The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 475–479

ON CONVERGENCE RATES IN THE CENTRAL LIMIT THEOREM[1]

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1. Introduction and summary. Let $X_1, X_2, \cdots$ be independent random variables with distribution functions $V_1, V_2, \cdots$, zero means and finite non-zero variances $\sigma_1^2, \sigma_2^2, \cdots$.

Set $s_n^2 = \sum_1^n \sigma_i^2$ and $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-t^2/2}\, dt$. Define

$$\psi_n(c) = \sum_1^n \int_{|x|>c} x^2\, dV_i(x). \tag{1.1}$$

According to the well-known Lindeberg-Feller Theorem [1] the condition

$$s_n^{-2}\psi_n(\xi s_n) \to 0 \quad \text{as} \quad n \to \infty \qquad \text{for all} \quad \xi > 0$$

is both necessary and sufficient in order that $P[(X_1 + \cdots + X_n)s_n^{-1} \leqq x] \to \Phi(x)$ uniformly in x as $n \to \infty$ and that

$$\max_{1 \leqq j \leqq n} \sigma_j s_n^{-1} \to 0 \quad \text{as} \quad n \to \infty.$$

Using the method of [3] and [4], it is shown that there exists an absolute constant K, independent of n and of the particular sequence $V_1, V_2, \cdots$ such that

$$\sup_{-\infty<x<\infty} |P[(X_1 + \cdots + X_n)s_n^{-1} \leqq x] - \Phi(x)| \leqq K s_n^{-3} \int_0^{s_n} \psi_n(u)\, du. \tag{1.2}$$

Some corollaries are deduced and the accuracy of this bound is investigated.

Using a truncation scheme, an absolute upper bound is also derived for $\sup_{-\infty<x<\infty} |P[(X_1 + \cdots + X_n)B_n^{-1} \leqq x] - \Phi(x)|$, where the assumption of finite variances is now dropped and B_n is a norming constant defined in (4.1).

2. Main results. Define

$$\Delta_n = \sup_{-\infty<x<\infty} |P[(X_1 + \cdots + X_n)s_n^{-1} \leqq x] - \Phi(x)|. \tag{2.1}$$

Then we have

THEOREM 1. *There exists an absolute constant K such that*

$$\Delta_n \leqq K s_n^{-3} \int_0^{s_n} \psi_n(u)\, du. \tag{2.2}$$

PROOF. The proof follows from a result of Petrov [4], i.e.,

$$\Delta_n \leqq s_n^{-2}[(1 + \tfrac{1}{3}\cdot 14(2\pi)^{-\frac{1}{2}})\psi_n(s_n) + 64C_0 e_n(s_n)] \tag{2.3}$$

where $e_n(c) = c^{-1} \sum_1^n \int_{|x| \leqq c} |x|^3\, dV_i$ and C_0 is the absolute constant in the Berry-Esséen theorem. To obtain (2.2) from (2.3) it suffices to note that

$$c^{-1} \int_0^c \psi_n(u)\, du = \psi_n(c) + e_n(c) \tag{2.4}$$

which is easily established by inverting the order of integration on the left side of (2.4).

Received 13 May 1968.

[1] Adapted from the author's doctoral dissertation, Columbia University, 1967.

The constant K may be taken to be $1 + \frac{1}{3}\cdot 14(2\pi)^{-\frac{1}{2}} + 32C_0$. To obtain this from (2.3), it is necessary to halve the constant 64 appearing in (2.3). This can be done using the estimate $E|X - EX|^3 \leqq 4E|X|^3$ which can be demonstrated with the aid of the identity $|X - EX|^3 - |X|^3 = (|X - EX| - |X|)(|X - EX|^2 + |X|\cdot|X - EX| + X^2)$ and the moment inequality $(E|X|^s)^{1/s} \geqq (E|X|^r)^{1/r}$ $(s > r)$.

COROLLARY 1. *Suppose for some* $0 < \delta < 1$, *some* $M < \infty$ *and some* $\sigma^2 > 0$ *we have* $E|X_j|^{2+\delta} \leqq M$ *and* $\sigma_j^2 \geqq \sigma^2$ *for all* j. *Assume, also, that*

$$\beta_j(c) \equiv \int_{|x|>c} |x|^{2+\delta}\, dV_j \rightarrow 0 \quad as \quad c \rightarrow \infty$$

uniformly in j. *Then* $\Delta_n = o(n^{-\delta/2})$ *as* $n \rightarrow \infty$.

PROOF. Letting $\beta(c) = \sup_j \beta_j(c)$, we have

$$\int_{|x|>c} x^2\, dV_j = \int_{|x|>c} |x|^{2+\delta} |x|^{-\delta}\, dV_j \leqq c^{-\delta}\beta(c).$$

In view of the fact that $n \leqq s_n^2\sigma^{-2}$, the right side of (2.2) is at most $K\sigma^{-2}s_n^{-1}\int_0^{s_n} u^{-\delta}\beta(u)\, du$.

Select $p > 1$ such that $p\delta < 1$ and q such that $1/p + 1/q = 1$. By Hölder's inequality,

$$s_n^{-1}\int_0^{s_n} u^{-\delta}\beta(u)\, du \leqq s_n^{-\delta}(1 - \delta p)^{-1/p}[s_n^{-1}\int_0^{s_n}\beta^q(u)\, du]^{1/q}.$$

The first factor is at most $(1 - \delta p)^{-1/p}\sigma^{-\delta}n^{-\delta/2}$. The second factor approaches zero by Kronecker's lemma. This proves Corollary 1.

COROLLARY 2. *There exists a sequence of random variables that satisfies the central limit theorem for which*

$$\Delta_n = o(1/s_n).$$

PROOF. Let V_i have a jump of $\frac{1}{2}$ at $\pm\sigma_i$, $i = 1, 2 \cdots$, where $\{\sigma_i\}$ is any sequence of positive constants for which $\sum_1^n \sigma_i^3$ remains bounded but $\sum_1^n \sigma_i^2 = s_n^2 \rightarrow \infty$ (e.g. $\sigma_i = i^{-\frac{1}{2}}$). Then

$$s_n^{-3}\int_0^{s_n}\psi_n(u)\, du = s_n^{-2}\sum_1^n\int_{|x|>s_n}|x|^2\, dV_i + s_n^{-3}\sum_1^n\int_{|x|\leqq s_n}|x|^3\, dV_i$$
$$= 0 + s_n^{-3}\sum_1^n \sigma_i^3.$$

This proves Corollary 2.

3. Some properties of the bound. Throughout this section B_n shall denote

$$Ks_n^{-3}\int_0^{s_n}\psi_n(u)\, du.$$

THEOREM 2. $B_n \rightarrow 0$ *as* $n \rightarrow \infty$ *is necessary and sufficient to ensure that* $\Delta_n \rightarrow 0$ and $\max_{1\leqq k\leqq n}\sigma_k s_n^{-1} \rightarrow 0$ *as* $n \rightarrow \infty$.

PROOF. Since $\psi_n(u)$ decreases with u, this follows from the Lindeberg-Feller theorem and the inequality

$$\begin{aligned} \epsilon s_n^{-2}\psi_n(\epsilon s_n) &\leqq s_n^{-3}\int_0^{\epsilon s_n}\psi_n(u)\, du \leqq B_nK^{-1} \\ &= s_n^{-3}\left(\int_0^{\epsilon s_n}\psi_n(u)\, du + \int_{\epsilon s_n}^{s_n}\psi_n(u)\, du\right) \\ &\leqq \epsilon + (1 - \epsilon)s_n^{-2}\psi_n(\epsilon s_n) \end{aligned} \tag{3.1}$$

for all $\epsilon > 0$.

THEOREM 3. *There exists a sequence of distribution functions for which B_n is sharp in the sense that $B_n\Delta_n^{-1}$ is bounded.*

PROOF. In [2], the sequence $\{V_j\}$ is considered, where $V_j(x) = (1 - e^{-j})G_1(x) + e^{-j}G_2(x)$, $j = 1, 2, \cdots$, $G_1(x)$ is the symmetric Bernoulli distribution with jumps at the points ± 1 and $G_2(x)$ is any absolutely continuous symmetric distribution function with unit variance and infinite moments of order $2 + \delta$. Moreover, it is shown that $s_n^{-2}\psi_n(\epsilon s_n) < (e - 1)^{-1}n^{-1}$ when $n > \epsilon^{-2}$ and that $\Delta_n > e^{-2}(2\pi n)^{-\frac{1}{2}}$ for n sufficiently large. Taking ϵ to be $2n^{-\frac{1}{2}}$ and using the right half of (3.1) we obtain that $K\Delta_n B_n^{-1} \geqq \frac{1}{3}e^{-2}(2\pi)^{-\frac{1}{2}}$ for n sufficiently large.

THEOREM 4. *Let $V_j = V$, $j = 1, 2, \cdots$, where $V_j(x)$ is a symmetric distribution function and the restriction of $1 - V(x)$ to $(0, \infty)$ varies regularly*[2] *with exponent $-e$, $2 < e < 3$. Then there is a slowly varying function $K(x)$ such that $B_n\Delta_n^{-1} \leqq K(n)$ for all n.*

PROOF.

$$P[|X_1 + \cdots + X_n| > t] \geqq \tfrac{1}{2}P[\max |X_j| > t]$$

for symmetric random variables [1] so that

$$1 - F_n(x) \equiv P[(X_1 + \cdots + X_n)(n\sigma^2)^{-\frac{1}{2}} > x] \geqq \tfrac{1}{4}(1 - V^n(x(n\sigma^2)^{\frac{1}{2}})). \tag{3.2}$$

On the other hand it is well-known that

$$1 - \Phi(x) < (2\pi)^{-\frac{1}{2}}e^{-x^2/2}x^{-1}. \tag{3.3}$$

Set $\sigma^2 = \int x^2\, dV$. As $\Delta_n \geqq |1 - F_n(t_n\sigma^{-1}) - (1 - \Phi(t_n\sigma^{-1}))|$ for any choice of $\{t_n\}$, our aim is to select the sequence $\{t_n\}$ increasing rapidly enough so that $1 - \Phi(t_n\sigma^{-1})$ is small in comparison to $1 - F_n(t_n\sigma^{-1})$ since, by (3.2),

$$[1 - \Phi(t_n\sigma^{-1})][1 - V^n(t_n n^{\frac{1}{2}})]^{-1} \to 0 \quad \text{as} \quad n \to \infty \quad \text{implies} \tag{3.4}$$

$$\Delta_n \geqq \tfrac{1}{2}(1 - F_n(t_n\sigma^{-1}) \geqq \tfrac{1}{8}(1 - V^n(t_n n^{\frac{1}{2}}) \quad \text{for } n \text{ sufficiently large.}$$

However, $\{t_n\}$ must increase slowly enough to keep $B_n[1 - V^n(t_n n^{\frac{1}{2}})]^{-1}$ slowly varying. Let

$$t_n = (2\sigma^2\gamma \ln n)^{\frac{1}{2}} \quad \text{where} \quad \gamma > \tfrac{1}{2}(e - 2). \tag{3.5}$$

The hypotheses imply that $1 - V(t) = t^{-e}L(t)(t > 0)$ with $L(t)$ slowly varying. Now,

$$\begin{aligned} 1 - V^n(t_n n^{\frac{1}{2}}) &= (1 - V(t_n n^{\frac{1}{2}}))(1 + V(t_n n^{\frac{1}{2}}) + \cdots + V^{n-1}(t_n n^{\frac{1}{2}})) \\ &\geqq nV^n(t_n n^{\frac{1}{2}})(t_n n^{\frac{1}{2}})^{-e}L(t_n n^{\frac{1}{2}}). \end{aligned} \tag{3.6}$$

Also,

$$V^n(t_n n^{\frac{1}{2}}) = (1 - L(t_n n^{\frac{1}{2}})t_n^{-e}n^{-e/2})^n = (1 + o(n^{-1}))^n \to 1$$

as $n \to \infty$ in view of (3.5). Then by (3.6)

$$1 - V^n(t_n n^{\frac{1}{2}}) \geqq S(n)n^{(2-e)/2} \quad \text{for } n \text{ sufficiently large,} \tag{3.7}$$

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[2] For the definitions of slowly and regularly varying functions and their properties, the reader is referred to [1].

where $S(n) \equiv \frac{1}{2}t_n^{-e}L(t_n n^{\frac{1}{2}})$ is a slowly varying function of n. Because of (3.3), (3.5) and (3.7) we have

$$[1 - \Phi(t_n\sigma^{-1})][1 - V^n(t_n n^{\frac{1}{2}})]^{-1} \leqq \sigma(2\pi)^{-\frac{1}{2}}t_n^{-1}S^{-1}(n)n^{(e-2)/2-\gamma}$$

for n sufficiently large so that by (3.4) and (3.7),

(3.8) $\Delta_n \geqq \frac{1}{8}S(n)n^{(2-e)/2}$ for n sufficiently large, say, $n \geqq n_0$.

But $\psi_n(c)n^{-1} = \int_{|x|>c} x^2\, dV = 2L(c)c^{2-e} + 4\int_c^\infty x^{1-e}L(x)\, dx$ and using the results in [1] it is seen that

$$\psi_n(c)n^{-1} = c^{2-e}N(c)$$

with $N(c)$ slowly varying. Then

$$B_nK^{-1} = \sigma^{-3}n^{-\frac{1}{2}}\int_0^{\sigma n^{\frac{1}{2}}} u^{2-e}N(u)\, du$$

and the same results show that

(3.9) $B_n = M(n)n^{(2-e)/2}$, with $M(n)$ slowly varying.

Then by (3.8) and (3.9)

$$B_n\Delta_n^{-1} \leqq 8M(n)(S(n))^{-1} \qquad (n \geqq n_0).$$

To complete the proof, set

$$K(x) = \max_{k=1,\cdots,n_0} B_n\Delta_n^{-1} \quad \text{if} \quad x \leqq n_0;$$
$$K(x) = 8M(x)(S(x))^{-1} \quad \text{if} \quad x > n_0.$$

4. The case without variances.

THEOREM 5. *Set*

$$U_i(c) = \int_{|x|\leqq c} |x|^2\, dV_i,$$
$$A_n(c) = \sum_1^n c\int_{|x|>c} |x|\, dV_i.$$

Assume

(4.1) $$B_n^2 = \sum_1^n U_i(B_n) > 0.$$

Define $G_n(x)$ *to be* $P[(X_1 + \cdots + X_n)B_n^{-1} \leqq x]$,

$$\Delta_n = \sup_{-\infty<x<\infty} |G_n(x) - \Phi(x)|.$$

Then there is an absolute constant K' *such that*

(4.2) $$\Delta_n \leqq K'B_n^{-3}\int_0^{B_n} A_n(u)\, du.$$

PROOF. Let $X_i' = X_i$ if $|X_i| \leqq B_n$ and 0 if not, $i = 1, 2, \cdots, n$. Set

(4.3) $$b_n(c) = c^{-1}\sum_1^n \int_{|x|\leqq c} |x|^3\, dV_i,$$

$$s_n^2 = \text{Var}\,(X_1' + \cdots + X_n').$$

Using an argument that parallels the one in [4], we obtain that

$$A_n(B_n)B_n^{-2} > \tfrac{3}{4} \quad \text{if} \quad s_n < B_n/2 \qquad \text{and}$$

$$B_n^2\Delta_n \leqq 1 + \tfrac{1}{3}\cdot 10(2\pi)^{-\frac{1}{2}}A_n(B_n) + 32C_0 b_n(B_n)$$

if $s_n \geqq \frac{1}{2}B_n$, where C_0 is the absolute constant in the Berry-Essén theorem.

The conclusion follows from the fact that

$$c^{-1}\int_0^c A_n(u)\,du = \tfrac{1}{2}(A_n(c) + b_n(c)). \tag{4.4}$$

Theorem 6 shows that Theorem 5 is non-vacuous, and gives sufficient conditions that the bound of Theorem 5 approach zero.

THEOREM 6. *If $V_1, V_2, \cdots = V$ where V is in the domain of attraction of the normal law and $B_n \to \infty$ as $n \to \infty$ then the bound approaches zero. Also, if V is continuous and in the domain of attraction of the normal law, then for n sufficiently large, there exist solutions $\{B_n\}$ of (4.1) so that $B_n \to \infty$.*

PROOF. The hypotheses imply that $U(c) \equiv \int_{|x|\leqq c} x^2\,dV$ is slowly varying. $U(c)c^{-2} \to 0$ as $c \to \infty$. There exists c_0 such that $U(c_0) > 0$.

Let n_0 be the smallest positive integer such that $U(c_0)c_0^{-2} > n_0^{-1}$. Then, if $U(c)$ is continuous, there is some $B > c_0$ such that $U(B)B^{-2} = n_0^{-1}$. Call it B_{n_0}. Inductively, if $U(B_n)B_n^{-2} = n^{-1}$, select $B_{n+1} > B_n$ so that $U(B_{n+1})B_{n+1}^{-2} = (n+1)^{-1}$. If $c_0 < B_n < M$ then $U(B_n)B_n^{-2} > U(c_0)M^{-2}$ so it must be that $B_n \to \infty$.

This proves the second assertion.

The bound is

$$\tfrac{1}{2}K'[\int_{|x|\leqq B_n} |x|^3\,dV[B_nU(B_n)]^{-1} + B_n\int_{|x|>B_n} |x|\,dV[U(B_n)]^{-1}]$$

because of (4.1)–(4.4).

Denote the terms in brackets by T_1 and T_2 respectively. For arbitrary $\epsilon > 0$,

$$T_1 \leqq \epsilon + [U(B_n) - U(\epsilon B_n)][U(B_n)]^{-1}.$$

Since U is slowly varying, $T_1 \to 0$ as $B_n \to \infty$.

Integrating by parts,

$$T_2 = -1 + B_n[U(B_n)]^{-1}\int_{B_n}^{\infty} U(x)\cdot x^{-2}\,dx.$$

The results of [1] on slowly varying functions show that

$$B[U(B)]^{-1}\int_B^{\infty} U(x)\cdot x^{-2}\,dx \to 1 \quad \text{as} \quad B \to \infty.$$

This proves the first assertion.

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