The Annals of Mathematical Statistics
1969, Vol. 40, No. 1, 71–78

ON ERLANG'S FORMULA[1]

By Lajos Takács

Case Western Reserve University

1. Introduction. The following mathematical model of telephone traffic has some importance in designing telephone exchanges.

In the time interval $(0, \infty)$ calls are arriving at a telephone exchange in accordance with a Poisson process of density λ, that is, if $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ denote the arrival times, then $\tau_n - \tau_{n-1} (n = 1, 2, \cdots; \tau_0 = 0)$ are mutually independent random variables having a common distribution function

$$F(x) = 1 - e^{-\lambda x} \quad \text{if} \quad x \geqq 0, \qquad = 0 \quad \text{if} \quad x < 0. \tag{1}$$

There are m available lines. If an arriving call finds a free line, then a connection is realized without delay. If every line is busy when a call arrives, the call is lost. The holding times are mutually independent, positive random variables having a common distribution function $H(x)$, and a finite expectation

$$\alpha = \int_0^\infty x \, dH(x). \tag{2}$$

The holding times are also independent of the arrival times and the initial state. The initial state is given by the number of busy lines at time $t = 0$ and by the remaining lengths of the holding times in progress at time $t = 0$.

If we choose the initial distribution in such a way that the process becomes stationary, then the probability that at time t the number of busy lines in k is given by Erlang's formula,

$$P_k = [(\lambda\alpha)^k/k!][\textstyle\sum_{j=0}^{m} (\lambda\alpha)^j/j!]^{-1} \tag{3}$$

for $k = 0, 1, \cdots, m$ and all $t \geqq 0$.

This formula has an interesting history. In 1917 A. K. Erlang [1] deduced formula (3) for the case when the holding times are constant α. While Erlang's result is correct, his proof is not complete. He has made use of a property of the process which is far from evident. Erlang noted also that if the holding times have an exponential distribution, that is,

$$H(x) = 1 - e^{-x/\alpha} \quad \text{for} \quad x \geqq 0, \qquad = 0 \quad \text{for} \quad x < 0, \tag{4}$$

then (3) is valid. If $H(x)$ is an exponential distribution function, then Erlang's proof is acceptable.

Received 16 October 1967.

[1] This research was sponsored by the National Science Foundation under Contract No. NSF-GP-7847.

Erlang's investigations automatically raised the problem of whether (3) remains valid also for an arbitrary $H(x)$. The answer is affirmative and this was established first in 1927 by A. E. Vaulot [11], and subsequently by F. Pollaczek [8], C. Palm [7], L. Kosten [5], K. Lundkvist [6], A. Y. Khinchin [4] and B. A. Sevastyanov [9]. For other appropriate references see R. Syski [10] pp. 271–278.

There is an essential difference between the cases of an exponential $H(x)$ and an arbitrary $H(x)$. If we denote by $\nu(t)$ the number of busy lines at time t, then $\{\nu(t), 0 \leqq t < \infty\}$ is a Markov process if and only if $H(x)$ is an exponential distribution function. If $H(x)$ is given by (4), then it is easy to show that

$$\lim_{t\to\infty} \mathbf{P}\{\nu(t) = k\} = P_k \tag{5}$$

exists, is independent of the distribution of $\nu(0)$, and $P_k(k = 0, 1, \cdots, m)$ is given by (3). If we assume that $\mathbf{P}\{\nu(0) = k\} = P_k(k = 0, 1, \cdots, m)$, then $\{\nu(t), 0 \leqq t < \infty\}$ becomes a stationary process for which $\mathbf{P}\{\nu(t) = k\} = P_k$ for all $t \geqq 0$. That is for a stationary process, $P_k(k = 0, 1, \cdots, m)$ is the probability that at time t the number of busy lines is k.

In the general case, $\{\nu(t), 0 \leqq t < \infty\}$ is not a Markov process. However, if we introduce auxiliary variables we can achieve that the process becomes Markovian. If $\nu(t) = k$ $(k = 1, 2, \cdots, m)$, then let $\chi_1(t), \chi_2(t), \cdots, \chi_k(t)$ be a random permutation of the remaining lengths of the k holding times in progress at time t. We suppose that all the $k!$ permutations are equally probable. Then the process $\{\nu(t), \chi_1(t), \cdots, \chi_{\nu(t)}(t); 0 \leqq t < \infty\}$ is a Markov process. It can be shown that

$$\lim_{t\to\infty} \mathbf{P}\{\nu(t) = k, \chi_1(t) \leqq x_1, \cdots, \chi_k(t) \leqq x_k\} = P_k \prod_{i=1}^{k} H^*(x_i) \tag{6}$$

for $k = 0, 1, \cdots, m$ and $x_1 \geqq 0, x_2 \geqq 0, \cdots, x_m \geqq 0$, where

$$H^*(x) = \begin{cases} \alpha^{-1} \int_0^x [1 - H(u)]\, du & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{7}$$

and P_k is defined by (3). The limit (6) is independent of the distribution of $(\nu(0), \chi_1(0), \cdots, \chi_{\nu(0)}(0))$. If we suppose that the distribution of $(\nu(0), \chi_1(0), \cdots, \chi_{\nu(0)}(0))$ agrees with the limiting distribution (6), then $\{\nu(t), \chi_1(t), \cdots, \chi_{\nu(t)}(t)\}$ becomes a stationary process for which $\nu(t)$ has the same distribution for all $t \geqq 0$, namely, $\mathbf{P}\{\nu(t) = k\} = P_k$ for $k = 0, 1, \cdots, m$ and all $t \geqq 0$.

We note that in some of the papers mentioned above only $\{P_k\}$ has been found. F. Pollaczek [8], K. Lundkvist [6] and A. Y. Khinchin [4] proved only Erlang's formula (3). C. Palm [7], L. Kosten [5], and B. A. Sevastyanov [9] proved also (6). Actually they interpreted $\chi_1(t), \cdots, \chi_{\nu(t)}(t)$ as the lengths of the past durations of the holding times in progress at time t. The interpretation used in this paper has also been used by S. Erlander [2].

As far as the proof of (6) is concerned, it consists of two parts. First, the ergodicity of the process should be proved. This is based on the use of rather deep theorems. Second, it requires the solution of a system of integral equations and the proof of the existence and the uniqueness of the solution. Most of the proofs mentioned above are intuitive and contain heuristic reasoning. From the mathematical point of view the proof of B. A. Sevastyanov [9] is the most satisfactory.

There is one thing that is worth special mention. The probability (3) is usually interpreted as the probability that an arriving call finds k lines busy in a stationary process. Let us denote by ν_n the number of lines that the nth arriving call finds busy. If we suppose that the process $\{\nu(t), \chi_1(t), \cdots, \chi_{\nu(t)}(t)\}$ is stationary, then the distribution of ν_n depends on n, that is, $\{\nu_n\}$ is not a stationary sequence. If we want to be exact, the latter interpretation of probability (3) fails. However, in most of the applications, our interest is in finding the probability that an arriving call finds k lines busy, or, in particular, all the m lines busy.

In what follows we shall study the limiting distribution of ν_n as $n \to \infty$. By choosing a suitable initial distribution we shall define a process for which ν_n has the same distribution for all $n = 1, 2, \cdots$, that is, for which $\{\nu_n\}$ is a stationary sequence.

It turns out that if we investigate the distribution of ν_n $(n = 1, 2, \cdots)$ instead of $\nu(t)(0 \leqq t < \infty)$, then everything becomes very simple. To prove the ergodicity of $\{\nu_n\}$ we need to refer only to an elementary theorem of recurrent events and to find the limiting distribution of ν_n as $n \to \infty$ we need to use only integration by parts.

It is interesting to note that by this shift of attention from the time-dependent behavior of the process to the arrival-dependent behavior, all mathematical difficulties disappear.

2. General holding times. Consider the mathematical model of telephone traffic formulated at the beginning of the Introduction. Let ν_n be the number of lines busy immediately before the arrival of the nth call. If $\nu_n = k$ $(k = 1, 2, \cdots, m)$, then let $(\chi_{n1}, \cdots, \chi_{nk})$ be a random permutation of the remaining lengths of the k holding times in progress at time τ_n. We suppose that all the $k!$ permutations are equally probable. Now we shall prove that the following theorem holds.

THEOREM. *We have*

$$(8) \quad \lim_{n\to\infty} \mathbf{P}\{\nu_n = k, \chi_{n1} > x_1, \cdots, \chi_{nk} > x_k\} = P_k \prod_{i=1}^{k} [1 - H^*(x_i)]$$

for $k = 0, 1, \cdots$ *and* $x_1 \geqq 0, x_2 \geqq 0, \cdots, x_m \geqq 0$, *where*

$$(9) \qquad P_k = [(\lambda\alpha)^k/k!][\textstyle\sum_{j=0}^{m} (\lambda\alpha)^j/(j!)]^{-1}$$

for $k = 0, 1, \cdots, m$ *and*

$$(10) \qquad H^*(x) = \alpha^{-1} \int_0^x [1 - H(u)]\, du \quad \textit{for} \quad x \geqq 0,$$
$$= 0 \qquad\qquad \textit{for} \quad x < 0.$$

The limit (8) *is independent of the initial state.*

PROOF. The vector sequence $\xi_n = (\nu_n, \chi_{n1}, \cdots, \chi_{n\nu_n})$, $(n = 1, 2, \cdots)$ is a discrete parameter Markov process. First we shall show that the distribution defined by (8) is a stationary distribution of the process, that is, if we assume that ξ_n has the distribution (8), then it follows that ξ_{n+1} has the same distribution. (8) is indeed a stationary distribution, if the following equations hold:

$$(11)\quad P_0 = P_0 \int_0^\infty H(y)e^{-\lambda y}\lambda\, dy + \cdots + P_{m-1} \int_0^\infty [H^*(y)]^{m-1}H(y)e^{-\lambda y}\lambda\, dy + P_m \int_0^\infty [H^*(y)]^m e^{-\lambda y}\lambda\, dy,$$

and for $k = 1, 2, \cdots, m$ and $x_1 \geqq 0, x_2 \geqq 0, \cdots, x_m \geqq 0$,

$$\begin{aligned}(12)\quad & P_k[1 - H^*(x_1)] \cdots [1 - H^*(x_k)] \\ &= k^{-1}P_{k-1} \int_0^\infty [1 - H^*(x_1 + y)] \cdots [1 - H^*(x_k + y)] \\ &\quad \cdot \{\textstyle\sum_{i=1}^k [1 - H(x_i + y)][1 - H^*(x_i + y)]^{-1}\}e^{-\lambda y}\lambda\, dy \\ &\quad + \cdots + P_{m-1} \int_0^\infty [1 - H^*(x_1 + y)] \cdots [1 - H^*(x_k + y)] \\ &\quad \cdot \{\tbinom{m-1}{k}[H^*(y)]^{m-1-k}H(y) + k^{-1}\tbinom{m-1}{k-1}[H^*(y)]^{m-k} \\ &\quad \cdot \textstyle\sum_{i=1}^k [1 - H(x_i + y)][1 - H^*(x_i + y)]^{-1}\}e^{-\lambda y}\lambda\, dy \\ &\quad + P_m\tbinom{m}{k} \int_0^\infty [1 - H^*(x_1 + y)] \cdots [1 - H^*(x_k + y)][H^*(y)]^{m-k}e^{-\lambda y}\lambda\, dy\end{aligned}$$

where $P_k (k = 0, 1, \cdots, m)$ is defined by (9).

In obtaining the right hand side of (11) we took into consideration that the event that the $(n + 1)$st call finds no busy line can occur in the following ways: $\tau_{n+1} - \tau_n = y$ where $0 \leqq y < \infty$, $\nu_n = r$ where $r = 0, 1, \cdots, m$ and the remaining lengths of the current holding times at τ_n and the length of the holding time beginning at τ_n (if $r < m$) are all $\leqq y$.

In obtaining the right hand side of (12) we took into consideration that the event that the $(n + 1)$st call finds k lines busy and the remaining lengths of the current holding times at time τ_{n+1} are greater than $x_1, x_2, \cdots, x_k$ respectively, can occur in the following ways: $\tau_{n+1} - \tau_n = y$ where $0 \leqq y < \infty$, $\nu_n = r$ where $r = k - 1, k, \cdots, m$ and among the remaining lengths of the current holding times at time τ_n and the length of the holding time beginning at τ_n (if $r < m$) k are greater than $x_1 + y, \cdots, x_k + y$ respectively and all the others are $\leqq y$.

Now we have to check whether the left hand sides of (11) and (12) are equal to the corresponding right hand sides or not. It is easy to show that both (11) and (12) hold for any $m = 1, 2, \cdots$ and $x_1 \geqq 0, \cdots, x_m \geqq 0$. It will be convenient to use the following abbreviations for fixed $x_1 \geqq 0, \cdots, x_m \geqq 0$,

$$(13)\quad A_j(y) = [1 - H^*(x_1 + y)] \cdots [1 - H^*(x_j + y)]$$

if $j = 1, 2, \cdots, m$, and

$$(14)\quad B_j(y) = [H^*(y)]^j$$

for $j = 0, 1, 2, \cdots$. Then we have

$$(15)\quad dA_j(y)/dy = -A_j(y)\alpha^{-1} \textstyle\sum_{i=1}^j [1 - H(x_i + y)][1 - H^*(x_i + y)]^{-1},$$

and

$$(16)\quad dB_j(y)/dy = j[1 - H(y)]\alpha^{-1}B_{j-1}(y).$$

By using the above notation, (11) and (12) can be written in the following equivalent forms:

$$(17)\quad 1 = \int_0^\infty H(y)e^{-\lambda y}\lambda\, dy + \cdots + (\lambda\alpha)^{m-1}[(m-1)!]^{-1} \cdot \int_0^\infty B_{m-1}(y)H(y)e^{-\lambda y}\lambda\, dy + (\lambda\alpha)^m(m!)^{-1}\int_0^\infty B_m(y)e^{-\lambda y}\lambda\, dy$$

and for $k = 1, 2, \cdots, m,$

$$(18)\quad \begin{aligned}(\lambda\alpha)^k(k!)^{-1}A_k(0) &= -(\lambda\alpha)^k(k!)^{-1}\int_0^\infty (dA_k(y)/dy)e^{-\lambda y}\, dy + \cdots \\ &+ (\lambda\alpha)^{m-1}[(m-1)!]^{-1}\int_0^\infty [\tbinom{m-1}{k}A_k(y)B_{m-k-1}(y)H(y) \\ &- \alpha k^{-1}\tbinom{m-1}{k-1}(dA_k(y)/dy)B_{m-k}(y)]e^{-\lambda y}\lambda\, dy \\ &+ (\lambda\alpha)^m(m!)^{-1}\tbinom{m}{k}\int_0^\infty A_k(y)B_{m-k}(y)e^{-\lambda y}\lambda\, dy.\end{aligned}$$

First, we prove (17). If $m = 1$, then (17) reduces to

$$(19)\quad \lambda\alpha\int_0^\infty B_1(y)e^{-\lambda y}\lambda\, dy = \int_0^\infty [1 - H(y)]e^{-\lambda y}\lambda\, dy = \alpha\int_0^\infty (dB_1(y)/dy)e^{-\lambda y}\lambda\, dy,$$

which can be seen to be true by integrating by parts. Now we shall prove by mathematical induction that (17) is true for all $m = 1, 2, \cdots$. Suppose that (17) is true for m $(m = 1, 2, \cdots)$. The difference between the right hand side of (17) for $m + 1$ and for m is

$$(20)\quad (\lambda\alpha)^{m+1}[(m+1)!]^{-1}[\int_0^\infty B_{m+1}(y)e^{-\lambda y}\lambda\, dy - \int_0^\infty (dB_{m+1}(y)/dy)e^{-\lambda y}\, dy] = 0,$$

which follows by integrating by parts. Thus (17) holds also for $m + 1$. Accordingly (17) is true for all $m = 1, 2, \cdots$.

Second, we prove (18). If $m = k$, then (18) reduces to

$$(21)\quad (\lambda\alpha)^k(k!)^{-1}A_k(0) = (\lambda\alpha)^k(k!)^{-1}[\int_0^\infty A_k(y)e^{-\lambda y}\lambda\, dy - \int_0^\infty (dA_k(y)/dy)e^{-\lambda y}\, dy]$$

which is evidently true. Now we shall prove by mathematical induction that (18) is true for all $m = k, k + 1, \cdots$. Suppose that (18) is true for m $(m = k, k + 1, \cdots)$. The difference between the right hand side of (18) for $m + 1$ and m is

$$(22)\quad \begin{aligned}(\lambda\alpha)^{m+1}[(m+1)!]^{-1}[\int_0^\infty A_k(y)B_{m+1-k}(y)e^{-\lambda y}\lambda\, dy \\ - \int_0^\infty (dA_k(y)/dy)B_{m+1-k}(y)e^{-\lambda y}\, dy \\ - \int_0^\infty A_k(y)(dB_{m+1-k}(y)/dy)e^{-\lambda y}\, dy] = 0\end{aligned}$$

which follows again by integrating by parts. Thus (18) holds also for $m + 1$. Accordingly (18) is true for all $m = k, k + 1, \cdots$.

We can conclude that $\xi_n = (\nu_n, \chi_{n1}, \cdots, \chi_{n\nu_n})$ $(n = 1, 2, \cdots)$ has a stationary distribution defined by (8). If we suppose that ξ_1 has the distribution defined by (8), then every ξ_n $(n = 1, 2, \cdots)$ will have the same distribution as ξ_1.

We observe that the event that an arriving call finds all the m lines free is a recurrent event. This event is aperiodic and it occurs at the nth arrival if $\nu_n = 0$. Consequently,

$$\lim_{n\to\infty} \mathbf{P}\{\nu_n = 0\} \tag{23}$$

exists, and is independent of the initial state. If, in particular, $\{\xi_n\}$ is the stationary sequence considered above, then we have seen that $\mathbf{P}\{\nu_n = 0\} = P_0$ for all $n = 1, 2, \cdots$. Thus it follows that the limit (23) is necessarily P_0. Accordingly, the recurrent event is persistent, and the mean recurrence time is $1/P_0$. (See W. Feller [3] Chapter XIII.) This fact implies that the limiting distribution of ξ_n is independent of the initial distribution. For any time when an arriving call finds all the m lines free, the future stochastic behavior of the process is the same independently of the past. In a particular case we found the limiting distribution of ξ_n, namely for the stationary process defined above. Hence, regardless of the initial distribution, ξ_n has the limiting distribution (8). This completes the proof of the Theorem.

Evidently (8) is the unique stationary distribution of $\{\xi_n\}$. We note that $\{\xi_n\}$ becomes a stationary sequence if we suppose that the distribution of $(\nu(0), \chi_1(0), \cdots, \chi_{\nu(0)}(0))$ is as follows: $\mathbf{P}\{\nu(0) = 0\} = 0$,

$$\begin{aligned} \mathbf{P}\{\nu(0) &= k, \chi_1(0) > x_1, \cdots, \chi_k(0) > x_k\} \\ &= k^{-1}P_{k-1}[1 - H^*(x_1)] \cdots [1 - H^*(x_k)] \\ &\quad \cdot \textstyle\sum_{i=1}^{k} [1 - H(x_i)][1 - H^*(x_i)]^{-1} \end{aligned} \tag{24}$$

for $k = 1, 2, \cdots, m - 1$ and

$$\begin{aligned} \mathbf{P}\{\nu(0) &= m, \chi_1(0) > x_1, \cdots, \chi_m(0) > x_m\} \\ &= [1 - H^*(x_1)] \cdots [1 - H^*(x_m)] \\ &\quad \cdot \{m^{-1}P_{m-1} \textstyle\sum_{i=1}^{m} [1 - H(x_i)][1 - H^*(x_i)]^{-1} + P_m\}. \end{aligned} \tag{25}$$

In this case the probability that the nth arriving call finds k lines busy is given by P_k ($k = 0, 1, \cdots, m$) for every $n = 1, 2, \cdots$.

3. Exponential holding times. If we suppose that the holding times have an exponential distribution

$$\begin{aligned} H(x) &= 1 - e^{-\mu x} && \text{if} \quad x \geqq 0, \\ &= 0 && \text{if} \quad x < 0, \end{aligned} \tag{26}$$

and ν_n denotes the number of busy lines immediately before the arrival of the nth call, then we can see easily that $\{\nu_n\}$ is a Markov chain. This Markov chain is homogeneous, irreducible and aperiodic. The transition probabilities $\mathbf{P}\{\nu_{n+1} = k \mid \nu_n = j\} = p_{jk}$ can be expressed as follows:

$$p_{jk} = \binom{j+1}{k} \int_0^\infty (1 - e^{-\mu x})^{j+1-k} e^{-k\mu x} e^{-\lambda x} \lambda \, dx \tag{27}$$

for $j = 0, 1, \cdots, m - 1$ and $p_{m,k} = p_{m-1,k}$.

In this case the limiting distribution $\lim_{n\to\infty} \mathbf{P}\{\nu_n = k\} = P_k$ $(k = 0, 1, \cdots, m)$ exists, is independent of the initial distribution and $\{P_k\}$ is the only stationary distribution. Now we are going to find $\{P_k\}$. We note that if we know the binomial moments

$$B_r = \sum_{k=r}^{m} \binom{k}{r} P_k \tag{28}$$

for $r = 0, 1, \cdots, m$, then $\{P_k\}$ can be obtained by

$$P_k = \sum_{r=k}^{m} (-1)^{r-k} \binom{r}{k} B_r \qquad (k = 0, 1, \cdots, m). \tag{29}$$

For the determination of the binomial moments $B_0, B_1, \cdots, B_m$ we can deduce a recurrence formula. If $\nu_n = j$ and $\tau_{n+1} - \tau_n = x$, then ν_{n+1} has a Bernoulli distribution $B(j + 1, e^{-\mu x})$ for $j = 0, 1, \cdots, m - 1$ and $B(m, e^{-\mu x})$ for $j = m$. Thus

$$\begin{aligned} &\mathbf{E}\{\tbinom{\nu_{n+1}}{r} \mid \nu_n = j\} \\ &= \tbinom{j+1}{r} \int_0^\infty e^{-r\mu x - \lambda x} \lambda\, dx = \tbinom{j+1}{r} \lambda(\lambda + r\mu)^{-1} \quad \text{if} \quad j = 0, 1, \cdots, m-1, \\ &= \tbinom{m}{r} \int_0^\infty e^{-r\mu x - \lambda x} \lambda\, dx = \tbinom{m}{r} \lambda(\lambda + r\mu)^{-1} \quad \text{if} \quad j = m. \end{aligned} \tag{30}$$

If $\{\nu_n\}$ has a stationary distribution $\{P_k\}$, then $\mathbf{E}\{\binom{\nu_n}{r}\} = B_r$ for all $n = 1, 2, \cdots$. Then by (30) we obtain that

$$(\lambda + r\mu)B_r = \lambda B_r + \lambda B_{r-1} - \lambda \binom{m}{r-1} B_m, \tag{31}$$

or, in a simpler form,

$$r\mu B_r = \lambda B_{r-1} - \lambda \binom{m}{r-1} B_m \tag{32}$$

for $r = 1, 2, \cdots, m$.

If we put (32) into (29) and we use (28), then we can write that

$$P_k = \lambda(\mu k)^{-1} P_{k-1} \tag{33}$$

for $k = 1, 2, \cdots, m$, whence it follows that

$$P_k = [(\lambda/\mu)^k/k!][\sum_{j=0}^{m} (\lambda/\mu)^j (j!)^{-1}]^{-1} \tag{34}$$

for $k = 0, 1, \cdots, m$. This is in agreement with (9) because now $\alpha = 1/\mu$ is the expectation of the holding times.

We note that if we assume that $\mathbf{P}\{\nu(0) = 0\} = 0$, $\mathbf{P}\{\nu(0) = k\} = P_{k-1}$ for $k = 1, 2, \cdots, m - 1$ and $\mathbf{P}\{\nu(0) = m\} = P_{m-1} + P_m$, then $\{\nu_n\}$ becomes a stationary Markov chain for which $\mathbf{P}\{\nu_n = k\} = P_k$ $(k = 0, 1, \cdots, m)$ for all $n = 1, 2, \cdots$. In this case $\{\nu(t), 0 \leqq t < \infty\}$ is not a stationary Markov process. Conversely, if $\{\nu(t), 0 \leqq t < \infty\}$ is a stationary Markov process, then $\{\nu_n\}$ is not a stationary sequence.

4. Infinitely many lines. Suppose that in the telephone exchange every arriving call realizes a connection without delay, that is, there is no lost call. In this case m, the number of available lines, is infinite. Denote by ν_n the number of busy lines immediately before the arrival of the nth call. The distribution of ν_n

can easily be found. If at time $t = 0$ there is no busy line, then we have

$$(35)\quad \mathbf{P}\{\nu_{n+1} = k\} = \binom{n}{k}\lambda^{n+1}(n!)^{-1}\int_0^\infty e^{-\lambda x}\left[\int_0^x H(u)\,du\right]^{n-k}\left\{\int_0^x [1 - H(u)]\,du\right\}^k dx.$$

In case of any other initial state, (35) needs only obvious modifications. Regardless of the initial distribution we have the following limit:

$$(36)\quad \lim_{n\to\infty} \mathbf{P}\{\nu_n = k\} = e^{-\lambda\alpha}(\lambda\alpha)^k(k!)^{-1}$$

for $k = 0, 1, 2, \cdots$.

If $\nu_n = k$, then let $\chi_{n1}, \cdots, \chi_{nk}$ be a random permutation of the remaining lengths of the k holding times in progress at the arrival of the nth call. Suppose that all the $k!$ permutations are equally probable. If we suppose that at time $t = 0$ there is no busy line, then for $x_1 \geqq 0, \cdots, x_k \geqq 0$ we have

$$(37)\quad \mathbf{P}\{\nu_{n+1} = k, \chi_{n+1,1} > x_1, \cdots, \chi_{n+1,k} > x_k\} = \binom{n}{k}\lambda^{n+1}(n!)^{-1}\int_0^\infty e^{-\lambda x}\left[\int_0^x H(u)\,du\right]^{n-k}\left\{\prod_{i=1}^k \int_0^x [1 - H(u + x_i)]\,du\right\} dx,$$

whence

$$(38)\quad \lim_{n\to\infty} \mathbf{P}\{\nu_n = k, \chi_{n1} > x_1, \cdots, \chi_{nk} > x_k\} = e^{-\lambda\alpha}(\lambda\alpha)^k(k!)^{-1}\prod_{i=1}^k [1 - H^*(x_i)],$$

where $H^*(x)$ is defined by (10). If we consider an arbitrary initial distribution, then (37) needs only obvious modifications and (38) remains valid regardless of the initial distribution.

REFERENCES

[1] Erlang, A. K. (1917). Solution of some problems in the theory of probabilities of significance in automatic telephone exchanges. (Danish) *Elektroteknikeren* **13** 5–13 [English translation in the *P. O. Elect. Eng. J.* **10** (1917–1918) 189–197.]

[2] Erlander, Sven (1967). A note on telephone traffic with losses. *J. Appl. Prob.* **4** 406–408.

[3] Feller, W. (1957). *An Introduction to Probability Theory and its Applications.* Vol. I. 2nd Edition. Wiley, New York.

[4] Khinchin, A. Y. (1963). Erlang's formulas in the theory of mass service. *Theor. Probability Appl.* **7** 320–325.

[5] Kosten, L. (1948–1949). On the validity of the Erlang and Engset loss formulae. *Het P.T.T. Bedriff* **2** 22–45.

[6] Lundkvist, K. (1953). General theory for telephone traffic. *Ericsson Technics* **9** No. 2 111–140.

[7] Palm, C. (1938). Analysis of the Erlang traffic formulae for busy-signal arrangements. *Ericsson Technics* No. 4 39–58.

[8] Pollaczek, F. (1932). Lösung eines geometrischen Wahrscheinlichkeitsproblems. *Math. Zeitschrift* **35** 230–278.

[9] Sevastyanov, B. A. (1957). An ergodic theorem for Markov processes and its application to telephone systems with refusals. *Theor. Probability Appl.* **2** 104–112.

[10] Syski, R. (1960). *Introduction to Congestion Theory in Telephone Systems.* Oliver and Boyd, Edinburgh.

[11] Vaulot, A. E. (1927). Extension des formules d'Erlang au cas où les durées des conversations suivent une loi quelconque. *Rev. gén. d'électricité* **22** 1164–1171.