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ON A THEOREM OF KARLIN REGARDING ADMISSIBLE ESTIMATES FOR EXPONENTIAL POPULATIONS

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1. Introduction. Let X be a random variable with the distribution function $dF_\omega = (\beta(\omega))^{-1} \exp(\omega x)\, d\mu(x)$, where μ is a σ-finite measure on the real line, and the parameter ω assumes values in the set $\Omega = \{\omega \mid \beta(\omega) < \infty\}$ which is an interval of the real line; let $\bar{\omega}$, $\underline{\omega}$ be the upper and lower end points respectively of Ω; $\bar{\omega}$, $\underline{\omega}$ may be finite or infinite and may or may not belong to Ω. This notation is the same as that of Karlin (1958), save for the trivial modification that the function $\beta(\omega)$ is the reciprocal of the function denoted by that symbol by Karlin. This change is made as the function defined by us is more convenient to deal with. x denotes a single observation of X. Karlin (1958) has considered the admissibility, with the squared error as the loss function, of linear estimates $x/(1 + \lambda)$ where $\lambda \geqq 0$, for the parameter $\theta = E_\omega(X) = \beta'(\omega)/\beta(\omega)$, and has proved the following results:

THEOREM 1.1. *If, ω' being any arbitrary interior point of Ω,*

$$\text{(a)} \quad \int_{\underline{\omega}}^{\omega'} \beta^\lambda(\omega)\, d\omega = \infty, \quad \textit{and} \quad (b) \quad \int_{\omega'}^{\bar{\omega}} \beta^\lambda(\omega)\, d\omega = \infty,$$

then the estimate $x/(1 + \lambda)$ is admissible for estimating $\theta = \beta(\omega)/\beta(\omega)$.

PROPOSITION 1.1. *The estimate $x/(1 + \lambda)$ is inadmissible for the parameter $\theta = \beta'(\omega)/\beta(\omega)$ if $\lambda < L_1$ or $\lambda > L_2$, where L_1 and L_2 are the infimum and supremum respectively as ω varies over Ω, of*

$$(1) \qquad I^2(\omega) = -(d/d\omega)\beta(\omega)/\beta'(\omega) = [\beta(\omega)\beta''(\omega) - \beta'^2(\omega)]/\beta'^2(\omega).$$

It is easily verified that the function $I^2(\omega)$ in (1) is identical with the function denoted by the same symbol by Karlin. Karlin has conjectured that the conditions in Theorem 1.1 are not merely sufficient, but are necessary also for the admissibility of $x/(1 + \lambda)$.

Whereas the criteria for admissibility in Theorem 1.1 depend on the behaviour of $\beta(\omega)$, only near the end points of Ω, $\underline{\omega}$ and $\bar{\omega}$, the criteria for inadmissibility in Proposition 1.1 depend on the variation of $\beta(\omega)$ over the whole interval Ω. It is therefore of interest to obtain criteria for inadmissibility of the estimate $x/(1 + \lambda)$, which depend on the behavior of $\beta(\omega)$ only at the end points of Ω. Proposition 2.1 proved in Section 2 of this paper gives such improved criteria and is the main result in this paper.

Then in Section 3, necessary conditions are obtained for the convergence and divergence of the two integrals which occur in Theorem 1.1, and using the improved criteria proved in Theorem 2.1, the range of values of λ, for which Karlin's conjecture remains open, is narrowed down.

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This question has also been considered recently by Richard Morton and M. Raghavachari (1966). Their result is included as a special case of our results.

2. Main result. For all $\omega \in \Omega$,

$$\beta(\omega) = \int_{-\infty}^{\infty} \exp(\omega x)\, d\mu(x) > 0,$$

and similarly,

$$\beta''(\omega) = \int_{-\infty}^{\infty} x^2 \exp(\omega x)\, d\mu(x) > 0. \tag{2}$$

We therefore have three possible cases.

(I) $\beta'(\omega)$ is always positive. In this case $\beta(\omega)$ is always increasing in Ω; it then follows from the definition of Ω that $\underline{\omega} = -\infty$, and further that $\mu[x < 0] = 0$ as otherwise $\beta(\omega)$ will $\rightarrow \infty$ as $\omega \rightarrow -\infty$.

(II) $\beta'(\omega)$ is always negative. In this case $\beta(\omega)$ is always decreasing, so that by the definition of Ω, $\bar{\omega} = +\infty$ and further $\mu[x > 0] = 0$.

(III) $\beta'(\omega)$ is negative initially but increases to a positive value. In this case $\beta(\omega)$ is a decreasing function at first until it reaches a minimum value and increases thereafter; $\bar{\omega}$ and $\underline{\omega}$ may be finite or infinite and $\mu[x < 0] > 0$ and also $\mu[x > 0] > 0$.

We now prove the following:

PROPOSITION 2.1. *The estimate* $x/(1 + \lambda)$ *is inadmissible for* $\theta = \beta'(\omega)/\beta(\omega)$,

(I) *when* $\beta'(\omega)$ *is positive for all* $\omega \in \Omega$, *if*,

(i) $\lambda > \limsup_{\omega\to-\infty} I^2(\omega)$ *or* (ii) $\lambda < \liminf_{\omega\to\bar{\omega}} I^2(\omega)$;

(II) *when* $\beta'(\omega)$ *is negative for all* $\omega \in \Omega$, *if*

(i) $\lambda > \limsup_{\omega\to+\infty} I^2(\omega)$ *or* (ii) $\lambda < \liminf_{\omega\to\underline{\omega}} I^2(\omega)$;

(III) *when* $\beta'(\omega)$ *assumes both positive and negative values, if*

(i) $\lambda < \liminf_{\omega\to\underline{\omega}} I^2(\omega)$ *or* (ii) $\lambda < \liminf_{\omega\to\bar{\omega}} I^2(\omega)$.

PROOF. We shall give the proof for case I in detail. The proof in the other case is on similar lines.

Taking the first condition in (I), let

$$\limsup_{\omega\to-\infty} I^2(\omega) = k_2, \tag{3}$$

and let λ be any number $> k_2$. Putting $\epsilon = \lambda - k_2 > 0$, it follows from (3) that we can find an ω_0 such that for all $\omega \leqq \omega_0$,

$$I^2(\omega) \leqq k_2 + \epsilon/2 = \lambda - \epsilon/2. \tag{4}$$

Putting

$$\gamma_0 = (1 + \lambda)^{-1}, \tag{5}$$

the estimate $\gamma_0 x$ will be inadmissible if there exist numbers $\gamma < 1$ and $a > 0$, such that for all $\omega \in \Omega$,

$$\int_0^a (\gamma x - \theta)^2 \exp(\omega x)\, d\mu(x) \leqq \int_0^a (\gamma_0 x - \theta)^2 \exp(\omega x)\, d\mu(x) \tag{6}$$

with the strict inequality holding for some $\omega \in \Omega$, for then the estimate $\gamma_0 x$ is uniformly inferior to the estimate $u(x)$ given by

$$u(x) = \gamma x, \qquad x \leqq a,$$
$$u(x) = \gamma_0 x, \qquad x > a.$$

We prove our result by showing that numbers γ and a exist for which (6) holds. Put,

$$R_\gamma = \int_0^a (\gamma x - \theta)^2 \exp(\omega x)\, d\mu(x), \tag{7}$$
$$\alpha(\omega) = \int_0^a \exp(\omega x)\, d\mu(x).$$

The value of a, will be fixed suitably later. In the following, where there is no scope for ambiguity we shall write for short, simply α, α', β, β' etc. in place of $\alpha(\omega)$, $\alpha'(\omega)$, $\cdots$, $\beta(\omega)$, $\beta'(\omega)$ etc.

Now two alternatives are possible, namely (A) there exists a number $a > 0$ such that $\mu\{x < a\} = 0$ or (B) there exists no number a such that $\mu\{x < a\} = 0$. Under alternative (A), the estimate $\gamma_0 \cdot x$ is obviously inadmissible, as it is uniformly inferior to the estimate

$$u(x) = \max[\gamma_0 x, a].$$

Hence under alternative (A), (i) of Case I is true. Next suppose that alternative (B) holds. From (7), we have, since $\theta = \beta'/\beta$

$$R_\gamma = \gamma^2\alpha'' - 2\gamma\alpha'\beta'/\beta + \beta'^2/\beta^2. \tag{8}$$

Hence R_γ is minimized for

$$\gamma = \alpha'\beta'/\alpha''\beta = \beta'/\beta \div \alpha''/\alpha'. \tag{9}$$

Putting $\rho(x) = \int_{a+}^\infty \exp(\omega x)\, d\mu(x)$, we have

$$\beta''/\beta' = (\alpha'' + \rho'')/(\alpha' + \rho'). \tag{10}$$

Now $\alpha'' = \int_0^a \exp(\omega x)\cdot x^2\, d\mu(x) \leqq a \int_0^a \exp(\omega x)\cdot x\, d\mu(x) = a\alpha'$

$$\therefore \alpha''/\alpha' \leqq a. \tag{11}$$

Similarly $\rho'' = \int_{a+}^\infty \exp(\omega x).\, x^2\, d\mu(x) \geqq a\rho'$

$$\therefore \rho'' \geqq a\rho'. \tag{12}$$

It follows from (10), (11) and (12) that

$$\beta''/\beta' \geqq \alpha''/\alpha' \quad \text{for all } \omega \in \Omega \quad \text{and all } a \geqq 0.$$

Hence in (9), using (1) and (4)

$$\alpha'\beta'/\alpha''\beta \geqq \beta'^2/\beta\cdot\beta'' = (1 + I^2(\omega))^{-1} \geqq (1 + \lambda - \epsilon/2)^{-1} \quad \text{for all } \omega \leqq \omega_0. \tag{13}$$

Next let,

$$\beta'(\omega_0)/\beta(\omega_0) = B_0, \tag{14}$$

and let a be given by,

(15) $$a = B_0(1 + \lambda - \epsilon/2).$$

Now, since $(d/d\omega)\beta'/\beta = (\beta\beta'' - \beta'^2)/\beta'^2 > 0$ for all ω, for all $\omega > \omega_0$, $\omega \varepsilon \Omega$, $\beta'/\beta > B_0$.

Hence using (11) and (15),

(16) $$\alpha'\beta'/\alpha''\beta > B_0/a = (1 + \lambda - \epsilon/2)^{-1} \quad \text{for all } \omega > \omega_0.$$

(13) and (16) imply that

$$\gamma_i = \inf_{\omega\epsilon\Omega} \alpha'\beta'/\alpha''\beta \geqq (1 + \lambda - \epsilon/2)^{-1} > (1 + \lambda)^{-1} = \gamma_0.$$

By an argument similar to that of Karlin, it follows that (6) holds if in its left hand side we substitute γ_i for γ. Thus the estimate

$$u(x) = \gamma_i x, \qquad x \leqq a,$$

$$u(x) = \gamma_0 x, \qquad x > a,$$

is uniformly superior to $\gamma_0 x$, and therefore the latter is inadmissible. The proof of (ii) in case (I), is on entirely similar lines except that to define $\alpha(\omega)$, we now cut off a segment from the upper end of the x-axis, i.e. we put

(17) $$\alpha(\omega) = \int_a^\infty \exp(\omega x)\, d\mu(x).$$

To avoid multiplicity of symbols we use the same symbol $\alpha(\omega)$ to denote this new function. Let

$$\lim \inf_{\omega\to\bar{\omega}} I^2(\omega) = K_1,$$

and let λ be any number $> K_1$. We now take ω_0, so that for all ω, $\omega_0 \leqq \omega < \bar{\omega}$,

(18) $$I^2(\omega) \geqq K_1 - \epsilon/2 = \lambda + \epsilon/2.$$

We again have two alternatives: (A′), there may exist a number $a > 0$, such that $\mu[x > a] = 0$, or (B′) there may exist no such number a. But under alternative (A′), it follows from the definition of Ω that $\bar{\omega} = +\infty$; it is then easily verified that $I^2(\omega) \to 0$ as $\to +\infty$. Hence (ii) of (I) is trivially true in this case as no λ satisfying the condition exists. Next suppose that alternative (B′) holds.

Defining $R\gamma$ as in (8), it is seen to be minimized as in (9), for

(19) $$\gamma_s = \beta'/\beta \div \alpha''/\alpha'.$$

By an argument similar to that from (9) to (12), we now obtain

$$\beta''/\beta' \leqq \alpha''/\alpha' \quad \text{for all } \omega \varepsilon \Omega, \quad \text{and all } a.$$

Hence for $\omega \geqq \omega_0$, $\omega \varepsilon \Omega$,

(20) $$\beta'\alpha'/\beta\cdot\alpha'' \leqq \beta'^2/\beta\beta' = (1 + I^2(\omega))^{-1} \leqq (1 + \lambda + \epsilon/2)^{-1} \quad \text{by (18)}.$$

Defining B_0 as in (14), and a by

(21) $$a = B_0(1 + \lambda + \epsilon/2),$$

we now have for $\omega < \omega_0$,

$$\beta'/\beta \leqq B_0, \qquad \alpha''/\alpha' \geqq a = B_0(1 + \lambda + \epsilon/2),$$

so that,

$$\beta'\alpha'/\beta\alpha'' \leqq (1 + \lambda + \epsilon/2)^{-1}. \tag{22}$$

By (20) and (22)

$$\gamma_s = \sup_{\omega\epsilon\Omega} \beta'\alpha'/\beta\alpha'' \leqq (1 + \lambda + \epsilon/2)^{-1} < (1 + \lambda)^{-1} = \gamma_0,$$

so that by an argument similar to the previous, the estimate

$$\begin{aligned} u(x) &= \gamma_s x, & x &\geqq a, \\ &= \gamma_0 \cdot x, & x &< a, \end{aligned}$$

is uniformly superior to $\gamma_0 x$ thus proving (ii) of (I).

The proofs in cases (II) and (III) are on identical lines except for minor changes to allow for the sign of β.

3. Karlin's integrals. Necessary conditions for the convergence or divergence of the integrals in Karlin's Theorem 1.1 are easily obtained. Let I_1 denote the integral in condition (a) and I_2 the integral in condition (b) of Theorem 1.1. As the same method is applicable to all the cases, we shall consider only case (I), i.e. β' always positive, and prove the following:

PROPOSITION 3.1. *If* $\beta'(\omega) > 0$ *for all* $\omega \varepsilon \Omega$, *then the integral* I_1,

(i) *converges if* $\lambda > \limsup_{\omega\to-\infty} I^2(\omega)$ *and*

(ii) *diverges if* $\lambda < \liminf_{\omega\to-\infty} I^2(\omega)$.

Similarly the integral I_2

(iii) *converges if* $\lambda < \liminf_{\omega\to\bar{\omega}} I^2(\omega)$, *and*

(iv) *diverges if* $\lambda > \limsup_{\omega\to\bar{\omega}} I^2(\omega)$.

Proof. Let

$$k_1 = \liminf_{\omega\to-\infty} I^2(\omega). \tag{23}$$

Suppose $k_1 > 0$, and let λ be any number less than k_1. Let

$$\lambda = k_1 - \epsilon, \qquad \epsilon > 0, \tag{24}$$

(23) implies that we can obtain ω_0 such that

$$I^2(\omega) \geqq k_1 - \epsilon/2 \quad \text{for all} \quad \omega \leqq \omega_0. \tag{25}$$

Hence by (1),

$$-(d/d\omega)\beta/\beta' \geqq \lambda + \epsilon/2 \quad \text{for} \quad \omega \leqq \omega_0. \tag{26}$$

Integrating both sides of (26) with respect to ω from an arbitrary point ω upto the point ω_0, we get on putting $C_0 = \beta(\omega_0)/\beta'(\omega_0) > 0$,

$$\beta/\beta' \geqq C_0 + (\lambda + \epsilon/2)(\omega_0 - \omega) \quad \text{for all } \omega \leqq \omega_0,$$

and hence

(27) $$\beta'/\beta \leqq [C_0 + (\lambda + \epsilon/2)(\omega_0 - \omega)]^{-1}.$$

Again integrating both sides of (27) with respect to ω, from an arbitrary point $\omega < \omega_0$ upto ω_0, we get, putting $\beta_0 = \beta(\omega_0)$,

$$\log \beta_0/\beta \leqq ((\lambda + \epsilon/2))^{-1} \log \{1 + (\lambda + \epsilon/2)C_0^{-1}(\omega_0 - \omega)\},$$

and hence

(28) $$\beta^\lambda \geqq \beta_0^\lambda\{1 + C_0^{-1}(\lambda + \epsilon/2)(\omega_0 - \omega)\}^{-\lambda/(\lambda+\epsilon/2)} \quad \text{for all } \omega \leqq \omega_0.$$

As the integral of the right hand side of (28) over $(-\infty, \omega_0)$ diverges, the integral of the left hand side, i.e. I_1 must also diverge. This proves (ii) of the proposition.

(i) is proved exactly similarly. Let

$$k_2 = \lim \sup_{\omega\to-\infty} I^2(\omega)$$

and let $\lambda = k_2 + \epsilon$, $\epsilon > 0$.

It is then shown that eventually,

(29) $$\beta \leqq \beta_0[1 + (\lambda - \epsilon/2)C_0^{-1}(\omega_0 - \omega)]^{-\lambda/(\lambda-\epsilon/2)}$$

and the convergence of the integral on the right hand side of (29) at $-\infty$, implies the convergence of I_1.

Next consider (iii). $\bar{\omega}$ may be $+\infty$ or may be finite. If $\bar{\omega}$ is $+\infty$, the proof is exactly similar to that given above, and need not be repeated here.

Suppose next that $\bar{\omega}$ is finite. As $\omega \to \bar{\omega}$, $\beta(\omega)$ being non-decreasing, may either (A) diverge to infinity or (B) converge to a finite limit. In case (A) $\log \beta$ also $\to \infty$, as $\omega \to \bar{\omega}$ and hence as $\bar{\omega}$ is finite, $(d/d\omega) \log \beta = \beta'/\beta$ must diverge to ∞ as $\omega \to \bar{\omega}$. In case (B), the definition of the set Ω, implies that $\beta(\omega) = \infty$ for any $\omega > \bar{\omega}$, and arbitrarily close to $\bar{\omega}$. Hence $\beta'(\omega)$ must $\to\infty$ as $\omega \to \bar{\omega}$. As $\beta(\omega)$ converges to a finite limit $\beta'/\beta \to \infty$ as $\omega \to \bar{\omega}$. Thus in both cases (A) and (B), as $\omega \to \bar{\omega}$, $\beta'/\beta \to \infty$ and hence $\beta/\beta' \to 0$.

Let

(30) $$K_1 = \lim \inf_{\omega\to\bar{\omega}} I^2(\omega)$$

and let λ be any number $< K_1$. Put

(31) $$\lambda = K_1 - \epsilon, \qquad \epsilon > 0.$$

(30) implies that we can find $\omega_0 < \bar{\omega}$, such that for all ω, $\omega_0 \leqq \omega < \bar{\omega}$,

(32) $$-(d/d\omega)\beta/\beta' = I^2(\omega) \geqq K_1 - \epsilon/2 = \lambda + \epsilon/2.$$

Integrating both sides of (32), with respect to ω, from an arbitrary point $\omega < \bar{\omega}$ to the point $\bar{\omega}$, we get, by the first mean value theorem of analysis that since $\beta/\beta' = 0$ at $\omega = \bar{\omega}-0$,

(33) $$\beta/\beta' \geqq (\lambda + \epsilon/2)(\bar{\omega} - \omega), \qquad \omega_0 \leqq \omega < \bar{\omega}.$$

On integrating both sides of (33) with respect to ω, from ω_0 to a point $\omega < \bar{\omega}$, we get after a little reduction

$$\beta^{\lambda} \leqq \beta_0^{\lambda}(\bar{\omega} - \omega_0)^{\lambda/(\lambda+\epsilon/2)}(\bar{\omega} - \omega)^{-\lambda/(\lambda+\epsilon/2)} \tag{34}$$

where $\beta_0 = \beta(\omega_0)$ as before.

As the integral of the right hand side of (34) converges, so does the integral of the left hand side, i.e. I_2. This proves (iii). (iv) is proved exactly similarly.

On combining Proposition 2.1 and 3.1, the range of values of λ for which Karlin's conjecture remains open is narrowed down. Thus at the end point $-\infty$, the integral I_1 can converge only if $\lambda \geqq k_1 = \liminf_{\omega\to-\infty} I^2(\omega)$ and if $\lambda > k_2 = \limsup_{\omega\to-\infty} I^2(\omega)$, then the estimate $x/(1 + \lambda)$ is proved inadmissible. Hence the conjecture remains open only for the range of values of λ given by

$$k_1 \leqq \lambda \leqq k_2 . \tag{35}$$

Similarly at the upper end point $\bar{\omega}$ the Karlin's conjecture remains open only for the range

$$K_1 \leqq \lambda \leqq K_2 , \tag{36}$$

K_1 and K_2 being respectively the inferior and superior limits of $I^2(\omega)$ as $\omega \to \bar{\omega}$. Since the estimate is inadmissible if $\lambda > k_2$ or $\lambda < K_1$, the range in which the conjecture remains open for I_1 is further reduced to

$$\max [k_1 , K_1] \leqq \lambda \leqq k_2 \tag{37}$$

and the range for I_2 to,

$$K_1 \leqq \lambda \leqq \min [k_2 , K_2]. \tag{38}$$

Cases in which $I^2(\omega)$ oscillates as ω tends to an end point $-\infty$, or $\bar{\omega}$, are rather exceptional. In most cases $I^2(\omega)$ converges as ω tends to an end point. In such cases, suppose,

$$\lim_{\omega\to-\infty} I^2(\omega) = k, \qquad \omega \lim_{\omega\to\bar{\omega}} I^2(\omega) = K.$$

Then if $\lambda < k$, I_1 diverges and if $\lambda > k$, $x/(1 + \lambda)$ is inadmissible; if $\lambda > K$, I_2 diverges and if $\lambda < K$, $x/(1 + \lambda)$ is inadmissible. In this case the conjecture remains open only if $K < k$, and then only for the two points $\lambda = k$, if I_1 converges and $\lambda = K$, if I_2 converges.

REMARK 3.1. The result proved by Richard Morton and Raghavachari (1966) is a special case of the above result. They have given a sufficient condition which ensures that the integrals I_1 and I_2 diverge for the critical values $\lambda = k$ and $\lambda = K$. It follows from our general result that for such distributions Karlin's conjecture must hold good.

We have dealt with only Case I in Proposition 3.1. It may be easily verified that in Case II, the results at the upper end point $+\infty$ correspond to (i) and (ii) and at the lower end point $\underline{\omega}$ to (iii) and (iv) of Proposition 3.1 and that in

Case III, the results at both end points $\bar{\omega}$, and $\underline{\omega}$ correspond to (iii) and (iv) of Proposition 3.1.

4. An illustration. The following is a simple illustration of the application of our results. Let the measure be such that $d\mu(x) = 0$ for $x < 0$ and for $x \geqq 0$, $d\mu(x) = \{\exp(-ax) - \exp(-bx)\} \cdot dx$, where $b > a > 0$. The distribution falls under Case I. It is easily found that $\beta(\omega) = [(a - \omega)(b - \omega)]^{-1}$, $\underline{\omega} = -\infty$, and $\bar{\omega} = a$. The integral I_1 diverges only for $\lambda \leqq \frac{1}{2}$ and I_2 diverges only for $\lambda \geqq 1$. Hence according to Karlin's conjecture, the estimate $x/(1 + \lambda)$ should be inadmissible for every $\lambda \geqq 0$. But it is easily found that in this case $L_1 = \frac{1}{2}$ and $L_2 = 1$, so that the inadmissibility of the estimate $x/(1 + \lambda)$ cannot be derived from Karlin's Proposition 1.1. But it immediately follows from our sharpened result, because in this case, as may be easily found $k = L_1 = \frac{1}{2}$ and $K = L_2 = 1$. Hence by Theorem 2.1 the estimate $x/(1 + \lambda)$ is inadmissible for $\lambda > k = \frac{1}{2}$ or $\lambda < K = 1$ and hence for every λ.

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