ &\qquad - 2m^d\textstyle\sum_{|n|\geqq a_i} |\mu_{t_i}(n+u) - \mu_{t_i}(n)| \geqq c/2 - c/8 - c/8 = c/4,\end{aligned}$$

by (3.36), (3.37), and (3.38).

In summary, ν satisfies (2.2), $t_i \to \infty$ and

$$\textstyle\sum_{n\varepsilon Z^d} \nu(-n)(\mu_{t_i}(n+u) - \mu_{t_i}(n)) \geqq c/4 > 0. \tag{3.41}$$

This contradicts (3.27) and completes the proof of the necessity of Theorem 1.

PROOF OF NECESSITY PART OF THEOREM 2. Suppose (2.4) doesn't hold. Then by the proof of the necessity part of Theorem 1, we can find a measure ν concentrated on $a \odot Z$ such that (2.2) holds for ν, but the equation

$$\lim_{t\to\infty} (\nu * \mu_t)(a \odot \Delta) = \lambda|a \odot \Delta| \tag{3.23}$$

doesn't hold.

We can construct a random counting measure N_A such that $EN_A = \nu(A)$, N_A is a Poisson distributed random variable and, for disjoint, $A_1, \cdots, A_n$, the random variables $N_{A_1}, \cdots, N_{A_n}$ are independent. Then $N_{a\odot\Delta}(t)$ is a Poisson distributed random variable with mean $(\nu * \mu_t)(a \odot \Delta)$. Since (3.23) doesn't hold $N_{a\odot\Delta}(t)$ can't possibly be asymptotically distributed as a Poisson variate with mean $\lambda|a \odot \Delta|$.

PROOF OF THEOREM 3. The main theorem of renewal theory implies that

$$\lim_{|x|\to\infty} EN_{x+\Delta} = \lambda. \tag{3.42}$$

Direct computations show that

$$\sup_{x\varepsilon X} EN^2_{x+\Delta} < \infty, \tag{3.43}$$

and

$$\lim_{|x-y|\to\infty} [EN_{x+\Delta}N_{y+\Delta} - EN_{x+\Delta}EN_{y+\Delta}] = 0. \tag{3.44}$$

It follows easily from (3.42)–(3.44) that

$$\lim_{m\to\infty} E((N_{-\Delta_m}m^{-1} - \lambda)^2) = 0 \tag{3.45}$$

uniformly for $x \varepsilon X$, and hence that (2.7) holds.

PROOF OF THEOREM 4. We begin with

LEMMA 3. *Suppose* (2.4) *holds and for some fixed $A \varepsilon \mathcal{B}$ with $|A| = 1$, that* (2.5) *holds. Then for this A and each $a > 0$ and compact subset C of X*

$$\begin{aligned}\lim_{t\to\infty} \textstyle\sum_{n\varepsilon Z^d} \sup_{x\varepsilon C} |(\varphi * \mu_t)(a \odot (n + x + A))&\\ - (\varphi * \mu_t)(a \odot (n + \Delta))| &= 0\end{aligned} \tag{3.46}$$

uniformly over all probability measures φ supported on X.

PROOF OF LEMMA 3. In proving (3.46) we need only consider the case $a = 1$, where (3.46) reduces to

$$(3.47)\quad \lim_{t\to\infty} \sum_{n\varepsilon Z^d} \sup_{x\varepsilon C} |(\varphi * \mu_t)(n + x + A) - (\varphi * \mu_t)(n + \Delta)| = 0$$

uniformly over all probability measures φ supported on X.

Let C_1 be a compact subset of X containing $C - \Delta$ and $-\Delta$. Then

$$\begin{aligned}
&\sum_{n\varepsilon Z^d} \sup_{x\varepsilon C} |(\varphi * \mu_t)(n + x + a) - (\varphi * \mu_t)(n + \Delta)| \\
&= \sum_{n\varepsilon Z^d} \sup_{x\varepsilon C} |\textstyle\int_X \varphi(dy)(\mu_t(n + x - y + A) - \mu_t(n - y + \Delta))| \\
&\leqq \textstyle\int_X \varphi(dy) \sum_{n\varepsilon Z^d} \sup_{x\varepsilon C} | \mu_t(n + x - y + A) - \mu_t(n - y + \Delta) \\
&= \sum_{k\varepsilon Z^d} \textstyle\int_{k+\Delta} \varphi(dy) \sum_{n\varepsilon Z^d} \sup_{x\varepsilon C} |\mu_t(n + x - y + A) - \mu_t(n - y + \Delta)| \\
&= \sum_{k\varepsilon Z^d} \textstyle\int_{k+\Delta} \varphi(dy) \sum_{n\varepsilon Z^d} \sup_{x\varepsilon X} |\mu_t(n + x - (y - k) + A) \\
&\quad - \mu_t(n - (y - k) + \Delta)| \leqq \sum_{k\varepsilon Z^d} \textstyle\int_{k+\Delta} \varphi(dy) \sum_{n\varepsilon Z^d} \sup_{x\varepsilon C_1} |\mu_t(n + x \\
&\quad + A) - \mu_t(n + \Delta)| + \sum_{k\varepsilon Z^d} \textstyle\int_{k+\Delta} \varphi(dy) \sum_{n\varepsilon Z^d} \sup_{x\varepsilon C_1} |\mu_t(n + x + \Delta) \\
&\quad - \mu_t(n + \Delta)| = \sum_{n\varepsilon Z^d} \sup_{x\varepsilon C_1} |\mu_t(n + x + A) - \mu_t(n + \Delta)| \\
&\quad + \sum_{n\varepsilon Z^d} \sup_{x\varepsilon C_1} |\mu_t(n + x + \Delta) - \mu_t(n + \Delta)| \to 0
\end{aligned}$$

as $t \to \infty$ uniformly over φ. This completes the proof of Lemma 3.

We return to the proof of Theorem 4. If $T = [0, \infty)$, then $\mu_t = \mu_{t-[t]}\mu_{[t]}$. Thus by Lemma 3 the general case is easily reduced to the random walk case. Assume then that $T = \{0, 1, 2, \cdots\}$.

We can write

$$(3.48)\quad \mu_1 = \tfrac{1}{2}(\varphi * \chi),$$

where φ and χ are probability measures supported on X, χ is non-degenerate and has finite covariance, and if some iterate of μ_1 is non-singular, then so is some iterate of χ. Note that

$$(3.49)\quad \mu_t = \mu_1^{(t)} = \sum_{s=0}^{t} \binom{t}{s} 2^{-t} \varphi^{(t-s)} * \chi^{(s)},$$

where $\chi^{(s)}$ denotes the s-fold convolution of χ with itself.

By using (3.49) and Lemma 3, we can reduce the general case to the case of finite covariance.

Suppose then that μ_1 has finite covariance. Let $\bar{\mu}_1$ and Σ denote its mean and covariance. Then by the local form of the central limit theorem (Stone [7]) for $A \varepsilon \mathfrak{A}$

$$(3.50)\quad \lim_{t\to\infty} [t^{d/2}\mu_t(x + A) - P_\Sigma((x - t\bar{\mu}_1)/t^{\frac{1}{2}})^{-1}|A|] = 0$$

uniformly for $x \varepsilon X$. If some μ_t is non-singular, then (3.50) is known to hold for

$A \varepsilon \mathfrak{B}$. Since $\Delta \varepsilon \mathfrak{A}$, (3.50) holds for $A = \Delta$. The conclusion of Theorem 4 follows very easily from (3.50).

PROOF OF THEOREM 5. Let $p(x)$, $x \varepsilon R^d$, denote the density of V. Then $Y(t)$ has density $t^{-d}p(t^{-1}y)$. We can find densities $p_k(x)$, $x \varepsilon R^d$, which are continuous and have compact support and are such that

$$\lim_{k\to\infty} \int_{R^d} |p_k(x) - p(x)|\, dx = 0. \tag{3.51}$$

Note that

$$\int_{R^d} |t^{-d}p_k(t^{-1}x) - t^{-d}p(t^{-1}x)|\, dx = \int_{R^d} |p_k(x) - p(x)|\, dx.$$

By (3.51) and (3.52) we can reduce the general case to the special case in which p is continuous and has compact support. Also in proving (2.5) we can take $a = 1$.

Equation (2.5) is now easily reduced to the result that for any compact subset C of R^d

$$\lim_{t\to\infty} t^{-d} \sum_{n \varepsilon Z^d} \sup_{x \varepsilon C} |p(t^{-1}(n + x)) - p(t^{-1}n)| = 0. \tag{3.53}$$

But (3.53) is clearly true whenever p is a continuous function having compact support.

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