The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 356–365

ON THE ADMISSIBILITY OF A RANDOMIZED SYMMETRICAL DESIGN FOR THE PROBLEM OF A ONE WAY CLASSIFICATION[1]

By R. H. Farrell

Cornell University

1. Introduction. The paper by Kiefer [2], this paper, and Farrell [1], have resulted from a desire to begin giving a theoretical background to the choice of experimental designs in the analysis of variance. Thus it has been our purpose to formulate a decision theory meaning of admissibility for randomized designs and to use this definition to evaluate various procedures.

The present paper, as well as Kiefer [2], deals with the question of obtaining randomized designs having good power locally about the hypothesis. We show that in the case of the one way classification a certain randomized design, followed by use of the appropriate analysis of variance test, is an admissible procedure. The power function of this procedure locally about zero was investigated by Kiefer, op. cit., who gave definitions of optimality (to be distinguished from admissibility) and applied his definitions to the one-, two-, and three-way classifications. It is the author's conclusion, based on the contents of the papers cited, that a partial theory of design can be developed from convexity considerations provided one is willing to use randomized designs and base his choice upon the partial ordering of power functions.

The present paper has a main non-mathematical conclusion. This is, the practical statistician demands more than that his procedure have optimum power locally about zero. When Kiefer's paper, op. cit., was evaluated prior to publication he received criticism that no one would want to use such randomized procedures. The referee of the present paper writes "Perhaps it would help if some small effort was made . . . to give an example where one might conceivably want to perform an experiment by taking all observations from one class." Yet, in spite of these objections the mathematics is quite clear (but hard), that locally, good power is obtained using randomized designs as described.

We begin with a formulation of the admissibility concept. We suppose throughout that N observations are to be taken. If there are I "factors" under consideration then a design consists in part of specification of a vector $n^T = (n_1, \cdots, n_I)$ of integers, the design vector, such that if $1 \leqq i \leqq I$, then $n_i \geqq 0$ and n_i observations are taken on the ith factor. Therefore $N = n_1 + n_2 + \cdots + n_I$. To each design vector n we may form the set $\mathcal{R}_n$ of all risk functions of tests which use the observations taken according to the design vector n. $\mathcal{R}_n$ is then a convex (usually compact) set. The number of possible non-randomized designs is then the number J of partitions of N into I non-negative integer summands. A randomized design

Received 28 March 1968; revised 18 November 1968.

[1] This research was supported in part by the Office of Naval Research under Contract Nonr 401(50). Reproduction in whole or in part is permitted for any purpose of the United States Government.

consists of specifying a probability vector $p^T = (p_1, \cdots, p_J)$ such that if $1 \leqq j \leqq J$ then the design vector $n^{(j)}$ is used with probability p_j. In order to completely specify the randomized design we must state the test function φ_j that will be used when observations are taken using the design vector $n^{(j)}$. The set $\mathcal{R}$ of all possible risk functions is then the convex hull of $\mathcal{R}_{n^{(1)}} \cup \cdots \cup \mathcal{R}_{n^{(J)}}$. We shall say that a randomized procedure is admissible if and only if the corresponding risk function in $\mathcal{R}$ is an admissible point of $\mathcal{R}$. We shall say occasionally that one risk point is as good as another meaning that as functions one function is everywhere less than or equal to the other function.

We wish to apply these definitions to the example of a one way classification with I classes. We assume that independently normally distributed random variables are observed such that if the design vector $n^T = (n_1, \cdots, n_I)$ is used and if $1 \leqq i \leqq I$ then n_i of the observations are normal (μ_i, σ^2) random variables. In order to simplify notations we will almost always speak in terms of the sufficient statistic $(n_1^{\frac{1}{2}}Y_1, \cdots, n_I^{\frac{1}{2}}Y_I)$, $(Z_1, \cdots, Z_I)$ such that $n_1^{\frac{1}{2}}Y_1, \cdots, n_I^{\frac{1}{2}}Y_I$, $Z_1, \cdots, Z_I$ are mutually independent random variables, if $1 \leqq i \leqq I$ and $n_i > 0$ then $n_i^{\frac{1}{2}} Y_i$ is normal $(n_i^{\frac{1}{2}} \mu_i, \sigma^2)$ and Z_i/σ^2 is $\chi^2_{n_i-1}$, while if $n_i = 0$ then $Y_i \equiv Z_i \equiv 0$. This convention will allow us to always think of a test function φ as a function of $2I$ variables. The normalization is so chosen that the normally distributed random variables always have variance σ^2.

The hypothesis to be tested is $\mu_1^2 + \cdots + \mu_I^2 = 0$; the alternative is $\mu_1^2 + \cdots + \mu_I^2 > 0$. When observations are taken according to a design n followed by use of the analysis of variance test based on the statistic $\sum_{i=1}^n (n_i^{\frac{1}{2}}Y_i)^2/\sum_{i=1}^n Z_i$, we shall let $\beta(\lambda; \alpha, n)$ denote the power of the test computed at the parameter point $\lambda = \frac{1}{2}\sum_{i=1}^n n_i\mu_i^2/\sigma^2$. The corresponding risk point in $\mathcal{R}_n$ is the function which equals α if $\lambda = 0$, which equals $1 - \beta(\lambda; \alpha, n)$ otherwise. More generally if the test $\psi(n_1^{\frac{1}{2}}Y_1, \cdots, n_I^{\frac{1}{2}}Y_I, Z_1, \cdots, Z_I)$ has power function $\beta(n_1^{\frac{1}{2}}\mu_1, \cdots, n_I^{\frac{1}{2}}\mu_I, \sigma; \psi)$ then the corresponding risk point in $\mathcal{R}_n$ is the function equal $\beta(0, \cdots, 0, \sigma; \psi)$ if $\lambda = 0$, and equal $1 - \beta(n_1\mu_1, \cdots, n_I\mu_I, \sigma; \psi)$ otherwise.

THEOREM. *Let $q^T = (q_1, \cdots, q_I)$ be a probability vector of I components. Suppose the randomized design, use the vector $(N, \cdot, \cdots, 0)$ with probability q_1, use the vector $(0, N, \cdots, 0)$ with probability $q_2, \cdots$, use the vector $(0, 0, \cdots, N)$ with probability q_I, followed in each case by a use of the analysis of variance test, is used.*

Let $n^{(1)}, \cdots, n^{(J)}, \varphi_1, \cdots, \varphi_J, p^T = (p_1, \cdots, p_J)$ be a randomized design which is as good as the design described in the preceding paragraph. Then each of $n^{(1)}, \cdots, n^{(J)}$ is one of the designs $(N, 0, \cdots, 0), \cdots, (0, 0, \cdots, N)$. Let $t_0 = 0$ and t_i of $n^{(1)}, \cdots, n^{(J)}$ be the design taking all observations in the ith class, $1 \leqq i \leqq I$. Let $p_{ij}^* = p_{t_0+\cdots+t_{i-1}+j}$, $\varphi_{ij}^* = \varphi_{t_0+\cdots+t_{i-1}+j}$, and $n^{(ij)*} = n^{(t_0+\cdots+t_{i-1}+j)}$ $1 \leqq i \leqq I$, $1 \leqq j \leqq t_i$, and suppose the designs $n^{(i1)*}, \cdots, n^{(it_i)*}$ each take all observations in the ith class, $1 \leqq i \leqq I$. Then if $1 \leqq i \leqq I$, $\sum_{j=1}^{t_i} p_{ij}^* = q_i$, $q_i^{-1} \sum_{j=1}^{t_i} p_{ij}^*\varphi_{ij}^*$ is the UMP size α analysis of variance test (except on a set of measure zero) of $\mu_i = 0$ against $\mu_i \neq 0$.

NOTE. In the statement of the theorem and in the sequel we let $J \geqq 1$ be an arbitrary integer rather than the number of partitions of N into I non-negative summands. This will allow us to assume that if $1 \leqq j \leqq J$ then $p_j > 0$.

2. Notation and lemmas. If a design vector n is used and the vector n has j non zero components then the corresponding analysis of variance test has j degrees of freedom in the numerator, $N - j$ degrees of freedom in the denominator, and non-centrailty parameter $\lambda = (\frac{1}{2})\sigma^{-2}(n_1\mu_1^2 + \cdots + n_I\mu_I^2)$. We will write for the power function of the size α $F_{j,N-j}$-test with non-centrality λ,

$$\beta(\lambda; \alpha, n) = \alpha + g_{j,N-j}(\alpha)\lambda + h_{j,N-j}(\alpha, \lambda), \tag{2.1}$$

where $g_{j,N-j}(\alpha)$ is the right derivative at zero of $\beta(\cdot\,; \alpha, n)$ and $h_{j,N-j}$ is the error term. Then

$$\sup\nolimits_{0<\lambda<\infty} \lambda^{-2}h_{j,N-j}(\alpha, \lambda) < \infty. \tag{2.2}$$

The following lemma is proven in Kiefer [2].

LEMMA 2.1. *Suppose $N_1 \leqq N_1'$, $N_1' + N_2' \leqq N_1 + N_2$, with at least one strict inequality. Then*

$$g_{N_1,N_2}(\alpha) > g_{N_1',N_2'}(\alpha).$$

In addition we need the following lemmas.

LEMMA 2.2. *Let $1 \leqq j \leqq N - 1$ and let $\alpha, \alpha_1, \cdots, \alpha_k$ be numbers between 0 and 1, and let $(p_1, \cdots, p_k)$ be a probability vector such that $\alpha = p_1\alpha_1 + \cdots + p_k\alpha_k$. Then $\sum_{i=1}^k p_i g_{j,N-j}(\alpha_j) \leqq g_{j,N-j}(\alpha)$. Strict inequality holds unless $\alpha_i \neq \alpha$ implies $p_i = 0$, $1 \leqq i \leqq k$.*

PROOF. Let ψ_i be the test function of the UMP one sided size α_i $F_{j,N-j}$ test for a random variable Y having a $F_{j,N-j}$ density. Then by (2.1) the power function of $\sum_{i=1}^k p_i\psi_i$ is

$$\begin{aligned}\textstyle\sum_{i=1}^k p_i(\alpha_i + g_{j,N-j}(\alpha_i)\lambda + h_{j,N-j}(\alpha_i, \lambda))&\\ = \alpha + (\textstyle\sum_{i=1}^k p_i g_{j,N-j}(\alpha_i))\lambda + \textstyle\sum_{i=1}^k p_i h_{j,N-j}(\alpha_i, \lambda).&\end{aligned}$$

Therefore the randomized test $p_1\psi_1 + \cdots + p_k\psi_k$ has size α. However there are UMP one sided tests based on a F-statistic, so that

$$\begin{aligned}\alpha + (\textstyle\sum_{i=1}^k p_i g_{j,N-j}(\alpha_i))\lambda + \textstyle\sum_{i=1}^k p_i h_{j,N-j}(\alpha_i, \lambda)&\\ \leqq \alpha + g_{j,N-j}(\alpha)\lambda + h_{j,N-j}(\alpha, \lambda).&\end{aligned} \tag{2.3}$$

Subtract α from both sides, divide by λ and let $\lambda \to 0+$. From this we obtain $\sum_{i=1}^k p_i g_{j,N-j}(\alpha_i) \leqq g_{j,N-j}(\alpha)$. If $p_1\psi_1 + \cdots + p_k\psi_k$ is essentially different from the analysis of variance test then strict inequality holds in (2.3) if $\lambda > 0$. It then follows by the argument used above that

$$\textstyle\sum_{i=1}^k p_i g_{j,N-j}(\alpha_i) < g_{j,N-j}(\alpha). \qquad \square$$

The first step of the proof in Section 3 replaces the given test functions by test functions that are invariant under sign and scale changes. The justification for this given in the next lemma.

LEMMA 2.3. *Let design vectors* $n^{(1)}, \cdots, n^{(J)}$ *be specified with* $\psi_1, \cdots, \psi_J$ *the associated test functions. Let* $p^T = (p_1, \cdots, p_J)$ *be the probability vector specifying the probabilities with which* $n^{(1)}, \cdots, n^{(J)}$ *will be selected. If* $1 \leqq j \leqq J$ *let* $n^{(j)T} = (n_{1j}, \cdots, n_{Ij})$ *and let the power function of* $n^{(j)}, \psi_j$ *be*

$$\beta(n_{1j}^{\frac{1}{2}}\mu_1, \cdots, n_{Ij}^{\frac{1}{2}}\mu_I, \sigma^2; \alpha_j, n^{(j)}, \psi_j).$$

Let $\alpha = \sum_{j=1}^J p_j\alpha_j$. *Suppose the power function* $\sum_{j=1}^J p_j\beta(n_{1j}^{\frac{1}{2}}\mu_1, \cdots, n_{Ij}^{\frac{1}{2}}\mu_I, \sigma^2; \alpha_j, n^{(j)}, \psi_j) \geqq \beta(\lambda; \alpha, n)$ *(See (2.1)). Then there exist test functions* $\psi_1^*, \cdots, \psi_J^*$ *which are invariant under sign and scale changes and such that*

(2.4) (i) *the randomized design* $n^{(1)}, \cdots, n^{(J)}, \psi_1^*, \cdots, \psi_J^*, (p_1, \cdots, p_J)$ *is* size α, *and for all parameter values,*

(ii) $\beta(\lambda; \alpha, n) \leqq \sum_{j=1}^J p_j\beta(n_{1j}^{\frac{1}{2}}\mu_1, \cdots, n_{Ij}^{\frac{1}{2}}\mu_I, \sigma^2; \alpha_j, n^{(j)}, \psi_j^*)$.

PROOF. We make an application of the development of the Hunt-Stein theory as presented in Lehmann [3].

$$\text{If } n \geqq 1 \text{ let } \gamma_n = \int_{1/n}^n dx/x. \tag{2.5}$$

Since analysis of variance tests are invariant under scale change, we find

$$\begin{aligned} \beta(\lambda; \alpha, n) &= \beta(x^2\lambda/x^2; \alpha, n) \\ &\leqq \sum_{j=1}^J p_j\beta(xn_{1j}^{\frac{1}{2}}\mu_1, \cdots, xn_{Ij}^{\frac{1}{2}}\mu_I, x^2\sigma^2; \alpha_j, n^{(j)}, \psi_j). \end{aligned} \tag{2.6}$$

Integrate both sides of (2.6) by the probability density $1/(\gamma_n x)$, $1/n \leqq x \leqq n$, and, by change of the order of integration together with a change of variable, we find

$$\beta(\lambda; \alpha, n) \leqq \sum_{j=1}^J p_j\beta(n_{1j}^{\frac{1}{2}}\mu_1, \cdots, n_{Ij}^{\frac{1}{2}}\mu_I, \sigma^2; \alpha_j, n^{(j)}, \psi_{nj}), \tag{2.7}$$

where

$$\begin{aligned} \text{if } 1 \leqq j \leqq J \text{ then } &\psi_{nj}(x_1, \cdots, x_N) \\ &= (1/\gamma_n) \int_{1/n}^n t^{-1}\psi_j(tx_1, \cdots, tx_N)\, dt. \end{aligned} \tag{2.8}$$

The functions ψ_{nj} only take values between 0 and 1. Thus we may choose a subsequence $\{\psi_{n1j}, n \geqq 1\}$ such that

$$\text{weak } \lim_{n\to\infty} \psi_{n1j} = \psi_{1j} \quad \text{exists}, \qquad 1 \leqq j \leqq J. \tag{2.9}$$

It follows from the discussion given by Lehmann, op. cit., Section 8.4, that ψ_{1j} is an almost invariant function and that we may choose ψ_{1j} to be a scale invariant function. See Lehmann, op. cit., Section 6.5. Further, for every choice of the parameters,

$$\begin{aligned} \lim_{n\to\infty} &\beta(n_{1j}^{\frac{1}{2}}\mu_1, \cdots, n_{Ij}^{\frac{1}{2}}\mu_I, \sigma^2; \alpha_j, n^{(j)}, \psi_{n1j}) \\ &= \beta(n_{1j}^{\frac{1}{2}}\mu_1, \cdots, n_{Ij}^{\frac{1}{2}}\mu_I, \sigma^2; \alpha_j, n^{(j)}, \psi_{1j}). \end{aligned} \tag{2.10}$$

Hence (2.4) holds for the tests $\psi_{11}, \cdots, \chi_{1J}$. By averaging over the 2^N possible sign changes of the values of $x_1, \cdots, x_N$, the tests $\psi_{11}, \cdots, \psi_{1J}$ may be replaced by sign and scale invariant tests $\psi_1^*, \cdots, \psi_J^*$ such that (2.4) holds. □

LEMMA 2.4. *Let* $Y_1, \cdots, Y_J$ *be independent normal* (η, τ^2) *random variables.*

Let ψ be a sign and scale invariant function of $y_1, \cdots, y_J$. Then there exists a measurable function ψ' of a real variable such that $E_{\eta,\tau}\psi(Y_1, \cdots, Y_J) = E_{\eta,\tau}\psi'(\bar{Y}^2/\sum_{j=1}^J (Y_j - \bar{Y})^2)$, $\bar{Y} = (Y_1 + \cdots + Y_J)/J$, holding for all choices of the parameters.

PROOF. $(\bar{Y}, \sum_{j=1}^J (Y_j - \bar{Y})^2)$ is a sufficient statistic. Therefore a function ψ'' of two real variables exists such that if $x > 0$, if $-\infty < \eta < \infty$, and if $\tau > 0$, then

$$\begin{aligned} E_{\eta,\tau}\psi''(x\bar{Y}, x^2 \textstyle\sum_{j=1}^J (Y_i - \bar{Y})^2) &= E_{\eta,\tau}\psi''(-x\bar{Y}, x^2 \textstyle\sum_{j=1}^J (Y_i - \bar{Y})^2) \\ &= E_{x\eta,x\tau}\psi''(\bar{Y}, \textstyle\sum_{j=1}^J (Y_i - \bar{Y})^2) \\ &= E_{x\eta,x\tau}\psi(Y_1, \cdots, Y_J) = E_{\eta,\lambda}\psi(Y_1, \cdots, Y_J) \\ &= E_{\eta,\tau}\psi''(\bar{Y}, \textstyle\sum_{j=1}^J (Y_j - \bar{Y})^2). \end{aligned}$$

Since the sufficient statistic is complete it follows that for each $x > 0$ and $\epsilon = \pm 1$, with probability one,

$$\psi''(\epsilon x\bar{Y}, x^2 \textstyle\sum_{j=1}^J (Y_j - \bar{Y})^2) = \psi''(\bar{Y}, \textstyle\sum_{j=1}^J (Y_j - \bar{Y})^2).$$

Therefore in the sense of Lehmann [2], Section 6.5, ψ'' is an almost invariant function. By Lehmann, op. cit., there exists an invariant measurable function ψ''' of two real variables such that if $-\infty < \eta < \infty$, if $\tau > 0$, then

$$E_{\eta,\tau}\psi''(\bar{Y}, \textstyle\sum_{j=1}^J (Y_j - \bar{Y})^2) = E_{\eta,\tau}\psi'''(\bar{Y}, \textstyle\sum_{j=1}^J (Y_j - \bar{Y})^2).$$

The invariance of ψ''' implies there is a measurable function ψ' of a single real variable such that $\psi'(x^2/y) = \psi'''(x, y)$, $-\infty < x < \infty$, $y > 0$. □

LEMMA 2.5. *Let the sign and scale invariant test function ψ' based on the design vector $m^T = (m_1, \cdots, m_I)$ have power function*

$$\alpha' + (\tfrac{1}{2}) \textstyle\sum_{r=1}^I \sum_{s=1}^I \delta_{rs}\sigma^{-2}m_r^{\frac{1}{2}}m_s^{\frac{1}{2}}\mu_r\mu_s + \textit{higher order terms}. \tag{2.11}$$

If $1 \leqq i \leqq I$, and $m_i > 0$, then

$$\delta_{ii} \leqq g_{1,N-1}(\alpha'). \tag{2.12}$$

PROOF. To prove this lemma, consider the test function ψ' as a test of $\mu_i = 0$ against $\mu_i \neq 0$ when the parameters are restricted to the subset of the parameter space $\mu_1 = \cdots = \mu_{i-1} = \mu_{i+1} = \cdots = \mu_I = 0$. Then assuming $m_i > 0$ we have m_i observations which are normal (μ_i, σ^2) and $N - m_i$ observations which are normal $(0, \sigma^2)$. It will be convenient to let these observations be $X_1, \cdots, X_{m_i}$ and $X_{m_i+1}, \cdots, X_N$ respectively. Define $\bar{X} = (X_1 + \cdots + X_{m_i})/m_i^{\frac{1}{2}}$, $S^2 = \sum_{i=1}^{m_i} ((X_i - \bar{X})/m_i^{\frac{1}{2}})^2$, and $T^2 = X_{m_i+1}^2 + \cdots + X_N^2$. The sufficient statistic is $(\bar{X}, U)$ with $U = (S^2 + T^2)^{\frac{1}{2}}$. By taking conditional expectation relative to the sufficient statistic we may replace ψ' by ψ'', a measurable function of two-variables such that

$$E_{\mu,\sigma}\psi'(X_1 \quad \cdots, X_N) = E_{\mu,\sigma}\psi''(\bar{X}, U).$$

As in the proof of Lemma 2.4 we may suppose that if $\epsilon = \pm 1$ and $x > 0$ then $\psi''(\epsilon xy, xu) = \psi''(y, u)$, $-\infty < y < \infty$, $u > 0$. Thus there is a function ψ''' of a single real variable such that $\psi'''(y^2/u^2) = \psi''(y, u)$. The statistic $(N-1)\bar{X}^2/U^2$ is a (non-central) $F_{1,N-1}$ statistic. The test $\psi'''(\bar{X}^2/U^2)$ is a size α' test based on the F-statistic $\bar{X}^2/U^2$ with parameter $\lambda = \frac{1}{2}m_i\mu_i^2/\sigma^2$. The slope of the power function at $\lambda = 0$ is (compare with the UMP size α test) $\leqq g_{1,N-1}(\alpha')$ while from (2.11) the slope is δ_{ii}. Therefore (2.12) is proven. □

LEMMA 2.6. *Let the sign and scale invariant function ψ' based on the design vector $m^T = (m_1, \cdots, m_I)$ have power function* (2.11). *Suppose m has k non-zero components, say $m_{a_1}, \cdots, m_{a_k}$. Then*

$$k^{-1}(\delta_{a_1a_1} + \cdots + \delta_{a_ka_k}) \leqq g_{k,N-k}(\alpha'). \tag{2.13}$$

PROOF. We apply a result due to Wald [4]. Suppose for simplicity that $m_1 \neq 0, \cdots, m_k \neq 0, m_{k+1} = \cdots = m_I = 0$. Let $\eta_i = m_i^{\frac{1}{2}}\mu_i$, $1 \leqq i \leqq I$, so that the parameters of the problem are the $1 \times k$ vector η, and σ. Write the power function as $\beta(\eta/\sigma; \alpha, m, \psi')$. If U is an $k \times k$ orthogonal matrix, then Wald's result asserts that

$$\int \beta(U\eta/\sigma; \alpha', m, \psi')\, dU \leqq \alpha' + \sigma^{-2}g_{k,N-k}(\alpha')(\eta_1^2 + \cdots + \eta_k^2) + \text{higher order terms}\cdot \tag{2.14}$$

If we write $D = (\delta_{rs})$ as the $k \times k$ matrix formed from $\delta_{11}, \cdots, \delta_{kk}$, then

$$\int UDU^T\, dU = (k^{-1} \operatorname{tr} D) \quad \text{identity matrix.} \tag{2.15}$$

Putting (2.11) with (2.14), using (2.15), we may write

$$\alpha' + \sigma^{-2}(k^{-1}\operatorname{tr} D)(\eta_1^2 + \cdots + \eta_k^2) \leqq \alpha' + \sigma^{-2}g_{k,N-k}(\alpha')(\eta_1^2 + \cdots + \eta_k^2) + \text{higher order terms.} \tag{2.16}$$

The assertion (2.13) now follows. □

3. Proof of the theorem. Let $n^{(1)}, \cdots, n^{(J)}, \varphi_1, \cdots, \varphi_J, p^T = (p_1, \cdots, p_J)$ be a randomized design which is as good as the design described in the statement of the theorem. The power function of the randomized design using the analysis of variance tests is

$$\alpha + g_{1,N-1}(\alpha) \textstyle\sum_{i=1}^I (Nq_i\mu_i^2/\sigma^2) + h(\mu_1, \cdots, \mu_I, \sigma), \tag{3.1}$$

where

$$h(\mu_1, \cdots, \mu_I, \sigma) = \textstyle\sum_{i=1}^I p_i h_{1,N-1}(\alpha; N\mu_i^2/\sigma^2). \tag{3.2}$$

By Lemma 2.3 we may replace the test functions $\varphi_1, \cdots, \varphi_J$ by test functions $\varphi_1', \cdots, \varphi_J'$ which are invariant under sign and scale changes such that if $1 \leqq j \leqq J$ then φ_j' has size α_j. Then by Lemma 2.3 if φ_j' has power function

$$\alpha_j + \sigma^{-2} \textstyle\sum_{r=1}^I \sum_{s=1}^I \delta_{rs}^j n_{rj}^{\frac{1}{2}} n_{sj}^{\frac{1}{2}} \mu_r\mu_s + \text{higher order terms,} \tag{3.3}$$

we obtain

$$\begin{aligned}(3.4)\quad &\alpha + g_{1,N-1}(\alpha) \textstyle\sum_{i=1}^{I} (Nq_i\mu_i^2/\sigma^2) + h(\mu_1, \cdots, \mu_I, \sigma)\\ &\leqq \textstyle\sum_{j=1}^{J} p_j\alpha_j + \sigma^{-2} \sum_{j=1}^{J} \sum_{r=1}^{I} \sum_{s=1}^{I} p_j\delta_{rs}^j n_{rj}^{\frac{1}{2}} n_{sj}^{\frac{1}{2}} \mu_r\mu_s + \text{higher order terms.}\end{aligned}$$

Since the randomized design $n^{(1)}, \cdots, n^{(J)}, \varphi_1, \cdots, \varphi_J, (p_1, \cdots, p_J)$ is as good as the design using the analysis of variance tests, we obtain from (3.4) that

$$(3.5)\qquad \alpha = \textstyle\sum_{j=1}^{J} p_j\alpha_j .$$

In (3.4) subtract α from both sides, set $\mu_1 = \cdots = \mu_{i-1} = \mu_{i+1} = \cdots = \mu_I = 0$, divide by μ_i and let $\mu_i \to 0+$. We obtain

$$(3.6)\qquad Nq_i\mu_i^2 g_{1,N-1}(\alpha) \leqq \textstyle\sum_{j=1}^{J} n_{ij}p_j\mu_i^2\delta_{ii}^j .$$

Using (2.12),

$$\begin{aligned}(3.7)\qquad &Nq_i\mu_i^2 g_{1,N-1}(\alpha) \leqq \textstyle\sum_{j=1}^{J} n_{ij}p_j\mu_i^2 g_{1,N-1}(\alpha_j);\\ &Ng_{1,N-1}(\alpha) \textstyle\sum_{i=1}^{I} q_i\mu_i^2 \leqq \sum_{i=1}^{I} \sum_{j=1}^{J} n_{ij}p_j\mu_i^2 g_{1,N-1}(\alpha_j).\end{aligned}$$

In (3.7) let $\mu_1 = \cdots = \mu_I = \mu \neq 0$. Then we obtain

$$(3.8)\qquad Ng_{1,N-1}(\alpha) \leqq N \textstyle\sum_{j=1}^{J} p_j g_{1,N-1}(\alpha_j) \leqq Ng_{1,N-1}(\alpha).$$

We may suppose without loss of generality that if $1 \leqq j \leqq J$ then $p_j \neq 0$. By Lemma 2.2 we must have strict inequality in (3.8) unless $\alpha_1 = \cdots = \alpha_J = \alpha$. Further, if for some $n_{ij} \neq 0$, $\delta_{ii}^j < g_{1,N-1}(\alpha)$, then strict inequality holds in (3.7) and hence in (3.8). That cannot be.

On the other hand, if $\delta_{ii}^j = g_{1,N-1}(\alpha)$, $1 \leqq i \leqq I$, $1 \leqq j \leqq J$, then, if k_j is the number of non-zero entries in $n^{(j)}$, by Lemma 2.6,

$$(3.9)\qquad k_j^{-1} \textstyle\sum_{\{i \mid n_{ij} \neq 0\}} \delta_{ii}^j \leqq g_{k_j, N-k_j}(\alpha) \leqq g_{1,N-1}(\alpha).$$

By Lemma 2.1, there must be strict inequality in (3.9) unless $k_j = 1$, $j = 1, \cdots, J$. Since by hypothesis the left side of (3.9) equals $g_{1,N-1}(\alpha)$, we have shown that $k_1 = \cdots = k_J = 1$ must hold.

We have shown that each test is a size α test and that each of the design vectors $n^{(1)}, \cdots, n^{(J)}$ specifies that all observations be taken in the same class. If $n^{(i)}$ and $n^{(j)}$ are the same then randomization between them is equivalent (as far as the power function is concerned) to using a randomized test function. Therefore we may define I randomized tests $\varphi_1^*, \cdots, \varphi_I^*$, as in the statement of the theorem, such that all N observations are in class i with probability p_i^* when φ_i^* is used, $1 \leqq i \leqq I$. Then the test φ_i^* has power function

$$(3.10)\qquad \beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \alpha, \varphi_i^*).$$

It follows that

$$\begin{aligned}(3.11)\quad &\alpha + g_{1,N-1}(\alpha)(\textstyle\sum_{i=1}^{I} Nq_i\mu_i^2/\sigma^2) + \sum_{i=1}^{I} q_i h_{1,N-1}(\alpha; N\mu_i^2/\sigma^2)\\ &\leqq \textstyle\sum_{i=1}^{I} p_i\beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \alpha, \varphi_i^*).\end{aligned}$$

Take $\mu_1 = \cdots = \mu_{i-1} = \mu_{i+1} = \cdots = \mu_I = 0$ and obtain

$$\alpha + \sigma^{-2}Nq_i\mu_i^2 g_{1,N-1}(\alpha) + \text{higher order terms} \leqq \alpha(1-p_i) + p_i\beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \alpha, \varphi_i^*). \tag{3.12}$$

As stated at the beginning of the section, φ_i^* is a sign and scale invariant test function. By Lemma 2.4 we may replace $\varphi_1^*, \cdots, \varphi_I^*$ by test functions $\varphi_1'', \cdots, \varphi_I''$ such that when the F-statistic is inserted the same power function results. Thus $\beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \alpha, \varphi_i^*)$ is a function only of $\eta_i = N\mu_i^2/\sigma^2$ and the derivative with respect to this variable evaluated at $\eta_i = 0$ has a value $\leqq g_{1,N-1}(\alpha)$. Thus from (3.12) we obtain

$$q_i \leqq p_i, \qquad 1 \leqq i \leqq I. \tag{3.13}$$

This implies $q_1 = p_1, \cdots, q_I = p_I$, since $1 = \sum_{i=1}^I q_i = \sum_{i=1}^I p_i$.

4. Proof of the theorem, continued. In order to establish the admissibility statement of the theorem we will show the randomized symmetrical analysis of variance test described in the theorem is a Bayes procedure relative to an absolutely continuous prior measure on the alternative. We begin by considering the Bayes character of tests based on an $F_{1,N-1}$ statistic.

LEMMA 4.1. *Let $n \geqq 1$ be an integer, $\sigma > 0$ and $-\infty < \mu < \infty$. Suppose constant $\sigma^{-n}|z|^{n-2}\exp(-[(x-\mu)^2 + z^2]/2\sigma^2) = f(x, z, \mu, \sigma^2)$ is a probability density in the two variables x, z. On the alternative let $\mu = \beta\sigma^2\eta$ and $\sigma^{-2} = \alpha + \eta^2$. Conditional on α, β let η have density function $(\alpha+\eta^2)^{-n/2}\exp(\beta^2\eta^2/2(\alpha+\eta^2)) = g_1(\eta, \alpha, \beta)$. Let α take values $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$, and let β have density function constant $e^{-\beta^2}$. On the hypothesis $\mu = 0$. Let $\sigma^{-2} = \alpha + \eta^2$ have, conditional on α, density $(\alpha+\eta^2)^{-n/2} = g_0(\eta, \alpha)$. Let α be distributed as above. Then*

$$\begin{aligned}&[\textstyle\sum_{n=1}^{\infty} 2^{-n}\int f(x, z, \beta\eta/(\alpha+\eta^2), \alpha+\eta^2)g_1(\eta, 1/n, \beta)\exp(-\beta^2)\,d\eta\,d\beta \\ &\cdot[\textstyle\sum_{n=1}^{\infty} 2^{-n}\int f(x, z, 0, \alpha+\eta^2)g_0(\eta, 1/n)\,d\eta]^{-1} = \int_{-\infty}^{\infty} e^{-\beta^2(1-x^2/2(x^2+z^2))}\,d\beta.\end{aligned} \tag{4.1}$$

The proof of Lemma 4.1 is an easy calculation which is left to the reader. It is to be observed that (4.1) is a strictly increasing strictly convex function of $x^2/(x^2+z^2)$.

For each α the map $\beta, \eta \to \beta\sigma^2\eta, (\alpha+\eta^2)^{-\frac{1}{2}} = \mu, \sigma$ preserves sets of zero Lebesgue measure. From this it is easy to see that the map $\alpha, \beta, \eta \to \mu, \sigma$ induces on μ, σ an absolutely continuous probability measure which is positive on every non-empty open set. We let g_3 be the density function of this measure, and, let g_2 be the density function of σ induced by the map $\alpha, \eta \to \sigma$ when $\mu = 0$ and η has (conditional) density $g_0(\eta, \alpha)$. The conditional density $g_3(\cdot, \sigma)(\int g_3(\mu, \sigma)\,d\mu)^{-1} = g_4(\cdot/\sigma)$ is thus everywhere positive.

On $\mathbb{R}_{I+1}$ we construct a probability density functions by, if $1 \leqq i \leqq I$, $-\infty < \mu_i < \infty, \sigma > 0$,

$$h(\mu_1, \cdots, \mu_I, \sigma) = (\textstyle\prod_{i=1}^I g_4(\mu_i/\sigma))\int g_3(\mu, \sigma)\,d\mu. \tag{4.2}$$

Let $\psi_1, \cdots, \psi_I$ be Borel measurable test functions of two variables $N^{\frac{1}{2}}Y$, Z so that

$$(4.3)\quad \begin{aligned}&\beta(N^{\frac{1}{2}}\mu, \sigma^2; \psi_i)\\ &\quad = \int\int \text{constant } \sigma^{-N} |z|^{N-2} \psi_i(x, z) \exp(-((x - N^{\frac{1}{2}}\mu)^2 + z^2)/2\sigma^2)\, dx\, dz.\end{aligned}$$

Using h as density over the alternative and g_2 as density over the hypohesis, with respective weights ν and $1 - \nu$, the Bayes risk of the randomized design $(N, 0, \cdots 0), (0, N, \cdots, 0), \cdots, (0, 0, \cdots, N), \psi_1, \cdots, \psi_I, q_1, \cdots, q_I$, is

$$(4.4)\quad \begin{aligned}&\nu \int \textstyle\sum_{i=1}^{I} q_i(1 - \beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \psi_i))h(\mu_1, \cdots, \mu_I, \sigma)\, d\mu_1 \cdots d\mu_I\, d\sigma\\ &\qquad + (1 - \nu) \int \textstyle\sum_{i=1}^{I} q_i\beta(0, \sigma^2; \psi_i)g_2(\sigma)\, d\sigma\\ &\quad = \textstyle\sum_{i=1}^{I} q_i[\nu \int\int\int (1 - \beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \psi_1))\}g_4(\mu_i \mid \sigma)g_3(\mu, \sigma)\, d\mu\, d\mu_i\, d\sigma\\ &\qquad + (1 - \nu) \int \beta(0, \sigma^2; \psi_i)g_2(\sigma)\, d\sigma].\end{aligned}$$

For the density νg_3 over the alternative and $(1 - \nu)g_2$ over the hypothesis the essentially unique Bayes test is a F-test. We let its Bayes risk be K. We have proven the following lemma.

LEMMA 4.2. *The randomized symmetrical analysis of variance test of the theorem has Bayes risk K relative to the densities νh over the alternative and $(1 - \nu)g_2$ over the hypothesis and is a Bayes procedure within the subset $\mathcal{R}^0$ of $\mathcal{R}$ consisting of risk points having the functional form*

$$(4.5)\qquad \textstyle\sum_{i=1}^{I} p_i^*\beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \psi_i^*).$$

Let n_0 be the design vector $n_0^{\ T} = (N, 0, \cdots, 0)$. Suppose the F-test Bayes relative to g_2, g_3 has size α. Then we may describe in the notation of (2.1) the power function of the randomized symmetrical analysis of variance test of the theorem as

$$(4.5)\qquad \textstyle\sum_{i=1}^{I} q_i\beta(N\mu_i^2/\sigma^2; \alpha, n_0).$$

The proof of Section 3 shows that if a randomized design is as good as the randomized symmetrical analysis of variance test then the test functions

$$(4.6)\qquad \text{if } 1 \leqq i \leqq I, \qquad \varphi_i^* = q_i^{-1} \textstyle\sum_{j=1}^{t_i} p_{ij}^*\varphi_{ij}^*$$

defined in the statement of the theorem must satisfy for all parameter values

$$(4.7)\qquad \textstyle\sum_{i=1}^{I} q_i\beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \varphi_i^*) \geqq \sum_{i=1}^{I} q_i\beta(N\mu_i^2/\sigma^2; \alpha, n_0)$$

and if $\mu_1 = \cdots = \mu_I = 0$ then

$$(4.8)\qquad \textstyle\sum_{i=1}^{I} q_i\beta(0, \sigma^2; \varphi_i^*) = \sum_{i=1}^{I} q_i\beta(0; \alpha, n_0) = \alpha.$$

That is to say,

$$(4.9)\quad \textstyle\sum_{j=1}^{J} p_j\beta(n_1^{\frac{1}{2}}\mu_1, \cdots, n_I^{\frac{1}{2}}\mu_I, \sigma^2; n^{(j)}, \varphi_j) = \sum_{i=1}^{I} q_i\beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \varphi_i^*),$$

so that a risk point as good as that given in (4.5) must be a point of $\mathcal{R}^0$.

The expression given in (4.5) is Bayes relative to νh, $(1 - \nu)g_2$ so that integrating the difference of the two sides of (4.7) gives

$$(4.10)\quad 0 = \int \textstyle\sum_{i=1}^{I} q_i(\beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \varphi_i^*) - \beta(N\mu_i^2/\sigma^2; \alpha, n_0)) \cdot h(\mu, \cdots, \mu_I, \sigma)\, d\mu_1 \cdots d\mu_I\, d\sigma.$$

The integrand of (4.10) is a continuous non-negative function and h is everywhere positive. Therefore, for all parameter values

$$(4.11)\quad \textstyle\sum_{i=1}^{I} q_i\beta(N^{\frac{1}{2}}\mu_i, \sigma^2; \varphi_i^*) = \sum_{i=1}^{I} q_i\beta(N\mu_i^2/\sigma^2; \alpha, n_0).$$

Let $\mu_1 \to \infty$, $\mu_2 \to \infty$, $\cdots$, $\mu_I \to \infty$. The right side of (4.11) tends to one in value. This implies

$$(4.12)\quad \lim_{\mu\to\infty} \beta(N^{\frac{1}{2}}\mu, \sigma^2; \varphi_i^*) = 1.$$

Fix the values $\mu_1, \cdots, \mu_{i-1}, \mu_{i+1}, \cdots, \mu_I$ and take two distinct values μ_{i1}, μ_{i2}. From (4.11) we obtain

$$(4.13)\quad \beta(N^{\frac{1}{2}}\mu_{i2}, \sigma^2; \varphi_i^*) - \beta(N^{\frac{1}{2}}\mu_{i1}, \sigma^2; \varphi_i^*) = \beta(N\mu_{i2}^2/\sigma^2; \alpha, n_0) - \beta(N\mu_{i1}/\sigma^2; \alpha, n_0).$$

Let $\mu_{i1} \to \infty$ and use (4.12) to obtain for all $\sigma > 0$, $-\infty < \mu < \infty$,

$$(4.14)\quad \beta(N^{\frac{1}{2}}\mu, \sigma^2; \varphi_i^*) = \beta(N\mu^2/\sigma^2; \alpha, n_0).$$

The test φ_i^* has the same power function as that of a UMP size α F-test and therefore φ_i^* is essentially equal the essentially unique Bayes procedure.

The proof of the theorem has been completed.

REFERENCES

[1] Farrell, R. H. (1968). On the admissibility at ∞, within the class of randomized designs, of balanced designs. *Ann. Math. Statist.* **39** 1978–1994.

[2] Kiefer, J. (1958). On the nonrandomized and randomized nonoptimality of symmetrical designs. *Ann. Math. Statist.* **29** 675–699.

[3] Lehmann, E. L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.

[4] Wald, A. (1942). On the power function of the analysis of variance test. *Ann. Math. Statist.* **13** 434–439.