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APPROXIMATION TO BAYES RISK IN SEQUENCES OF NON-FINITE GAMES[1]

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1. Introduction. This paper is concerned with the product of a game. The main result is the demonstration of a sequence strategy for player II which results in average risk across the first n plays approaching uniformly the Bayes envelope evaluated at the empirical distribution of player I's first n moves.

We consider a two-person game where player I chooses an $\epsilon \varepsilon M$ and player II chooses $\delta \varepsilon N$ with loss $L(\epsilon, \delta) \geqq 0$. A compact notation is provided by defining the set of loss functions $N^* = \{L(\cdot, \delta) \mid \delta \varepsilon N\}$. We let σ denote a generic element of N^* and $\epsilon\sigma$ denote σ evaluated at ϵ. This extends to operator notation $w\sigma = \int \epsilon\sigma w(d\epsilon)$ for measures w on M. The Bayes envelope is defined by

$$R(p) = \inf \{p\sigma \mid \sigma \varepsilon N^*\}$$

where p is a probability measure on M (a mixed strategy for player I).

We suppose that this game occurs repeatedly, ϵ_i represents player I's move at the ith stage, and G_{i-1}, the empirical distribution of $\epsilon_1, \cdots, \epsilon_{i-1}$, is known to player II before he makes his move at the ith stage, $i \geqq 2$. In this paper we let G_0 denote the zero measure and demonstrate sequence strategies $\boldsymbol{\delta} = (\sigma_1, \sigma_2, \cdots)$ for player II, where σ_i depends upon G_{i-1} and some artificial randomization, such that $n^{-1} \sum_1^n E(\epsilon_i\sigma_i) - R(G_n) \to 0$ as $n \to \infty$ uniformly in $\boldsymbol{\epsilon}$.

The notion of using the Bayes envelope as an asymptotic standard in a set of statistical decision problems, with statistical information on G_n replacing knowledge of G_{i-1} in the sequence case, is due to Robbins [11]. Since Robbins' original investigation, procedures which achieve the Bayes envelope asymptotically have been demonstrated and rates of convergence investigated for sequences and sets of a variety of statistical decision problem components, [3], [4], [5], [6], [8], [9], [10], [12], [13], [14], [15], [16], [17] and [18]. However, this paper treats the problem at a game theoretic level and is more closely related to the studies [1], [2], and [7]. This paper depends heavily upon the notation and ideas of Hannan [7], but an effort is made to keep the presentation self-contained.

2. The main result. We impose the following condition on the component game:

(A1) $\quad N^*$ is sequentially compact under pointwise convergence.

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Under (A1), an application of Fatou's lemma shows that $\inf \{p\sigma \mid \sigma \in N^*\}$ is attained for each probability measure p and we denote an infimizing σ by $\sigma(p)$. The domain of definition of $\sigma(\cdot)$ can be extended to all finite measures w by defining $\sigma(w)$ to be $\sigma(\cdot)$ evaluated at the normalized w if w is not the zero measure and arbitrary otherwise. This ensures that $\sigma(\cdot)$ is positive homogeneous; that is, $\sigma(kw) = \sigma(w)$ for all $k > 0$ and finite measures w.

Hannan [7], p. 129, investigates the natural procedure $\boldsymbol{\sigma}^*$ where $\sigma_i^* = \sigma(G_{i-1})$, $i \geqq 1$, and shows that

$$\lim_{t \downarrow 0} \epsilon(\sigma(p) - \sigma(p + t\epsilon)) = 0 \quad \text{uniformly in} \quad \epsilon, p$$

is sufficient for $n^{-1} \sum_1^n \epsilon_i \sigma_i^* - R(G_n) \to 0$ as $n \to \infty$ uniformly in $\boldsymbol{\epsilon}$. Also the inadequacy of $\boldsymbol{\sigma}^*$ is illustrated for certain finite and nonfinite component games. One of the main results of the Hannan paper is the demonstration in the finite M case of a procedure, which at the ith stage plays Bayes versus a random perturbation of G_{i-1}, whose average risk achieves the Bayes envelope asymptotically. We now extend this technique to a more general case.

To do this we impose a boundedness condition on the component game

$$\text{(A2)} \qquad \sup \{\epsilon\sigma \mid \epsilon \in M, \sigma \in N^*\} = B < \infty$$

and define the real valued function

$$\text{(1)} \qquad d(\epsilon, \epsilon') = \sup \{|\epsilon\sigma - \epsilon'\sigma| \mid \sigma \in N^*\}.$$

Clearly, (M, d) is a pseudo-metric space; and, if loss equivalent player I moves are identified, it constitutes a metric space.

Let $J_i \geqq 1$ be a non-decreasing integer valued sequence, $A = \{a_1, a_2, \cdots\} \subset M$, and $A_i = \{a_1, \cdots, a_{J_i}\}$, $i \geqq 1$. Corresponding to each sequence $\boldsymbol{\epsilon}$ is a sequence $\boldsymbol{\epsilon}' = (\epsilon_1', \epsilon_2', \cdots)$ where ϵ_i' is an element of A_i closest to ϵ_i in the metric d. We let G_i' denote the empirical distribution of $\epsilon_1', \cdots, \epsilon_i'$ and $E_i' = iG_i'$. The artificial randomization will be provided by independent and identically distributed uniform [0, 1] random variables $Z_1, Z_2, \cdots$. Consider a procedure $\boldsymbol{\sigma}$ where σ_1 is arbitrary and

$$\text{(2)} \qquad \sigma_i = \sigma(E_{i-1}' + H_{i-1}\mathbf{Z}_{i-1}), \qquad i \geqq 2.$$

Here $H_i > 0$ is a non-decreasing sequence of constants and $\mathbf{Z}_i$ to be interpreted as the measure placing mass Z_j on a_j, $j = 1, \cdots, J_i$. We first develop a bound for $\sum_1^n \epsilon_i\sigma_i - nR(G_n)$ and then give some applications.

We make the decomposition

$$\text{(3)} \qquad \sum_1^n \epsilon_i\sigma_i - nR(G_n) = B_n + C_n + D_n$$

where

$$B_n = \sum_1^n \epsilon_i'\sigma_i - nR(G_n'), \qquad C_n = \sum_1^n (\epsilon_i\sigma_i - \epsilon_i'\sigma_i),$$

and

$$D_n = n(R(G_n') - R(G_n)).$$

Clearly, $C_n \leqq \sum_1^n d(\epsilon_i, \epsilon_i')$ and $n(R(G_n') - R(G_n)) = \inf_\sigma \sum_1^n \epsilon_i'\sigma -$

$\inf_\sigma \sum_1^n \epsilon_i\sigma \leqq \sum_1^n d(\epsilon_i, \epsilon_i')$ so that

$$\max\{|C_n|, |D_n|\} \leqq \sum_1^n d(\epsilon_i, \epsilon_i'). \tag{4}$$

The term B_n can be treated using the following identity in $\epsilon_i \varepsilon M$, $\sigma_i \varepsilon N^*$ ((6.5) of [7]):

$$\sum_1^n \epsilon_i\sigma_i \equiv E_n\sigma_{n+1} + \sum_1^n E_{i-1}(\sigma_i - \sigma_{i+1}) + \sum_1^n \epsilon_i(\sigma_i - \sigma_{i+1}), \tag{5}$$

where $E_i = iG_i$. It follows from (5) that

$$B_n = E_n'(\sigma_{n+1} - \sigma(E_n')) + \sum_1^n E_i'(\sigma_i - \sigma_{i+1}). \tag{6}$$

Since $E_i'(\sigma_i - \sigma_{i+1}) \geqq (E_i' - (E_i' + H_i\mathbf{Z}_i))(\sigma_i - \sigma_{i+1})$,

$$\sum_1^n E_i'(\sigma_i - \sigma_{i+1}) \geqq -\sum_1^n H_i\mathbf{Z}_i(\sigma_i - \sigma_{i+1})$$
$$= H_n\mathbf{Z}_n\sigma_{n+1} - H_1\mathbf{Z}_1\sigma_1 - \sum_2^n (H_i\mathbf{Z}_i - H_{i-1}\mathbf{Z}_{i-1})\sigma_i.$$

Using the fact that H_i and J_i are non-decreasing we obtain

$$\sum_1^n E_i'(\sigma_i - \sigma_{i+1}) \geqq -BH_1J_1 - \sum_2^n (H_iJ_i - H_{i-1}J_{i-1})B = -BH_nJ_n.$$

Also, $E_n'(\sigma_{n+1} - \sigma(E_n')) \geqq 0$ so (6) yields

$$B_n \geqq -BH_nJ_n. \tag{7}$$

We note that $E_n'(\sigma_{n+1} - \sigma(E_n')) \leqq -H_n\mathbf{Z}_n(\sigma_{n+1} - \sigma(E_n'))$; and, similarly, $E_{i-1}'(\sigma_i - \sigma_{i+1}) \leqq -H_{i-1}\mathbf{Z}_{i-1}(\sigma_i - \sigma_{i+1})$. Application of these inequalities to (5) followed by summation by parts leads to

$$B_n \leqq BH_nJ_n + \sum_1^n \epsilon_i'(\sigma_i - \sigma_{i+1}). \tag{8}$$

We define $S_n = \{i \mid 1 \leqq i \leqq n, J_i = J_{i-1}\}, J_0 = 0$, and note that the cardinality of S_n is at least $n - J_n$. Therefore, we can weaken (8) to

$$B_n \leqq BH_nJ_n + BJ_n + \sum_{S_n} \epsilon_i'(\sigma_i - \sigma_{i+1}). \tag{9}$$

The above results combine to give the useful inequality

$$-BH_nJ_n - 2\sum_1^n d(\epsilon_i, \epsilon_i') \leqq \sum_1^n \epsilon_i\sigma_i - nR(G_n) \leqq BH_nJ_n + BJ_n + 2\sum_1^n d(\epsilon_i, \epsilon_i') + \sum_{S_n} \epsilon_i'(\sigma_i - \sigma_{i+1}). \tag{10}$$

The expectation of the last term in (10) can be bounded using a lemma due to Hannan [7], p. 131. It implies that if w, w' are two measures on $\{a_1, a_2, \cdots, a_J\}$ and $\mathbf{Z}$ is uniform on $[0, 1]^J$ then for each $j = 1, \cdots, J$,

$$|E(a_j(\sigma(w + \mathbf{Z}) - \sigma(w' + \mathbf{Z})))| \leqq B\sum_1^J |w_j - w_j'|.$$

Setting $w = H_{i-1}^{-1}E_{i-1}'$ and $w' = H_i^{-1}E_i'$ and requiring that iH_i^{-1} is non-decreasing in i, we have that $\sum_1^J |w_j - w_j'| \leqq 2H_i^{-1}$; and, therefore,

$$|E(\sum_{S_n} \epsilon_i'(\sigma_i - \sigma_{i+1}))| \leqq 2B\sum_1^n H_i^{-1}. \tag{11}$$

(The measurability problem ignored in stating (11) is taken up in Section 4.) We summarize (10) and (11) in

THEOREM 1. *If the component game satisfies* (A1) *and* (A2), *then the sequence strategy* $\boldsymbol{\sigma}$ *with* σ_1 *arbitrary and* σ_i *given by* (2), $i \geqq 2$, *with* H_i *and* iH_i^{-1} *non-decreasing, results in*

$$(12)\qquad \begin{aligned} -BH_nJ_n - 2\textstyle\sum_1^n d(\epsilon_i, \epsilon_i') \\ \leqq \textstyle\sum_1^n E(\epsilon_i\sigma_i) - nR(G_n) \\ \leqq B\{H_nJ_n + J_n + 2\textstyle\sum_1^n H_i^{-1}\} + 2\textstyle\sum_1^n d(\epsilon_i, \epsilon_i') \end{aligned}$$

where E can be interpreted as either the lower or upper integral.

The first application of the theorem is to establish the uniform $O(n^{-\frac{1}{2}})$ convergence in the finite M case which was proved previously by Hannan [7], p. 134.

COROLLARY 1. *Under* (A1) *and* (A2) *with* $H_i = i^{\frac{1}{2}}$, $A_i = M$ *finite,*

$$(13)\qquad n^{-1}\textstyle\sum_1^n E(\epsilon_i\sigma_i) - R(G_n) = O(n^{-\frac{1}{2}}) \quad \textit{uniformly in} \quad \boldsymbol{\epsilon}.$$

An application in the non-finite case is provided by

COROLLARY 2. *Under* (A1) *and* (A2) *with* (M, d) *a totally bounded metric space, there exist choices of* H_i, J_i, *and* A *such that*

$$(14)\qquad n^{-1}\textstyle\sum_1^n E(\epsilon_i\sigma_i) - R(G_n) = o(1) \quad \textit{uniformly in} \quad \boldsymbol{\epsilon}.$$

PROOF. Since (M, d) is totally bounded there exists a countable subset $A = \{a_1, a_2, \cdots\}$ such that $d(\epsilon, A_i) \to 0$ as $J_i \to \infty$ uniformly in ϵ. Therefore, with this choice of A and $n^{-1}H_nJ_n \to 0$, $n^{-1}\sum_1^n H_i^{-1} \to 0$ and $J_n \to \infty$ as $n \to \infty$, (14) follows from (12).

In practice, a rate of convergence can be obtained by balancing the terms making up the bounds in (12) through choice of A and the sequences H_i and J_i.

EXAMPLE. Consider the game of absolute deviation on the unit square (see, [7], p. 130, where it is shown that $\sup\{n^{-1}\sum_1^n \epsilon_i\sigma_i^* - R(G_n) \mid \boldsymbol{\epsilon} \in [0, 1]^\infty\} = \frac{1}{4}$). Here $d(\epsilon, \epsilon') = |\epsilon - \epsilon'|$; and we let $H_i = i^a$, $J_i = [i^b]$, $i \geqq 1$, where $a, b \in (0, 1)$ are yet to be specified. For the set $A = \{a_1, a_2, \cdots\}$ we take the points $\{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \cdots\}$; that is, $a_i = b_{kj}$ where $i = 2^{k-1} + j - 1$, $b_{kj} = (2j-1)2^{-k}$, $1 \leqq j \leqq 2^{k-1}$, $k \geqq 1$. With this choice $d(\epsilon_i, \epsilon_i') \leqq 2J_i^{-1}$ and (12) implies

$$n^{-1}\textstyle\sum_1^n E(\epsilon_i\sigma_i) - R(G_n) = O(n^{-1+a+b} + n^{-a} + n^{-b})$$

uniformly in $\boldsymbol{\epsilon}$. The choice $a = b = \frac{1}{3}$ balances the bound yielding $O(n^{-\frac{1}{3}})$.

3. Games with countable M. In Corollary 2 we have given an application of (12) to the case (M, d) is a totally bounded metric space. We now demonstrate a sequence strategy achieving $O(n^{-\frac{1}{2}})$ uniformly in $\boldsymbol{\epsilon}$ in a case where $M = \{1, 2, \cdots\}$ is countable and (M, d) is a bounded but not totally bounded metric space.

The two conditions on the component game that we impose are (A1) and

$$(\text{A3})\qquad \sup\{\|\sigma\|_1 \mid \sigma \in N^*\} = B < \infty$$

where $\| \ \|_1$ denotes the l_1 sequence norm. (It is sufficient for the set of loss functions N^* to be a bounded, closed set in the l_1 space.) It follows from (A1) and Fatou's lemma that for all $w \varepsilon m^+$, the set of bounded sequences with non-negative components, $\inf \{w\sigma \mid \sigma \varepsilon N^*\}$ is attained. As before, we let $\sigma(w)$ be a positive homogeneous determination of the infimizer. With $Z = (Z_1, Z_2, \cdots)$ where the Z_j are independent uniform [0, 1] random variables, we investigate the randomized procedure $\boldsymbol{\sigma}$ where σ_1 is arbitrary and

$$\sigma_i = \sigma(E_{i-1} + H_{i-1}Z), \qquad i \geqq 2. \tag{15}$$

In (15) the sequence of constants $H_i \geqq 1$ is such that H_i and iH_i^{-1} are non-decreasing and Z is interpreted as the measure that places mass Z_j on j, $j \varepsilon M$.

THEOREM 2. *If the component game has countable M and satisfies* (A1) *and* (A3), *then the procedure given by* (15) *results in*

$$-BH_n \leqq \textstyle\sum_1^n E(\epsilon_i\sigma_i) - nR(G_n) \leqq B\{H_n + 2\sum_1^n H_i^{-1}\}. \tag{16}$$

PROOF. We use the identity (5) and proceed as in the development of (7) and (8) to obtain

$$\textstyle\sum_1^n \epsilon_i\sigma_i - nR(G_n) \geqq -BH_n \tag{17}$$

and

$$\textstyle\sum_1^n \epsilon_i\sigma_i - nR(G_n) \leqq BH_n + \sum_1^n \epsilon_i(\sigma_i - \sigma_{i+1}). \tag{18}$$

The expectation of the last term in (18) is bounded by a direct extension of Lemma 2 [7]. Here we state and prove the needed specialization of that extension.

LEMMA. *Under the assumptions of Theorem* 2,

$$|E(\epsilon(\sigma(\omega + Z) - \sigma(w' + Z)))| \leqq B\|w - w'\|_1 \tag{19}$$

for all $\epsilon \varepsilon M$ *and* $w, w' \varepsilon m^+$.

PROOF. With $\mathfrak{X} = [0, 1]^\infty$ we write

$$E(\epsilon\sigma(w' + Z)) = \int_{\mathfrak{X}} \epsilon\sigma(w' + z)\, d\mu(z) = \int_{T\mathfrak{X}} \epsilon\sigma(w + v)\, d\nu(v)$$

where $\nu = \mu T^{-1}$ is the measure induced by the transformation $v = Tz = w' - w + z$. Therefore,

$$E(\epsilon(\sigma(w + Z) - \sigma(w' + Z))) \leqq \int_{\mathfrak{X}} \epsilon\sigma(w + z)\, d\mu(z) - \int_{\mathfrak{X} \cap T\mathfrak{X}} \epsilon\sigma(w + v)\, d\nu(v)$$

and, since the restrictions of μ and ν to $\mathfrak{X} \cap T\mathfrak{X}$ are equal,

$$E(\epsilon(\sigma(w + Z) - \sigma(w' + Z))) \leqq B\mu(\mathfrak{X} - T\mathfrak{X}). \tag{20}$$

Since $z \varepsilon \mathfrak{X} - T\mathfrak{X}$ if and only if $z_j \varepsilon [0, 1]$ for all j and $z_j - w_j' + w_j \not\varepsilon [0, 1]$ for some j, it follows that

$$\begin{aligned} \mu(\mathfrak{X} - T\mathfrak{X}) &\leqq \textstyle\sum_1^\infty \mu[Z_j < w_j' - w_j \text{ or } Z_j > 1 + w_j' - w_j] \\ &\leqq \textstyle\sum_1^\infty \{(w_j' - w_j)^+ + (w_j - w_j')^+\} \\ &= \|w - w'\|_1. \end{aligned} \tag{21}$$

The proof is completed by applying (21) to (20) and then interchanging the roles of w and w'.

Returning to the proof of the theorem we apply the lemma with the specification $w = H_{i-1}^{-1} E_{i-1}$, $w' = H_i^{-1} E_i$. Here $\|w - w'\|_1 \leqq 2H_i^{-1}$ so the proof is complete.

The choice $H_i = i^{\frac{1}{2}}$ in (15) yields a sequence strategy achieving $O(n^{-\frac{1}{2}})$ uniformly in $\boldsymbol{\varepsilon}$.

4. Concluding remarks. Hannan ([7], Appendix) has related average risk convergence to $R(G_n)$ and convergence to $R(G)$ in the case where player I repeatedly uses the same mixed strategy G to generate a move ϵ. If the ϵ_i are independent and identically distributed G and $\mathbf{E}$ denotes expectation with respect to G^∞, then for each $\sigma \varepsilon N^*$ and each i, $\mathbf{E}(\epsilon_i \sigma) \geqq R(G)$. Therefore, with sequence strategies and artificial randomization, $R(G) \leqq \mathbf{E}[n^{-1} \sum_1^n E(\epsilon_i \sigma_i)]$, where the joint measurability of $\epsilon_i \sigma_i$ is assumed to allow the interchange of $\mathbf{E}$ and E. Since quite generally $\mathbf{E}[R(G_n)] \leqq R(G)$, for example, see [5], Remark 3, we have

$$(22) \quad 0 \leqq \mathbf{E}[n^{-1} \textstyle\sum_1^n E(\epsilon_i \sigma_i)] - R(G) \leqq \mathbf{E}[n^{-1} \textstyle\sum_1^n E(\epsilon_i \sigma_i) - R(G_n)].$$

Hence, a sequence strategy which conditional on $\boldsymbol{\varepsilon}$ has average risk approaching $R(G_n)$ uniformly in $\boldsymbol{\varepsilon}$ has average risk approaching $R(G)$ uniformly in G, and the convergence of $\mathbf{E}[R(G_n)]$ to $R(G)$ follows as a corollary.

We conclude with a brief discussion of the hypotheses of Theorems 1 and 2. In Section 2 the boundedness condition (A2) is necessary for uniform convergence results. However, (A1) is not so essential; in fact, to derive (10) we need only assume that $\inf \{w\sigma \mid \sigma \varepsilon N^*\}$ is attained for each discrete measure w with finite support. This is the case if N is a compact topological space and each section $L(\epsilon, \cdot)$ is continuous. We have carried the stronger assumption (A1) because under it, it is possible to demonstrate a determination of $\sigma(\cdot)$ which makes $\epsilon_i'(\sigma_i - \sigma_{i+1})$, $i \varepsilon S_n$, measurable; and, therefore, $E(\epsilon_i'(\sigma_i - \sigma_{i+1}))$ of (11) meaningful, while this has not been accomplished under weaker conditions. This demonstration is given after the treatment of measurability in the countable M case of Section 3.

If M is countable and the component game satisfies (A1) and (A3), it is possible to demonstrate a determination of $\sigma(\cdot)$ such that for each $\epsilon \varepsilon M$, $\epsilon\sigma(w)$ is a measurable function of w; that is, $\epsilon\sigma(w)$ is a measurable function from the infinite product of the non-negative reals (with Borel σ-field) to the non-negative reals. For this purpose it is convenient to use the notation $\sigma_\epsilon = \epsilon\sigma$ and define for each $w \varepsilon m^+$

$$B_1(w) = \{\sigma \varepsilon N^* \mid \sigma \text{ is Bayes versus } w\}$$

and

$$\tau_1(w) = \min \{\sigma_1 \mid \sigma \varepsilon B_1(w)\}.$$

Continuing we let

$$B_{j+1}(w) = \{\sigma \varepsilon B_1(w) \mid \sigma_i = \tau_i(w),\ i = 1, \cdots, j\}$$

and

$$\tau_{j+1}(w) = \min \{\sigma_{j+1} \mid \sigma \varepsilon B_{j+1}(w)\}, \qquad j = 1, 2, \cdots.$$

From (A1) it follows that $\bigcap_j B_j(w)$ is non-empty so there exist $\sigma \varepsilon N^*$ such that $\sigma_j = \tau_j(w)$ for all $j = 1, 2, \cdots$. For each $w \varepsilon m^+$ we choose such a σ and call it $\sigma(w)$. As defined $\sigma(\cdot)$ is positive homogeneous on m^+ and has measurable coordinates.

We can let A_i play the role of M in the preceding paragraph and proceed sequentially to make a determination of $\sigma(\cdot)$ which ensures that each $E(\epsilon_i'(\sigma_i - \sigma_{i+1}))$, $i \varepsilon S_n$, is meaningful. In case M is finite such a determination results in the measurability of actual losses.

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