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ON MOMENTS OF THE MAXIMUM OF NORMED PARTIAL SUMS[1]

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1. Introduction and summary. Let $X, X_1, X_2, \cdots$ be independent random variables with $E(X_n) = 0 \quad (n \geqq 1)$, and put $S_n = X_1 + \cdots + X_n \quad (n \geqq 1)$. Marcinkiewicz and Zygmund [5] and Wiener [8] have shown that if the X's have a common distribution, then

$$E\{\sup_n |S_n/n|\} < \infty \tag{1}$$

provided that

$$E\{|X|U(|X|)\} < \infty, \tag{2}$$

where we have put $U(x) = \max(1, \log x)$ $(U_2(x) = U(U(x))$, etc.). Burkholder [2] has extended this result by showing that (1), (2), and

$$E\{\sup_n |X_n/n|\} < \infty, \tag{3}$$

are equivalent. More recently, motivated by certain optimal stopping problems Teicher [7] and Bickel [1] under various assumptions on the distributions of $X_1, X_2, \cdots$ have shown that

$$E\{\sup_n c_n|S_n|^\alpha\} < \infty \tag{4}$$

for certain sequences (c_n) and positive constants α. The interesting special case

$$c_n = (nU_2(n))^{-\alpha/2} \tag{5}$$

is *not* covered by the results of these authors.

This note gives a method which seems suitable for proving statements like (4) in a variety of cases. The method involves modifications of standard techniques used in the study of the law of the iterated logarithm. In particular, for each $\alpha = 1, 2, \cdots$ we are able to establish necessary and sufficient conditions for (4) when the X's are identically distributed and the sequence (c_n) satisfies (5). In Section 2 we state and prove one such theorem. Section 3 is devoted to explaining in somewhat more detail the scope of our results and their relation to the previously mentioned literature.

2. A maximal theorem.

THEOREM 1. *Let $X, X_1, X_2, \cdots$ be independent, identically distributed random variables with $EX = 0$. The following statements are equivalent*:

$$E\{X^2(U(|X|)/U_2(|X|))\} < \infty; \tag{6}$$

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(7) $$E\{\sup_n (nU_2(n))^{-1}S_n^2\} < \infty;$$

(8) $$E\{\sup_n (nU_2(n))^{-1}X_n^2\} < \infty.$$

PROOF. We shall show that (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (6). Suppose initially that the distribution of X is symmetric and $EX^2 = 1$. Put

(9) $$c_n = (nU_2(n))^{-1}, \qquad b_n = n^{\frac{1}{2}}(U_2(n))^{-\frac{1}{2}} \quad (n \geqq 1),$$

and define

$$X_n' = X_n I\{|X_n| \leqq b_n\}, \qquad X_n'' = X_n - X_n';$$
$$S_n' = \textstyle\sum_1^n X_k', \qquad S_n'' = S_n - S_n'.$$

To prove (7) it suffices to show

(10) $$E\{\sup_n c_n|S_n'|^2\} < \infty$$

and

(11) $$E\{\sup_n c_n|S_n''|^2\} < \infty.$$

Now

$$E\{\sup c_n|S_n''|^2\} \leqq E\{\textstyle\sum_1^\infty c_k^{\frac{1}{2}}|X_k''|\}^2 \leqq \sum_1^\infty c_k E|X_k''|^2 + 2(\sum_1^\infty c_k^{\frac{1}{2}} E|X_k''|)^2,$$

and from (6)

$$\begin{aligned}
&\textstyle\sum_{k=1}^\infty c_k E|X_k''|^2 \\
&\quad = \textstyle\sum_{k=1}^\infty c_k \sum_{j=k}^\infty \int_{\{b_j<|X|\leqq b_{j+1}\}} X^2 \\
&\quad = \textstyle\sum_{j=1}^\infty \sum_{k=1}^j c_k \int_{\{b_j<|X|\leqq b_{j+1}\}} X^2 \leqq \text{const.} \sum_{j=1}^\infty U(j)/U_2(j) \int_{\{b_j<|X|\leqq b_{j+1}\}} X_2 \\
&\quad \leqq \text{const.}\ E\{X^2(U(|X|)/U_2(|X|))\} < \infty.
\end{aligned}$$

Similarly

$$\textstyle\sum_1^\infty c_k^{\frac{1}{2}} E|X_k''| \leqq \text{const.}\ EX^2 < \infty,$$

and (11) follows. To prove (10) it suffices to show that

(12) $$\int_{x_0}^\infty uP\{\sup c_n^{\frac{1}{2}}|S_n'| > u\}\, du < \infty$$

for some $x_0 > 0$. For each $k = 0, \cdots$ let n_k be the largest integer $\leqq 3^k$. Writing $e_n = c_n^{\frac{1}{2}}$, we have by Levy's inequality

(13) $$\begin{aligned} P\{\sup e_n|S_n'| > u\} &\leqq \textstyle\sum_{k=0}^\infty P\{e_{n_k} \sup_{n_k \leqq n < n_{k+1}} |S_n'| > u\} \\ &\leqq 4 \textstyle\sum_{k=0}^\infty P\{e_{n_k} S'_{n_{k+1}} > u\}. \end{aligned}$$

We now use the fact that if $|Z| \leqq b$, then for any $t > 0$ for which $tb \leqq 1$

$$E\{\exp(tZ)\} \leqq \exp\{tEZ + t^2EZ^2\},$$

and Chebyshev's inequality

$$(P\{S_n' > x\} \leqq \exp(-tx) \textstyle\prod_1^n E\{\exp(tX_k')\}, \quad t > 0)$$

to obtain

$$\log P\{S'_{n_{k+1}} > e_{n_k}^{-1}u\} \leqq -te_{n_k}^{-1}u + t^2 n_{k+1} \quad (0 < t \leqq b_{n_{k+1}}^{-1},\ k = 0, 1, \cdots).$$

Setting $t = b_{n_{k+1}}^{-1}$, we have

$$\text{(14)} \qquad \log P\{S'_{n_{k+1}} > e_{n_k}^{-1}u\} \leqq -K_1(u - K_2)U_2(n_{k+1}),$$

where $K_1, K_2, \cdots$ denote constants, the exact values of which are of no interest. Taking x_0 to satisfy $K_1(x_0 - K_2) \geqq 2$, we have from (12)–(14)

$$\begin{aligned}
&\textstyle\int_{x_0}^{\infty} uP\{\sup_n e_n|S_n'| > u\}\, du \\
&\quad \leqq K_3 \textstyle\sum_{k=1}^{\infty} \int_{x_0}^{\infty} u \exp\{-K_1(u - K_2)U_2(n_k)\}\, du \\
&\quad \leqq K_3 \textstyle\sum_{k=1}^{\infty} \int_{x_0}^{\infty} u \exp\{-K_1(u - K_2)\log k\} \exp\{-K_1(u - K_2)U_2(3)\}\, du \\
&\quad \leqq K_3 \textstyle\sum_{k=1}^{\infty} k^{-2} \int_{x_0}^{\infty} u \exp\{-K_1(u - K_2)U_2(3)\}\, du < \infty.
\end{aligned}$$

This proves that (6) $\Rightarrow$ (7) for symmetrically distributed X. In general, let $X_1^{(s)}, X_2^{(s)}, \cdots$ be iid and independent of $X_1, X_2, \cdots$ with

$$P\{X_1^{(s)} \leqq x\} = P\{X \leqq x\} \quad (-\infty < x < \infty).$$

Let $S_n^{(s)} = \sum_1^n X_k^{(s)}$. Then (see Loève [3], p. 263, or Bickel [1])

$$\begin{aligned}
E\{\sup c_n|S_n|^2\} &= E\{\sup c_n|S_n - E(S_n^{(s)} \mid X_1, X_2, \cdots)|^2\} \\
&\leqq E\{\sup_n c_n E[|S_n - S_n^{(s)}|^2 \mid X_1, X_2, \cdots]\} \\
&\leqq E\{E[\sup_n c_n|S_n - S_n^{(s)}|^2 \mid X_1, X_2, \cdots]\} \\
&= E\{\sup_n c_n|S_n - S_n^{(s)}|^2\} < \infty
\end{aligned}$$

by our previous result.

To show that (7) $\Rightarrow$ (8) we merely observe that

$$c_n X_n^2 = c_n(S_n - S_{n-1})^2 \leqq 2(c_n S_n^2 + c_{n-1}S_{n-1}^2).$$

Suppose now that (8) is satisfied. Then

$$\textstyle\sum_{k=1}^{\infty} P\{\sup_n c_n X_n^2 > k\} < \infty,$$

or equivalently

$$\textstyle\sum_{k=1}^{\infty} (1 - \prod_{n=1}^{\infty} F(c_n^{-1}k)) < \infty,$$

where we have let F denote the distribution function of X^2 and have assumed, as we may by a change of scale, that $F(1) > 0$. Hence

$$\begin{aligned}
&\textstyle\int_1^{\infty}\int_1^{\infty} (1 - F(xU_2(x)y))\, dy\, dx \\
&\qquad \leqq \textstyle\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (1 - F(c_n^{-1}k)) \leqq -\sum_{k=1}^{\infty} \log \prod_{n=1}^{\infty} F(c_n^{-1}k) \\
&\qquad \leqq \text{const.} \textstyle\sum_{k=1}^{\infty} (1 - \prod_{n=1}^{\infty} F(c_n^{-1}k)) < \infty.
\end{aligned}$$

Setting $u = xU_2(x)y$, we obtain

$$\textstyle\int_1^{\infty} \int_{xU_2(x)}^{\infty} (1 - F(u))\, du (xU_2(x))^{-1}\, dx < \infty.$$

If we let φ denote the inverse of the function $x \to xU_2(x)$, we have by Fubini's theorem

$$\int_1^\infty [\int_1^{\varphi(u)} (xU_2(x))^{-1} dx](1 - F(u)) du < \infty. \tag{15}$$

Since $\varphi(u) \sim (u/U_2(u))$ $(u \to \infty)$ and $\int_1^t (xU_2(x))^{-1} dx \sim (U(t)/U_2(t))$ $(t \to \infty)$, it follows that (15) is equivalent to

$$\int_1^\infty (U(u)/U_2(u))(1 - F(u)) du < \infty,$$

which in turn is equivalent to (6).

3. Remarks. Relatively straightforward modifications of the proof of Theorem 1 lead to various other results, a few of which are summarized below.

Let $X, X_1, X_2, \cdots$ be independent random variables with $EX_n = 0$ $(n \geqq 1)$.

(16) If the X's are identically distributed, $\alpha = 1$, and (c_n) satisfies (5), then (4) is equivalent to

$$E(X^2) < \infty.$$

(17) If the X's are identically distributed, $\alpha = 3, 4, \cdots$, and (c_n) satisfies (5), then (4) is equivalent to

$$E|X|^\alpha < \infty.$$

(18) If the X's are identically distributed, $\alpha = 2$, $c_n = (nU(n))^{-1}$, then (4) is equivalent to

$$E(X^2U_2(|X|)) < \infty.$$

(19) If the X's are identically distributed and $c_n = (nU_2(n))^{-\frac{1}{2}}$, then

$$E\{\exp (t \sup_n c_n|S_n|)\} < \infty$$

for some $t > 0$ if and only if

$$E\{\exp (t|X|)\} < \infty$$

for some $t > 0$.

The result (18) improves on Teicher's theorem [7] in the sense that with the sequence (c_n) of (18) Teicher requires that

$$E\{X^2U(|X|)\} < \infty \tag{20}$$

to insure (4). In this regard note that even (6) is weaker than (20). Moreover, our methods apply in the non-identically distributed case, whereas Teicher's, which depend on the Wiener ergodic theorem [8], do not. (19) in part generalizes a result of Freedman [4].

It is interesting to compare our results with those of Marcinkiewicz and Zygmund [5] in the special case $\alpha = 2$. For future reference we state the elementary

(21) Lemma (Marcinkiewicz and Zygmund). If $x_1, x_2, \cdots$ is any sequence

of real numbers and $a_1, a_2, \cdots$ a non-increasing sequence of positive numbers, then

$$\sup_n |a_n \sum_1^n x_k| \leqq 2 \sup |\sum_1^n a_k x_k|.$$

The proof, which is omitted, is similar to that of the closely related Kronecker lemma. If $c_n \downarrow$ and $\sum_1^\infty c_n E X_n^2 < \infty$, to prove (4) it suffices by (21) to prove

$$E\{\sup_n |\sum_1^n c_k^{\frac{1}{2}} X_k|^2\} \leqq \text{const.} \sum_1^\infty c_k E X_k^2, \tag{22}$$

which is what Marcinkiewicz and Zygmund do (see their Theorems 1 and 7). (In the case $\alpha = 2$, Bickel's method likewise proves (22).) Moreover, when applicable, this idea leads to elegant proofs. For example, if $X_1, X_2, \cdots$ are independent and symmetrically distributed, then with $e_k = c_k^{\frac{1}{2}}$ we have by Levy's inequality

$$\begin{aligned} E\{\max_{1 \leqq k \leqq n} |\sum_1^k e_{k'} X_{k'}|^2\} &= \int_0^\infty P\{\max_{1 \leqq k \leqq n} |\sum_1^k e_{k'} X_{k'}| > u^{\frac{1}{2}}\} \, du \\ &\leqq 2 \int_0^\infty P\{|\sum_1^n e_k X_k| > u^{\frac{1}{2}}\} \, du = 2 \sum_1^n c_k E X_k^2, \end{aligned}$$

from which (22) follows by monotone convergence. Symmetrization as in the proof of Theorem 1 proves (22) in general. Truncation and a similar calculation provide an easy proof that (2) $\Rightarrow$ (1) in the identically distributed case. (The method of Section 2 completes the proof of the equivalence of (1), (2), and (3).) However, under the assumptions of, say, (18) the right hand side of (22) is $+\infty$, and in fact

$$P\{\sup_n [U_2(n)]^{-\frac{1}{2}} |\sum_1^n e_k X_k| = +\infty\} = 1. \tag{23}$$

To prove (23) observe that by the Lindeberg-Feller theorem (some calculation is required to verify the Lindeberg condition)

$$[U_2(n)]^{-\frac{1}{2}} \sum_1^n e_k X_k$$

converges in law to the standard normal random variable; (23) follows by the Kolmogorov 0–1 law (see, e.g. [6]). Thus the method of Marcinkiewicz and Zygmund does not without essential modification prove (18).

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