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INTERPOLATION OF HOMOGENEOUS RANDOM FIELDS ON DISCRETE GROUPS[1]

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0. Introduction and summary. Interpolation and extrapolation of stationary stochastic processes has been extensively studied by Kolmogorov, Wiener and Masani, Krein and many others. Rozanov [3] has formulated many of their results and some of his own in a very neat form. In this paper some of the basic concepts and theorems related to interpolation are investigated in the more general setting of homogeneous random fields on locally compact abelian groups.

In the stationary case, the regularity and singularity of the process is determined by its behavior on the class of intervals $(-\infty, t]$. Here, since the group is not necessarily ordered, this class is replaced by an arbitrary family, I, of non-empty Borel sets of the group. Regularity and singularity are then defined in terms of the behavior of the field on the sets of I.

Theorems 4.1 and 5.1 generalize Kolmogorov's minimality problem [1] and an interpolation problem studied by Yaglom [6] to groups. Theorem 4.1 is also seen to include the result of Wang Shou-Jen on interpolation in R_K [5]. The family I_∞, introduced in Section 5, provides a natural generalization of the intervals $(-\infty, t]$ for certain processes.

1. Background. Let G be a locally compact abelian (LCA) group and G^* the dual group of G. Then G^* is also a LCA group under the compact-open topology ([2], [4]). Because of the duality between G and G^* (Pontrjagin's duality theorem [2], [4]) we will denote the characters of G by (g, x), $g \varepsilon G$, $x \varepsilon G^*$. The Borel field, $\mathcal{B}$, of G is the minimal σ-field generated by the closed subsets of G. Similarly, $\mathcal{B}^*$ is the Borel field of G^*. If G is discrete, G^* is compact and the characters of G^* are orthonormal in $L(G^*)$, the space of all complex-valued, absolutely integrable, $\mathcal{B}^*$-measurable functions.

Let $(\Omega, \mathcal{E}, P)$ be a probability space. For all $g \varepsilon G$, let $X(g) \varepsilon L_2(\Omega, \mathcal{E}, P)$. Then $X(g)$ is a random variable on Ω with finite second moment. We will assume that the first moment vanishes for all g. $L_2(\Omega)$ is a Hilbert space with the inner product

$$(1) \qquad (f_1, f_2) = Ef_1\bar{f}_2 = \int_\Omega f_1(\omega)\overline{f_2(\omega)}P(d\omega), \qquad f_1, f_2 \varepsilon L_2,$$

where E is the mathematical expectation. Let $H(X)$ be the linear completion of

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$\{X(g):g \varepsilon G\}$ in $L_2(\Omega)$. If the correlation function

$$B(g, g') = (X(g), X(g')) \tag{2}$$

depends only on $g - g'$, $X(g)$ will be called a homogeneous random field (HRF) on G.

The correlation function of a HRF is a positive-definite function on G. If $B(g)$ is continuous on G there exists (Bochner's theorem for LCA groups [4]) a unique, finite, nonnegative regular measure F, called the spectral measure of $X(g)$, on $\mathcal{B}^*$ such that for $g \varepsilon G$,

$$B(g) = \int_{G^*} (g, x)F(dx). \tag{3}$$

Then, [8]

$$X(g) = \int_{G^*} (g, x)Z(dx) \tag{4}$$

where Z is an orthogonal measure on $\mathcal{B}^*$ such that

$$(Z(E), Z(E')) = F(E \cap E') \tag{5}$$

for all Borel sets E, E' of G^*. Z is called the spectral stochastic measure.

Henceforth, we will assume that $X(g)$ has a continuous correlation function. The Radon-Nikodym derivative $f(x)$ of F with respect to the Haar measure $x(E)$ on G^*, if it exists, is called the spectral density of $X(g)$.

2. Linear transformations. Denote by $L_2(F)$ the Hilbert space of complex-valued measurable functions $p(x)$ on G^* which are square-integrable with respect to the measure F. Then

LEMMA 2.1. *There exists an isometric correspondence between the elements of $H(X)$ and those of $L_2(F)$ given by*

$$p(x) \leftrightarrow h = \int_{G^*} p(x)Z(dx). \tag{6}$$

PROOF. Define

$$T: (g, x) \leftrightarrow X(g), \qquad g \varepsilon G.$$

Then T is easily seen to be an isometry on the set of characters onto $\{X(g):g \varepsilon G\}$. T is extended to an isometry on the linear hulls of these sets and hence to an isometry on their closures. The closure of the latter set is $H(X)$ and the closure of the former is $L_2(F)$. Hence the proof is complete.

A HRF $Y(g)$ is said to be obtained by a linear transformation from the HRF $X(g)$ if there exists a $p(x) \varepsilon L_2(F)$ such that for all $g \varepsilon G$

$$Y(g) = \int_{G^*} p(x)(g, x)Z(dx) \tag{7}$$

when Z is the spectral stochastic measure of X. Following Rozanov [3] we easily prove

THEOREM 2.1. *Let $X(g)$ and $Y(g)$ be* HRF *on G which are mutually homogeneously correlated. Then* (7) *holds if, and only if, for all $E \varepsilon \mathcal{B}^*$*

$$F_{YY}(E) = \int_E |p(x)|^2 F(dx), \qquad F_{YX}(E) = \int_E p(x)F(dx) \tag{8}$$

where F_{YY} *is the spectral measure of* $Y(g)$ *and* F_{YX} *is the measure associated with*

$$B_{YX}(g) = (Y(g + g'), X(g')) \quad \textit{by } (3). \tag{9}$$

Another set of necessary and sufficient conditions for (7) is given by the following theorem.

THEOREM 2.2. *Let* $X(g)$ *and* $Y(g)$ *be* HRF *on* G. *Then* (7) *holds if, and only if* $H(Y) \subset H(X)$ *and* $X(g)$, $Y(g)$ *are mutually homogeneously correlated.*

PROOF. If (7) holds, the necessity is clear. Assume now that for all $g \varepsilon G$, $Y(g) \varepsilon H(X)$. Then by Lemma 2.1, for all g there exists a function $p(x, g) \varepsilon L_2(F)$ such that

$$Y(g) = \int_{G^*} p(x, g) Z(dx). \tag{10}$$

Since $Y(g)$ and $X(g)$ are mutually homogeneously correlated

$$Y(g), X(g')) = (Y(0), X(g' - g)).$$

Hence, using (10) and (4) we have

$$\int_{G^*} p(x, g)(-g', x) F(dx) = \int_{G^*} p(x, 0)(g - g', x) F(dx).$$

Then

$$\int_{G^*} (-g', x)[p(x, g) - p(x, 0)(g, x)] F(dx) = 0, \qquad g' \varepsilon G,$$

so that for $g \varepsilon G$ ([4], p. 17) $p(x, g) = p(x, 0)(g, x)$. Hence, letting $p(x) = p(x, 0)$, (7) holds.

3. Regularity and singularity. Now let I be any family of non-empty Borel sets of G. The HRF $Y(g)$ is said to be I-subordinate to the HRF $X(g)$ if

(i) $Y(g)$ comes from $X(g)$ by a linear transformation,

(ii) $H(Y; A) \subset H(X; A)$ for all $A \varepsilon I$ where $H(X; A)$ is the Hilbert space generated by $X(g)$, $g \varepsilon A$. A HRF $X(g)$ is called I-singular (deterministic with respect to I) if for all $A \varepsilon I$, $H(X; A) = H(X)$. That is,

$$S = \bigcap_{A \varepsilon I} H(X; A) = H(X). \tag{11}$$

The field is called I-regular (purely non-deterministic with respect to I) if $S = (0)$.

If the family I is closed under translations for all $g \varepsilon G$, (i.e., if $A \varepsilon I$, $g \varepsilon G$, then $A + g = \{g' + g : g' \varepsilon A\} \varepsilon I$) then we have the following decomposition theorem.

THEOREM 3.1. *Let* $X(g)$ *and* I *be as above. Then there exists a unique decomposition of* $X(g)$ *in the form*

$$X(g) = Y(g) + W(g) \tag{12}$$

where

(i) $Y = Y(I)$ *and* $W = W(I)$ *are* HRF *on* G,

(ii) Y *and* W *are* I*-subordinate to* X,

(iii) *Y and W are uncorrelated,*
(iv) *Y is I-regular and W is I-singular.*

PROOF. Since I is closed under translations it is easily shown that $U_g S = S$ where U_g is the unitary shift-operator:

$$U_g : X(g') \to X(g + g').$$

Define

$$W(g) = X'(g, S) \varepsilon S, \qquad Y(g) = X(g) - X'(g, S) \perp S$$

where $X'(g, S)$ is the projection of $X(g)$ onto S. Let $H(W) = S$. Then $H(Y) = H(X) - S$, the orthogonal complement of S with respect to $H(X)$.

The details of the proof are easily supply supplied (see [3], p. 54).

4. I_0-regularity and singularity. Suppose that G is a discrete group and I_0 the family of complements of singletons of G. Then a HRF $X(g)$ is either I_0-singular or I_0-regular. Kolmogorov's theorem on the minimality of a stationary stochastic sequence [1] is extended to groups by

THEOREM 4.1. *Let G be a discrete* LCA *group and $X(g)$ a* HRF *on G. Then $X(g)$ is I_0-regular if, and only if, the spectral density $f(x)$ exists and its reciprocal belongs to $L(G^*)$.*

PROOF. Set

$$p(x) = 1/f(x), \quad f(x) > 0,$$
$$= 0, \qquad f(x) = 0.$$

Then $p(x) \varepsilon L_2(F)$ and if for $g \varepsilon G$, we set

$$Y(g) = \int_{G^*} (g, x) p(x) Z(dx),$$

then $Y(g)$ is obtained from $X(g)$ by a linear transformation. Moreover

$$\|Y(g)\|^2 = \int_{G^*} |p(x)|^2 F(dx) \neq 0.$$

If we let $A(g) = \{g' \varepsilon G : g' \neq g\} \varepsilon I$, then, using the compactness of G^* we have

$$\begin{aligned}(Y(g), X(g')) &= \int_{G^*} (g, x) p(x) (-g', x) f(x) dx \\ &= \int_{G^*} (g - g', x)\, dx \\ &= 0\end{aligned}$$

so that $Y(g)$ is orthogonal to $X(g')$, $g' \neq g$. Thus for $g \varepsilon G$

$$0 \neq Y(g) \varepsilon H(X; A(g)).$$

Clearly then $S \neq H(X)$ and so $X(g)$ is not I_0-singular. Hence $X(g)$ is I_0-regular.

Conversely, if $X(g)$ is I_0-regular we may write

$$X(g) = X'(g) + W(g) \tag{13}$$

where

$$X'(g) \varepsilon H(X; A(g)), \qquad W(g) \perp H(X; A(g)).$$

It is easily calculated that

$$(14)\qquad \begin{aligned}(W(g), W(g')) &= 0, \quad g \neq g', \\ &= d^2 > 0, \quad g = g',\end{aligned} \quad \text{and} \quad \begin{aligned}(W(g), X(g')) &= 0, \quad g \neq g', \\ &= d^2, \quad g = g'.\end{aligned}$$

Then $W(g)$ is a HRF and is mutually homogeneously correlated with $X(g)$. By Theorem 2.2 there exists a function $p(x) \varepsilon L_2(F)$ such that $W(g) = \int_{G^*} (g, x)p(x)Z(dx)$. Moreover, from (8) $F_{WX}(E) = \int_E p(x)F(dx)$. From (14) it is seen that $(W(g), X(g')) = d^2 \int_{G^*} (g - g', x)\, dx$. But we also have $(W(g), X(g')) = \int_{G^*} (g - g', x)F_{WX}(dx)$ so that $F_{WX}(E) = d^2x(E)$. Clearly $x(E) = 1/d^2 \int_E p(x)F(dx)$. Thus the derivative $dx/F(dx) = p(x)/d^2$ is finite almost everywhere with respect to F and hence is finite almost everywhere with respect to the Haar measure x. Then $f(x)$ exists, is positive almost everywhere and $1/f(x) \varepsilon L(G^*)$.

If σ^2 is the error of interpolation, we have

$$\begin{aligned}\sigma^2 &= \|X(g) - X'(g)\|^2 \\ &= \|W(g)\|^2 \\ &= d^2 \\ &= p(x)f(x).\end{aligned}$$

However, we also have

$$\begin{aligned}\|W(g)\|^2 &= \int_{G^*} F_{WW}(dx) \\ &= \int_{G^*} |p(x)|^2 F(dx) \\ &= \int_{G^*} |d^2/f(x)|^2 F(dx) \\ &= d^4 \int_{G^*} (1/f(x))\, dx.\end{aligned}$$

Hence

$$(15)\qquad \sigma^2 = d^2 = \left(\int_{G^*} (1/f(x))\, dx\right)^{-1}$$

Then,

$$X'(g) = \int_{G^*} (1 - \sigma^2/f(x))(g, x)Z(dx).$$

5. I_∞-regularity and singularity. Assume that G is discrete and that the spectral density, $f(x)$, exists. Let I_∞ be the family of complements of finite subsets of G. Set $I = \{A\}$ where $A \varepsilon I_\infty$ and $G - A = \{g_0, \cdots, g_n\}$. Then

THEOREM 5.1. *$X(g)$ is I-non-singular if, and only if, there exists a non-zero trigonometric polynomial $p(x)$ on G^* of the form*

$$(16)\qquad p(x) = \sum_{k=0}^{n} c_k(g_k, x)$$

such that

$$(17)\qquad |p(x)|^2/f(x) \varepsilon L(G^*).$$

PROOF. It is clear that the HRF $X(g)$ is I-non-singular means that there is a $g \not\in A$, say g_0, such that $X(g_0) \not\in H(X; A)$. Then we may write (see (13)).

$$X(g_0) = X'(g) + W(g_0) \tag{18}$$

where $W(g_0) \neq 0$.

In $L_2(F)$ this becomes

$$(g_0, x) = q'(x) + q(x) \tag{19}$$

where $$q'(x) \in L_2(A), \qquad q(x) \perp L_2(A).$$

($L_2(A)$ is the linear completion of $\{(g, x): g \in A\}$ in $L_2(F)$.) Then for all $g \in A$

$$\int_{G^*} q(x)(-g, x) f(x)\, dx = (q(x), (g, x)) = 0,$$

so that

$$f(x)q(x) = \sum_{k=0}^{n} a(k)(g_k, x)$$

where

$$a(k) = \int_{G^*} q(x) f(x)(-g_k, x)\, dx.$$

Since $a(0) = \|q(x)\|^2 \neq 0$, $p(x) = f(x)q(x)$, is a non-zero polynomial of the desired form. Moreover,

$$\int_{G^*} (|p(x)|^2/f(x))\, dx = \int_{G^*} |q(x)|^2 f(x)\, dx = \|q(x)\|^2$$

so that (17) holds.

Conversely, suppose there exists a non-zero trigonometric polynomial satisfying (16) and (17). We may assume that the coefficient of (g_0, x) is 1. Let (18) and (19) hold. If $q(x)$ can be shown to be non-zero, then $X(g_0) \not\in H(X; A)$ and $X(g)$ is not I-singular.

Following Yaglom [6] we let Q be the set of all $h(x) \in L_2(F)$ such that $h(x) \perp L_2(A)$ and $\|h(x)\|^2 - (h(x), (g_0 x))$. Since, for $h(x) \in Q$, $\|(g_0, x) - h(x)\|^2 = 1 - \|h(x)\|^2$, $q(x)$ is that element of Q of maximum norm. Let c be a constant such that $0 < 1/c = \int_{G^*} (|p(x)|^2/f(x))\, dx < \infty$ and set $h(x) = cp(x)/f(x)$. Then $h(x)$ is easily shown to be a non-zero element of Q. Thus $\|q(x)\| \geqq \|h(x)\|$ and $q(x) \neq 0$. This completes the proof.

The proof of the following result is immediate.

COROLLARY. *A HRF $X(g)$ is I-singular if, and only if,*

$$|p(x)|^2/f(x) \not\in L(G^*) \tag{20}$$

for all non-zero trigonometric polynomials $g_0, g_1, \cdots, g_n$.

This will be true if the spectral density $f(x)$ has zeros of sufficiently high orders. It is also possible to derive a formula for the error of interpolation in this case. It turns out that

$$\sigma^2 = \|X(g_0) - X'(g_0)\|^2 = (\min \int_{G^*} (|p(x)|^2/f(x))\, dx)^{-1}$$

where the minimum is taken over those trigonometric polynomials $p(x)$ in $g_0, \cdots, g_n$ whose coefficient of (g_0, x) is unity. This is simply the group analogue of the result obtained by Yaglom [6].

The main result of this section follows easily.

THEOREM 5.2. *A* HRF $X(g)$ *on a discrete* LCA *group is* I_∞*-singular if, and only if,* $|p(x)|^2/f(x) \notin L(G^*)$ *for all non-zero trigonometric polynomials* $p(x)$ *on* G^*. $X(g)$ *is* I_∞*-regular if, and only if for all* $g_0 \in G$, *there is a trigonometric polynomial* $p(x) \in L_2(F)$ *which is not orthogonal to* (g_0, x) *and is such that* $|p(x)|^2/f(x) \in L(G^*)$.

PROOF. $X(g)$ is I_∞-singular whenever the field is completely determined when $X(g)$ is known at all but a finite number of elements of G, say $g_0, \cdots, g_n$. By the previous corollary then, $|p(x)|^2/f(x)$ is not integrable for all non-zero trigonometric polynomials in $g_0, \cdots, g_n$. Since $g_0, \cdots, g_n$ and n were arbitrary the first part of the theorem is clear.

$X(g)$ is I_∞-regular means that $S(I_\infty) = (0)$. This can happen only if for all g in G, there is a subset $A \in I_\infty$ such that $X(g) \notin H(X; A)$. Clearly $g \in G - A = \{g_0, \cdots, g_n\}$. Assume $g = g_0$. By the preceding corollary, there exists a non-zero trigonometric polynomial $p(x)$ in $g_0, \cdots, g_n$ such that $|p(x)|^2/f(x) \in L(G^*)$. Moreover $(p(x), (g_0, x))$ cannot be zero since $p(x)$ is non-zero.

Conversely, if $p(x)$ satisfies the conditions of the theorem let $p(x)$ have the form $\sum_{k=0}^{n} c_k(g_k, x)$. Set $A = G - \{g_0, \cdots, g_n\}$. Then $p(x)$ is a non-zero trigonometric polynomial in $g_0, \cdots, g_n$. Since $|p(x)|^2/f(x) \in L(G^*)$, $X(g_0) \notin H(X; A)$ and the field is regular.

COROLLARY. *If* $X(g)$ *is* I_∞*-singular it is* I_0*-singular and if* I_0*-regular it is* I_∞*-regular.*

6. Concluding remarks. When G is the additive group of integers, G is a LCA group. We may compactify G by adding the point at infinity, ∞. The neighborhood system of ∞ consists of all subsets of $G \cup \{\infty\}$ containing the point ∞ whose complements, as subsets of G, are finite. We then have

$$S = S(I_\infty) = \bigcap_{t,s} H(X; (-\infty, t) \cup (s, \infty)).$$

Ordinary regularity and singularity are determined by

$$S' = \bigcap_t H(X; (-\infty, t)).$$

There are many stochastic processes for which $S = S'$. For these, the concepts of I_∞-regularity and singularity coincide with the ordinary concepts. It would be interesting to find necessary and sufficient conditions for the equality of S and S'. In the following three cases this relation holds:

(i) when $X(t) = X(-t)$,

(ii) when $X(t)$ has a spectral density such that there exists positive constants m and M such that $m \leqq f(x) \leqq M$ a.e.,

(iii) when there exists $T > 0$ such that for $t \geqq T$, $B(t) = 0$.

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