The Annals of Mathematical Statistics
1969, Vol. 40, No. 4, 1325–1343

A CLASS OF RANK ORDER TESTS FOR A GENERAL LINEAR HYPOTHESIS[1]

BY MADAN LAL PURI[2] AND PRANAB KUMAR SEN

Indiana University and University of North Carolina at Chapel Hill

0. Summary. For a general multivariate linear hypothesis testing problem, a class of permutationally (conditionally) distribution-free tests is proposed and studied. The asymptotic distribution theory of the proposed class of test statistics is studied along with a generalization of the elegant results of Hájek (1968) to the multistatistics and multivariate situations. Asymptotic power and optimality of the proposed tests are established and a characterization of the multivariate multisample location problem [cf. Puri and Sen (1966)] in terms of the proposed linear hypothesis is also considered.

1. Introduction. Consider a (double) sequence of stochastic matrices

$$\mathbf{E}_\nu = (\mathbf{X}_{\nu 1}, \cdots, \mathbf{X}_{\nu N_\nu}), \qquad 1 \leqq \nu < \infty,$$

where $\mathbf{X}'_{\nu i} = (X_{\nu i}^{(1)}, \cdots, X_{\nu i}^{(p)})$, $i = 1, \cdots, N_\nu$ are independent stochastic vectors having continuous cumulative distribution functions (cdf) $F_{\nu i}(\mathbf{x})$, $\mathbf{x} \,\varepsilon\, R^p$, $i = 1, \cdots, N_\nu$ respectively. It is assumed that

$$(1.1) \quad F_{\nu i}(\mathbf{x}) = F(\mathbf{x} - \boldsymbol{\alpha} - \boldsymbol{\beta}\mathbf{c}_{\nu i}), \qquad 1 \leqq i \leqq N_\nu, \qquad 1 \leqq \nu < \infty,$$

where $\boldsymbol{\alpha}' = (\alpha_1, \cdots, \alpha_p)$ and $\boldsymbol{\beta} = ((\beta_{jk}))$, $j = 1, \cdots, p$; $k = 1, \cdots, q$ are unknown parameters and $\mathbf{c}'_{\nu i} = (c_{\nu i}^{(1)}, \cdots, c_{\nu i}^{(q)})$, $i = 1, \cdots, N_\nu$ are known *regression constants* (*vectors*). Having observed $\mathbf{E}_\nu$ and assuming some conditions on F and $\mathbf{c}_{\nu i}$'s (to be stated in Section 2), we desire to test the null hypothesis of no regression, that is

$$(1.2) \qquad H_0: \boldsymbol{\beta} = \mathbf{0}^{p \times q},$$

against the set of alternatives that $\boldsymbol{\beta}$ is non-null.

A variety of tests for H_0 in (1.2), based on the assumed normality of $F(\mathbf{x})$, are available in the literature [cf. Anderson (1958, Chapter 8) and Rao (1965, Chapter 8)]. However the likelihood ratio test is one of the most adopted ones. In this paper, the assumption of multinormality of $F(\mathbf{x})$ is relaxed and a class of Chernoff-Savage-Hájek-type of rank-order tests is proposed and studied. These tests are valid for all continuous $F(\mathbf{x})$. In Section 2, along with the preliminary notions, these rank-order statistics are defined. Section 3 is concerned with certain

Received 15 February 1968; revised 26 February 1969.

[1] Research supported by (a) the National Institutes of Health, Public Health Service, Grant GM-12868 and (b) the Office of Naval Research, contract Nonr-285(38). Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

[2] Part of this work was done while the author had a visiting faculty appointment at the University of North Carolina at Chapel Hill.

permutational invariance structure of the joint distribution of $\mathbf{E}_\nu$ (when H_0 in (1.2) holds), and this is then utilized for the construction of permutationally distribution-free tests for H_0 in (1.2). In Section 4, by a generalization of Hájek's (1968) results, the asymptotic joint normality of the proposed rank-order statistics is established. Section 5 deals with the asymptotic power properties of the proposed tests for nearby alternatives, and Section 6 is concerned with the asymptotic optimality of the proposed tests, granted certain conditions on $F(\mathbf{x})$. In the last section, the relationship of the multivariate multisample location problem with the linear hypothesis in (1.1) and (1.2) is studied.

2. Preliminary notions and fundamental assumptions. Let $R_{\nu i}^{(j)}$ be the rank of $X_{\nu i}^{(j)}$ among $X_{\nu 1}^{(j)}, \cdots, X_{\nu N_\nu}^{(j)}$, that is,

$$R_{\nu i}^{(j)} = \sum_{i'=1}^{N_\nu} u(X_{\nu i}^{(j)} - X_{\nu i'}^{(j)}), \qquad i = 1, \cdots, N_\nu; \qquad j = 1, \cdots, p, \tag{2.1}$$

where $u(x)$ is equal to 1 or 0 according as $x \geqq 0$ or not. [Since $F(\mathbf{x})$ in (1.1) is assumed to be continuous, ties among $X_{\nu 1}^{(j)}, \cdots, X_{\nu N_\nu}^{(j)}$ can be neglected, in probability, for $j = 1, \cdots, p$.] Consider now p sets of score functions

$$\{a_{\nu j}(i), 1 \leqq i \leqq N_\nu\}, \qquad j = 1, \cdots, p, \qquad (\text{where } 1 \leqq \nu < \infty) \tag{2.2}$$

generated by functions $\{\varphi_j(t), 0 < t < 1\}, j = 1, \cdots, p$, in either of the following three ways:

$$a_{\nu j}(i) = \varphi_j(i/[N_\nu + 1]), \qquad 1 \leqq i \leqq N_\nu, \qquad j = 1, \cdots, p; \tag{2.3}$$

$$a_{\nu j}(i) = E\{\varphi_j(U_\nu^{(i)})\}, \qquad 1 \leqq i \leqq N_\nu, \qquad j = 1, \cdots, p; \tag{2.4}$$

$$a_{\nu j}(i) = N_\nu \int_{(i-1)/N_\nu}^{i/N_\nu} \varphi_j(t)\, dt, \qquad 1 \leqq i \leqq N_\nu, \qquad j = 1, \cdots, p; \tag{2.5}$$

where $U_\nu^{(1)} \leqq \cdots \leqq U_\nu^{(N_\nu)}$ denote an ordered sample of N_ν observations from the rectangular distribution over (0, 1).

Consider now the random variables

$$S_{\nu,jk} = \sum_{i=1}^{N_\nu} c_{\nu i}^{(k)} a_{\nu j}(R_{\nu i}^{(j)}), \qquad j = 1, \cdots, p, \qquad k = 1, \cdots, q. \tag{2.6}$$

Our proposed test for H_0 in (1.2) is based on the stochastic matrix

$$\mathbf{S}_\nu = ((S_{\nu,jk})). \tag{2.7}$$

In Section 3 we shall see that for the construction of some permutationally (conditionally) distribution-free tests for H_0, we require (for small samples) only the conditions that (i) the sum of product matrix in the regression constants is positive definite and (ii) the sum of product matrix of the scores (defined in (2.11)) is also positive definite. However, for large sample sizes, in order to justify the second condition and simplify the asymptotic distribution theory of the test statistic, we are required to impose certain regularity conditions on the regression constants as well as on the score functions. These are stated below.

ASSUMPTION I (Hájek). It is possible to express

$$\varphi_j(t) = \varphi_{j,1}(t) - \varphi_{j,2}(t), \qquad j = 1, \cdots, p; \qquad 0 < t < 1 \tag{2.8}$$

where the $\varphi_{j,k}(t)$, $k = 1, 2, j = 1, \cdots, p$ are all non-decreasing, square integrable and absolutely continuous in $(0, 1)$.

ASSUMPTION II (Hoeffding). In addition to (2.8)

$$(2.9) \quad \int_0^1 |\varphi_{j,k}(t)| \, [t(1-t)]^{-\frac{1}{2}} \, dt < \infty, \qquad \text{for all} \quad k = 1, 2; \qquad j = 1, \cdots, p.$$

It may be noted that for the asymptotic (multi-) normality of the elements of $\mathbf{S}_\nu$, the first assumption due to Hájek (1968) is sufficient. However, for the study of the asymptotic power properties of the proposed tests, we require to simplify the expression for $E(\mathbf{S}_\nu)$, and for this we require the slightly more restrictive assumption (2.9), due to Hoeffding (1968).

Let

$$(2.10) \quad \bar{\varphi}_j = \int_0^1 \varphi_j(t) \, dt, j = 1, \cdots, p; \qquad H_\nu(\mathbf{x}) = N_\nu^{-1} \textstyle\sum_{i=1}^{N_\nu} F_{\nu i}(\mathbf{x}).$$

The univariate marginal of the jth variate and the bivariate joint distribution of the (j, j')th variates corresponding to the cdf $H_\nu(\mathbf{x})$ are denoted by $H_{\nu[j]}(x)$ and $H_{\nu[j,j']}(x, y)$, respectively, for $j \neq j' = 1, \cdots, p$. Let

$$(2.11) \quad \lambda_{jj'}(H_\nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_j(H_{\nu[j]}(x)) \varphi_{j'}(H_{\nu[j']}(y)) \cdot dH_{\nu[j,j']}(x, y) - \bar{\varphi}_j \bar{\varphi}_{j'}, \qquad j, j' = 1, \cdots, p;$$

$$(2.12) \quad \mathbf{\Lambda}(H_\nu) = ((\lambda_{jj'}(H_\nu)))$$

ASSUMPTION III. $\mathbf{\Lambda}(H_\nu)$ is positive definite for all $\nu \geqq \nu_0$.

It may be noted that $\lambda_{jj}(H_\nu) = \int_0^1 [\varphi_j(t) - \bar{\varphi}_j]^2 \, dt = \lambda_{jj}$ is independent of H_ν for $j = 1, \cdots, p$. Further, under the Assumptions V and VII (or VII (a)) to follow, when (1.1) holds,

$$(2.13) \quad \lim_{\nu \to \infty} \mathbf{\Lambda}(H_\nu) = \mathbf{\Lambda}(F),$$

where $\mathbf{\Lambda}(F)$ is defined as in (2.12) with H_ν replaced by F. Thus, we may write

ASSUMPTION III (a). Under (1.1), (2.16), (2.19) [or (2.20)], Assumption III holds if

$$(2.14) \quad \mathbf{\Lambda}(F) \text{ is positive definite.}$$

ASSUMPTION IV. $\mathbf{c}_{\nu i}$, $i = 1, \cdots, N_\nu$ satisfy the condition

$$(2.15) \quad N_\nu \bar{\mathbf{c}}_\nu = \textstyle\sum_{i=1}^{N_\nu} \mathbf{c}_{\nu i} = \mathbf{0}^{q \times 1} \quad \text{for all} \quad 1 \leqq \nu < \infty,$$

(this can always be done by replacing $\boldsymbol{\alpha}$ in (1.1) by $\boldsymbol{\alpha}_\nu = \boldsymbol{\alpha} + \boldsymbol{\beta} \bar{c}_\nu$.)

ASSUMPTION V.

$$(2.15) \quad \text{Sup}_\nu \{ \textstyle\sum_{i=1}^{N_\nu} [c_{\nu i}^{(k)}]^2 \} < \infty \quad \text{for all} \quad k = 1, \cdots, q.$$

Let us also define a $q \times q$ matrix

$$(2.17) \quad \mathbf{C}_\nu = \textstyle\sum_{i=1}^{N_\nu} \mathbf{c}_{\nu i} \mathbf{c}_{\nu i}' = ((\sum_{i=1}^{N_\nu} c_{\nu i}^{(k)} c_{\nu i}^{(k')}))k, k' = 1, \cdots, q.$$

ASSUMPTION VI.

$$(2.18) \quad \text{Rank} [\mathbf{C}_\nu] = q (\geqq 1) \quad \text{for all} \quad 1 \leqq \nu < \infty.$$

[In fact, by reparameterization in (1.1), $\mathbf{C}_\nu$ can always be made of full rank. Hence, (2.18) imposes no further restriction on (1.1).]

ASSUMPTION VII (Hájek). For every $\epsilon > 0$ there exists a value of ν, say ν_ϵ, such that for $\nu \geqq \nu_\epsilon$

$$(2.19) \qquad \sum_{i=1}^{N_\nu} (C_{\nu i}^{(k)})^2 > \epsilon N_\nu[\mathrm{Max}_{1\leqq i \leqq N_\nu} [C_{\nu i}^{(k)}]^2], \quad k = 1, \cdots, q,$$

where of course we assume that N_ν is a non-decreasing function of ν with $\lim_{\nu\to\infty} N_\nu = \infty$.

In fact, following Hájek (1968, Theorem 2.2), we may relax the Assumption VII a bit more when the $\varphi_{j,k}$ have all bounded second derivatives.

ASSUMPTION VII (a) (Noether).

$$(2.20) \qquad \max_{1\leqq k\leqq q} \max_{1\leqq i\leqq N_\nu} |c_{\nu i}^{(k)}|/\{\sum_{i=1}^{N_\nu} [c_{\nu i}^{(k)}]^2\}^{\frac{1}{2}} = o(1).$$

[Note that VII $\Rightarrow$ VII (a) but not conversely.]

Now, corresponding to the matrix $\mathbf{\Lambda}(H_\nu)$, we define its sample measure $M_\nu = ((m_{\nu,jj'}))$ by

$$(2.21) \qquad m_{\nu,jj'} = (N_\nu - 1)^{-1}\{\sum_{i=1}^{N_\nu} a_{\nu j}(R_{\nu i}^{(j)})a_{\nu j'}(R_{\nu i}^{(j')}) - N_\nu \bar{a}_{\nu j}\bar{a}_{\nu j'}\};$$

$$(2.22) \qquad \bar{a}_{\nu j} = N_\nu^{-1} \sum_{i=1}^{N_\nu} a_{\nu j}(i), \quad j = 1, \cdots, p.$$

We also define a $pq \times pq$ matrix

$$(2.23) \qquad \mathbf{D}_\nu = \mathbf{M}_\nu \otimes \mathbf{C}_\nu = ((d_{\nu,jk;j'k'})),$$

where $\otimes$ refers to the Kronecker-product of the two matrices $\mathbf{M}_\nu$ and $\mathbf{C}_\nu$. By (2.18) and (2.23), we obtain

$$(2.24) \qquad \mathrm{rank}\,[\mathbf{D}_\nu] = (\mathrm{rank}\,[\mathbf{M}_\nu])q.$$

Thus, if $\mathbf{M}_\nu$ is of rank r, the rank of $\mathbf{D}_\nu$ is $rq\,(r \leqq p)$. For the time being we assume that $\mathbf{M}_\nu$ is of rank p and denote its reciprocal by $\mathbf{M}_\nu^{-1} = ((m_\nu^{jj'}))$. Also, we denote the reciprocal of $\mathbf{C}_\nu$ by $\mathbf{C}_\nu^{-1} = ((c_\nu^{kk'}))$. Then, we have

$$(2.25) \qquad D_\nu^{-1} = \mathbf{M}_\nu^{-1} \otimes \mathbf{C}_\nu^{-1} = ((d_\nu^{jk,j'k'})) = ((m_\nu^{jj'} \cdot c_\nu^{kk'})).$$

Our proposed test statistic is

$$(2.26) \qquad \mathcal{L}_\nu = \sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} d_\nu^{jk,j'k'} S_{\nu,jk} S_{\nu,j'k'},$$

and its rationality will be made clear in the next section. If $\mathbf{M}_\nu$ is of rank $p'\,(<p)$, we may work with either a generalized inverse of $\mathbf{M}_\nu$ and define $\mathcal{L}_\nu$ with D_ν^{-1} replaced by $\mathbf{M}_\nu^* \otimes \mathbf{C}_\nu^{-1}$ (where $\mathbf{M}_\nu^*$ is a generalized inverse of $\mathbf{M}_\nu$), or we may work with a subset of p' variates for which the corresponding minor of $\mathbf{M}_\nu$ is positive-definite, and define $\mathcal{L}_\nu$ in terms of only those $S_{\nu,jk}$'s for which j belongs to this subset. However, it will be seen later on that under Assumption III (or III (a)), $\mathbf{M}_\nu$ is positive definite, in probability, as $\nu \to \infty$. Hence, for large ν, $\mathcal{L}_\nu$ can be defined as in (2.26), in probability.

REMARK 1. For better use, we write $a_{\nu j}(i)$ in (2.2)–(2.5) as

$$(2.27) \qquad \varphi_{\nu,j}(1 + [tN_\nu]/(N_\nu + 1)) \quad 0 < t < 1, \; j = 1, \cdots, p,$$

where $[s]$ denotes the largest integer contained in s. Then, from Lemma a and Lemma b of Hájek and Šidák (1967, pp. 164–65), it follows that

$$(2.28)\quad \lim_{\nu\to\infty} \int_0^1 [\varphi_{\nu,j}(\{1 + [tN_\nu]\}/\{N_\nu + 1\}) - \varphi_j(t)]^2\, dt = 0, \quad j = 1, \cdots, p,$$

while for (2.4), (2.28) follows from the results of Hoeffding (1953).

REMARK 2. We define

$$(2.29)\qquad \mathbf{\Gamma} = \mathbf{\Lambda}(H_\nu) \otimes \mathbf{C}_\nu, \qquad \|\Gamma_\nu\| = \det \Gamma_\nu,$$

where det A stands for the determinant of a square matrix A. From Assumption III and VI, it follows that for $\nu \geqq \nu_0$,

$$(2.30)\qquad \operatorname{rank}[\mathbf{\Gamma}] = pq.$$

Moreover, from (2.29), (2.11) and (2.16), it follows that

$$(2.31)\qquad \sup_\nu \|\Gamma_\nu\| \leqq (\textstyle\prod_{j=1}^p \lambda_{jj})[\sup_\nu \{\prod_{k=1}^q C_{\nu,kk}\}] < \infty$$

where in (2.31) we use the well-known property of a positive definite (symmetric) matrix $\mathbf{A}$ (of order p)

$$(2.32)\qquad \|A\| \leqq \textstyle\prod_{i=1}^p a_{ii}.$$

Moreover, if $\lim_{\nu\to\infty} \mathbf{C}_\nu = \mathbf{C}$ exists, then under (1.1) we have

$$(2.33)\qquad \lim_{\nu\to\infty} \mathbf{\Gamma}_\nu = \mathbf{\Gamma} = \mathbf{\Lambda}(F) \otimes \mathbf{C}.$$

3. Permutation distribution of S_ν and the rationality of $\mathfrak{L}_\nu$. Under H_0 in (1.2), $\mathbf{E}_\nu$ is composed of N_ν independent and identically distributed random vectors. Hence, the joint distribution of $\mathbf{E}_\nu$ (which is the product of the N_ν distributions of $\mathbf{X}_{\nu 1}, \cdots, \mathbf{X}_{\nu N_\nu}$) remains invariant under the $N_\nu!$ permutations of the N_ν vectors $\mathbf{X}_{\nu 1}, \cdots, \mathbf{X}_{\nu N_\nu}$ among themselves when (1.2) holds. We now consider the basic rank-permutation principle, which is essentially due to Chatterjee and Sen (1964, 1966), and is discussed in detail by Puri and Sen (1966).

We define a $p \times N_\nu$ rank matrix $\mathbf{R}_\nu$

$$(3.1)\qquad \mathbf{R}_\nu = \begin{pmatrix} R_{\nu 1}^{(1)} & \cdots & R_{\nu N_\nu}^{(1)} \\ \vdots & \ddots & \vdots \\ R_{\nu 1}^{(p)} & \cdots & R_{\nu N_\nu}^{(p)} \end{pmatrix} = (\mathbf{R}_{\nu 1}, \cdots, \mathbf{R}_{\nu N_\nu}),$$

where $\mathbf{R}_{\nu i} = (R_{\nu i}^{(1)}, \cdots, R_{\nu i}^{(p)})'$ for $i = 1, \cdots, N_\nu$. Each row of $\mathbf{R}_\nu$ is a permutation of the numbers $1, \cdots, N_\nu$, there being in all $(N_\nu!)^p$ possible realizations of $\mathbf{R}_\nu$. Let us rearrange the columns of $\mathbf{R}_\nu$ in such a way that the first row has elements $1, \cdots, N_\nu$ in the natural order, and denote the corresponding matrix by $\mathbf{R}_\nu^*$. $\mathbf{R}_\nu$ is said to be permutationally equivalent to a matrix $\mathbf{R}_\nu^{(1)}$, if it is possible to obtain $\mathbf{R}_\nu^{(1)}$ from $\mathbf{R}_\nu$ only by permutations of the columns of the latter. Thus, corresponding to each $\mathbf{R}_\nu^*$, there will be a set $\sum(\mathbf{R}_\nu)$ of $N_\nu!$ possible realizations of $\mathbf{R}_\nu$, such that any member of this set will be permutationally equivalent to $\mathbf{R}_\nu^*$. Now, the probability distribution of $\mathbf{R}_\nu$ over the $(N_\nu!)^p$ possible realizations will depend on $F(\mathbf{x})$, even when H_0 in (1.2) holds (unless $F(\mathbf{x})$ has coordinate-wise independent marginals). Thus, rank-statistics, like $\mathbf{S}_\nu$ in (2.7), are, in general, not distribution-free under H_0 in (1.2). However, given a particular set $\sum(\mathbf{R}_\nu^*)$ (of N_ν ! realizations), the conditional distribution of $\mathbf{R}_\nu$ over the N_ν ! permutations of the columns of $\mathbf{R}_\nu^*$ would be uniform under H_0 in (1.2), i.e.,

$$(3.2)\quad P\{\mathbf{R}_\nu = \mathbf{r}_\nu \mid \textstyle\sum(\mathbf{R}_\nu^*),\ H_0\} = 1/N_\nu ! \qquad \text{for all}\quad \mathbf{r}_\nu \varepsilon \textstyle\sum(\mathbf{R}_\nu^*),$$

whatever be $F(\mathbf{x})$. Let us denote by $\mathcal{P}_\nu$ the permutational (conditional) probability measure generated by the conditional law in (3.2). Then, by routine computations (along the lines of Puri and Sen (1966)), we obtain that

$$(3.3)\qquad E\{\mathbf{S}_\nu \mid \mathcal{P}_\nu\} = \mathbf{0}^{p\times q}.$$

$$(3.4)\quad E\{S_{\nu,jk}\, S_{\nu,j'k'} \mid \mathcal{P}_\nu\} = d_{\nu,jk;j'k'} \quad \text{for } j,\ j' = 1, \cdots, p;\ k,\ k' = 1, \cdots, q,$$

where $d_{\nu,jk;j'k'}$'s are defined by (2.23).

Since $\mathbf{S}_\nu$ is a stochastic matrix, for actual test construction it is more convenient to use a real valued function of $\mathbf{S}_\nu$ as a test-statistic. By an adaption of the same arguments as in Chatterjee and Sen (1966) and Puri and Sen (1966), we may work with a positive-semi-definite quadratic form in the pq elements of $\mathbf{S}_\nu$, where the discriminant of the quadratic form is the inverse of the matrix $\mathbf{D}_\nu$, which has the elements $d_{\nu,jk;j'k'}$ defined by (2.23) and (3.4). This leads to the test statistic $\mathcal{L}_\nu$, defined by (2.26). $\mathcal{L}_\nu$ will be stochastically large if $\mathbf{S}_\nu$ is stochastically different from $\mathbf{0}$. For small values of ν (i.e., N_ν), the conditional distribution of $\mathcal{L}_\nu$, given $\sum(R_\nu^*)$, can be obtained with the aid of (3.2), and a conditionally distribution-free test for H_0 in (1.2) based on $\mathcal{L}_\nu$ can be constructed. This, however, requires the evaluation of the N_ν ! realizations of $\mathbf{S}_\nu$ (under $\mathcal{P}_\nu$), while $\mathbf{D}_\nu$ is $\mathcal{P}_\nu$-invariant. The task becomes prohibitively laborious and for large values of ν, we are forced to consider the following large sample approach in which we approximate the true permutation distribution of $\mathcal{L}_\nu$ by a χ^2 distribution with pq degrees of freedom (df). For this purpose, we consider first the following theorem.

THEOREM 3.1. *When the scores are defined by* (2.3), (2.4) *or* (2.5) *and the score functions satisfy the Assumption* I [in (2.8)], $\mathbf{M}_\nu - \mathbf{\Lambda}(H_\nu) \to_p \mathbf{0}_p$, *where* $\mathbf{\Lambda}(H_\nu)$ *and* $\mathbf{M}_\nu$ *are defined by* (2.11) *and* (2.21) *respectively.*

PROOF. We shall sketch the proof only for the scores in (2.3), while the cases with (2.4) or (2.5) will follow in a similar manner. By (2.21), (2.22), and (2.28), it follows that as $\nu \to \infty$

$$(3.5)\quad m_{\nu jj} \to \lambda_{jj}(H_\nu) = \lambda_{jj} = \int_0^1 [\varphi_j(t) - \varphi_j]^2\, dt, \quad j = 1, \cdots, p.$$

So, we require only to show that for $j \neq j'$, $m_{\nu,jj'} - \lambda_{jj'}(H_\nu) \to_p 0$. Now, as in Lemma 5.1 of Hájek (1968), we have for any $\epsilon > 0$,

$$(3.6)\qquad \varphi_j(t) = \Psi_j(t) + \varphi_{j,1}(t) - \varphi_{j,2}(t), \quad j = 1, \cdots, p,$$

where Ψ_j's are polynomials, $\varphi_{j,k}$'s are non-decreasing (in t), and

$$(3.7)\qquad \textstyle\sum_{k=1}^2 \int_0^1 \varphi_{j,k}^2(t)\, dt < \epsilon \qquad \text{for all } j = 1, \cdots, p.$$

Then, we may write $m_{\nu,jj'}$ as

$$
\begin{aligned}
(3.8)\qquad & (N_\nu - 1)^{-1} \textstyle\sum_{i=1}^{N_\nu} \\
& \cdot[\Psi_j(R_{\nu i}^{(j)}/(N_\nu + 1)) + \varphi_{j,1}(R_{\nu i}^{(j)}/(N_\nu + 1)) - \varphi_{j,2}(R_{\nu i}^{(j)}/(N_\nu + 1))] \\
& \cdot[\Psi_{j'}(R_{\nu i}^{(j')}/(N_\nu + 1)) - \varphi_{j',1}(R_{\nu i}^{(j')}/(N_\nu + 1)) - \varphi_{j',2}(R_{\nu i}^{(j')}/(N_\nu + 1))] \\
& - N_\nu^{-1}[\textstyle\sum_{i=1}^{N_\nu} \{\varphi_j(\mathrm{i}/(N_\nu + 1)) + \varphi_{j,1}(i/(N_\nu + 1)) - \varphi_{j,2}(i/(N_\nu + 1))\}] \\
& \cdot[\textstyle\sum_{i=1}^{N_\nu} \{\Psi_{j'}(i/(N_\nu + 1)) + \varphi_{j',1}(i/(\mathrm{N}_\nu + 1)) - \varphi_{j'\prime 2}(i/(N_\nu + 1))\}] \\
& = (N_\nu - 1)^{-1}\{\textstyle\sum_{i=1}^{N_\nu} \Psi_j(R_{\nu i}^{(j)}/(N_\nu + 1))\Psi_j(R_{\nu i}^{(j')}/(N_\nu + 1)) \\
& - N_\nu^{-1}[\textstyle\sum_{i=1}^{N_\nu} \Psi_j(i/(N_\nu + 1))][\textstyle\sum_{i=1}^{N_\nu} \Psi_{j'}(i/(N_\nu + 1))]\} + R_\nu,
\end{aligned}
$$

where using (2.28), (3.7) and the Cauchy-Schwarz inequality, it can be readily shown that

$$(3.9)\qquad |R_\nu| \leqq k\epsilon,$$

where k is a finite positive quantity. Since the Ψ_j's are polynomials, the proof of Theorem 4.2 of Puri and Sen (1966) can be readily extended to show that the first term on the right-hand side of (3.8) is *asymptotically stochastically equivalent*[3] to $\lambda_{jj'}(H_\nu)$ (as $\nu \to \infty$). Then the proof is terminated by letting ϵ be arbitrarily small.

COROLLARY 3.1. *Under Assumption* I, $\mathbf{D}_\nu$, *defined by* (2.23), *is stochastically equivalent* (*as* $\nu \to \infty$) *to* $\mathbf{\Gamma}_\nu$, *defined by* (2.29). *Hence, under Assumptions* III *and* VI, $\mathbf{D}_\nu$ *is positive definite, in probability, as* $\nu \to \infty$.

The proof is a direct consequence of (2.23), (2.29) and Theorem 3.1.

THEOREM 3.2[4]. *Under* $\mathcal{P}_\nu$, *the joint distribution of the elements of* $\mathbf{S}_\nu$ *converges, in probability,* (*as* $\nu \to \infty$) *to a multinormal distribution with null means and variance-covariances given by* (3.4).

PROOF. For any given $\ell = (\ell_1, \cdots, \ell_p)'$, define the vector

$$(3.10)\qquad \mathbf{Z}_\nu = (\textstyle\sum_{k=1}^{q} \ell_k S_{\nu,jk}, j = 1, \cdots, p),$$

which, by (2.6), can be written as

$$(3.11)\qquad (\textstyle\sum_{i=1}^{N_\nu} a_{\nu i}(R_{\nu i}^{(j)}) \sum_{k=1}^{q} \ell_k c_{\nu i}^{(k)}, j = 1, \cdots, p).$$

Now, $c_{\nu i}^* = \sum_{k=1}^{q} \ell_k c_{\nu i}^{(k)}, i = 1, \cdots, N_\nu$ satisfies the Noether condition [cf. (2.20)], and by (2.16) and (2.18), $\sup_\nu \sum_{i=1}^{N_\nu} (c_{\nu i}^*)^2 < \infty$. Also, by Theorem 3.1, $\mathbf{M}_\nu$, defined by (2.24), is positive definite, in probability, as $\nu \to \infty$, when Assumption III holds. Hence, it is easy to show that the condition (7.2) of Theorem 7.1

[3] A sequence of random variables $\{X_\nu\}$ is said to be asymptotically stochastically equivalent to another sequence of random variables $\{Y_\nu\}$ if $|X_\nu - Y_\nu| \to_p 0$ as $\nu \to \infty$.

[4] We say that the distribution of X_ν is asymptotically multinormal with mean vector $\boldsymbol{\mu}_\nu$ and dispersion matrix $\boldsymbol{\Sigma}_\nu$, if for every $\boldsymbol{\ell} \neq 0$, the distribution of $\boldsymbol{\ell}'(\mathbf{X}_\nu - \boldsymbol{\mu}_\nu)/[\boldsymbol{\ell}' \boldsymbol{\Sigma}_\nu \boldsymbol{\ell}]^{1/2}$ converges to a standard normal distribution when $\nu \to \infty$.

of Hájek (1961) is satisfied, in probability, as $\nu \to \infty$. Hence the theorem follows by an appeal to Theorem 7.1 of Hájek (1961). □

By virtue of the preceding theorem, we arrive at the following theorem through a few simple steps which are omitted.

THEOREM 3.3. *Under the conditions* (I) *to* (VII) *of Section* 2, $\mathcal{L}_\nu$, *defined by* (2.26), *has a permutation distribution* (*under* $\mathcal{P}_\nu$) *asymptotically converging, in probability, to a chi-square distribution with pq degrees of freedom.*

Hence we have the following large sample test procedure:

$$\text{if } \mathcal{L}_\nu \begin{cases} \geqq \chi^2_{pq,\alpha}, & \text{reject } H_0 \text{ in (1.2);} \\ < \chi^2_{pq,\alpha}, & \text{accept } H_0, \end{cases} \tag{3.12}$$

where $P\{\chi_t^2 \geqq \chi^2_{t,\alpha}\} = \alpha\,(0 < \alpha < 1)$, the level of significance.

4. Asymptotic multinormality of $\mathbf{S}_\nu$. In this section, generalizing the results of Hájek (1968), we shall derive the asymptotic distribution of $\mathbf{S}_\nu$, without restricting ourselves to the model (1.1). Also, as a particular case of special interest, we shall subsequently consider the model (1.1) and simplify the expression for the parameters of the asymptotic distribution of $\mathbf{S}_\nu$. We define $H_\nu(\mathbf{x})$ as in (2.10) and its univariate and bivariate distributions as in the line following (2.10). Also, let $F_{\nu i[j]}(x)$ and $F_{\nu i[j,j']}(x, y)$ be the univariate (jth) and bivariate ((j, j')th) marginals corresponding to $F_{\nu i}(\mathbf{x})$, for $j \neq j' = 1, \cdots, p$. We define

$$\begin{aligned} A_{jj(i;rs)} = {} & \int\int_{-\infty<x<y<\infty} F_{\nu i[j]}(x)[1 - F_{\nu i[j]}(y)] \\ & \cdot \varphi_j'(H_{\nu[j]}(x))\varphi_j'(H_{\nu[j]}(y))\, dF_{\nu r[j]}(x)\, dF_{\nu s[j]}(y) \\ & + \int\int_{-\infty<x<y<\infty} F_{\nu i[j]}(x)[1 - F_{\nu i[j]}(y)] \\ & \cdot \varphi_j'(H_{\nu[j]}(x))\varphi_j'(H_{\nu[j]}(y))\, dF_{\nu s[j]}(x)\, dF_{\nu r[j]}(y), \end{aligned} \tag{4.1}$$

for $i, r, s = 1, \cdots, N_\nu$, and $j = 1, \cdots, p$;

$$\begin{aligned} A_{jj'(i;rs)} = {} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_{\nu i[j,j']}(x, y) - F_{\nu i[j]}(x)F_{\nu i[j']}(y)] \\ & \cdot \varphi_j'(H_{\nu[j]}(x))\varphi_{j'}'(y))\, dF_{\nu r[j]}\, dF_{\nu s[j']}(y), \end{aligned} \tag{4.2}$$

for $j \neq j' = 1, \cdots, p$ and $i, r, s = 1, \cdots, N_\nu$; the subscript ν in (4.1) and (4.2) is understood. Let then

$$\sigma_{\nu,jk;j'k'} = (1/N_\nu^2) \sum_{i=1}^{N_\nu} \sum_{r=1}^{N_\nu} \sum_{s=1}^{N_\nu} (c_{\nu r}^{(k)} - c_{\nu i}^{(k)})(c_{\nu s}^{(k')} - c_{\nu i}^{(k')}) A_{jj'(i;rs)}, \tag{4.3}$$

for $j, j' = 1, \cdots, p$ and $k, k' = 1, \cdots, q$; the corresponding pq x pq matrix is denoted by $\mathbf{\Sigma}_\nu$. Throughout this section it will be assumed that

$$\mathbf{\Sigma}_\nu \text{ is positive definite and } \|\Sigma_\nu\| = \det \mathbf{\Sigma}_\nu < \infty. \tag{4.4}$$

Also, whenever we refer to the model (1.1) in this section, we shall assume that $F(\mathbf{x})$ in (1.1) is absolutely continuous having a continuous density function $f(\mathbf{x})$, $\mathbf{x} \in R^p$. It is to be noted that if we restrict ourselves to the model (1.1), we have the following lemma which shows that (4.4) holds under the assumptions I, III and IV of section 2.

LEMMA 4.1. *Under* (1.1), *and Assumptions* V *and* VII, $\mathbf{\Sigma}_\nu - \mathbf{\Lambda}(F) \otimes \mathbf{C}_\nu \to \mathbf{0}^{pq\times pq}$ *as* $\nu \to \infty$. *Hence under* (2.14) *and* (2.18), $\mathbf{\Sigma}_\nu$ *is positive definite in the limit as* $\nu \to \infty$.

PROOF. By virtue of (2.16) and (2.19) [or (2.20)], $\max_{1\leq i\leq N_\nu} \max_{1\leq k\leq q} |c_{\nu i}^{(k)}| \to 0$ as $\nu \to \infty$, and hence

(4.5) $$\lim_{\nu\to\infty} \{\max_{1\leq i\leq N_\nu} \sup_{\mathbf{x}\varepsilon R^p} |F_{\nu i}(\mathbf{x}) - F(\mathbf{x})|\} = 0.$$

From (4.1), (4.2), (4.3) and some routine computations, it follows that

(4.6) $$\lim_{\nu\to\infty} \{\max_{1\leq i,r,s\leq N_\nu} \max_{1\leq j,j'\leq p} [A_{jj'(i;rs)} - \lambda_{jj'}(F)]\} = 0,$$

where $\lambda_{jj'}(F)$ is defined by (2.11). From (4.3), (4.6), (2.16) and (2.19) [or (2.20)], we obtain after some simplifications that

(4.7) $$\sigma_{\nu,jk;j'k'} - C_{\nu,kk'}\lambda_{jj'}(F) \to 0 \quad \text{as } \nu \to \infty,$$

for all $j, j' = 1, \cdots, p$; $k, k' = 1, \cdots, q$. This completes the first part of the proof. The second part follows from (2.14), (2.18), (4.7) and some well-known results on limits of sequences. □

Let us now introduce the following notations. Let

(4.8) $$B_{j(i',i)}(X_{\nu i}^{(j)}) = \int_{-\infty}^{\infty} [u(x - X_{\nu i}^{(j)}) - F_{\nu i[j]}(x)]\varphi_j'(H_{\nu[j]}(x))\, dF'_{\nu i[j]}(x)$$

for $j = 1, \cdots, p$; $i, i' = 1, \cdots, N_\nu$. Also let

(4.9) $$Z_{\nu i(jk)} = (1/N_\nu)\sum_{r=1}^{N_\nu} (c_{\nu r}^{(k)} - c_{\nu i}^{(k)})B_{j(r\ i)}(X_{\nu i}^{(j)}),$$

for $j = 1, \cdots, p$; $k = 1, \cdots, q$ and $i = 1, \cdots, N_\nu$. Finally let

(4.10) $$Z_{\nu,jk} = \sum_{i=1}^{N_\nu} Z_{\nu i(jk)} \quad \text{for} \quad j = 1, \cdots, p; \quad k = 1, \cdots, q.$$

Straight forward but somewhat lengthy computations yield that

(4.11) $$\text{Cov}\{Z_{\nu,jk}, Z_{\nu,j'k'}\} = \sigma_{\nu,jk,j'k'}, \quad \text{for all } j, j' = 1, \cdots, p; k, k' = 1, \cdots, q$$

where $\sigma_{\nu,jk,j',k'}$'s are defined by (4.3).

To study the asymptotic distribution theory of $\mathbf{S}_\nu$, we shall now make use of a very elegant result by Hájek (1968) which establishes the convergence (in quadratic mean) of the elements of $\mathbf{S}_\nu$ to the elements $Z_{\nu,jk}$, $j = 1, \cdots, p$, $k = 1, \cdots, q$. Hájek (1968) has actually considered the univariate situation with one regression variable (i.e., $p = q = 1$) and established this convergence. Since, for the marginal distribution of $S_{\nu,jk}$ we are confronted with his situation, we obtain from his Theorem 4.2 and Lemma 5.1 that under the Assumptions I and V of Section 2.

(4.12) $$\sup_\nu [N_\nu\{E[(S_{\nu,jk} - ES_{\nu,jk}) - Z_{\nu,jk}]^2\}] < \infty,$$

for all $j = 1, \cdots, p$, $k = 1, \cdots, q$. We utilize (4.12) to establish a comparatively stronger result. With this end in view, we consider the following lemmas.

LEMMA 4.2. *Let* $\mathbf{a}_\nu = (a_{\nu 1}, \cdots, a_{\nu t})$ *and* $\mathbf{b}_\nu = (b_{\nu 1}, \cdots, b_{\nu t})$ *be two stochastic* $t(\geqq 2)$*-vectors, such that* $\text{Var}(a_{\nu j} - b_{\nu j}) \to 0$ *as* $\nu \to \infty$ *for all* $j = 1, \cdots, t$. *Then*

$$(4.13) \quad [\text{Cov}(a_{\nu j}, a_{\nu \ell}) - \text{Cov}(b_{\nu j}, b_{\nu \ell})] \to 0 \quad \text{for all} \quad j, \ell = 1, \cdots, t.$$

The proof follows by expressing $\text{Cov}(a_{\nu j}, a_{\nu \ell}) = \text{Cov}(b_{\nu j}, b_{\nu \ell}) + \text{Cov}(b_{\nu j}, a_{\nu \ell} - b_{\nu \ell}) + \text{Cov}(a_{\nu j} - b_{\nu j}, b_{\nu \ell}) + \text{Cov}(a_{\nu j} - b_{\nu j}, a_{\nu \ell} - b_{\nu \ell})$, applying Cauchy-Schwarz inequality to the second, third and fourth terms on the right hand side, and making use of the conditions stated in the lemma.

By virtue of (4.11), (4.12) and Lemma 4.2, it follows that

$$(4.14) \quad \lim_{\nu \to \infty} \{[\text{Cov}(S_{\nu, jk}, S_{\nu, j'k'}) - \sigma_{\nu, jk, j'k'}]\} = 0, \quad \text{for all} \quad j, j' = 1, \cdots, p;\ k; k' = 1, \cdots, q.$$

An immediate consequence of Lemma 4.2 is the following.

LEMMA 4.3. *If* $\mathbf{a}_\nu$ *has asymptotically a multinormal distribution with mean vector* $\boldsymbol{\alpha}_\nu$ *and a distribution matrix* $\mathbf{A}_\nu$ *and if for each*

$$j(= 1, \cdots, t), \quad \text{Var}(a_{\nu j} - b_{\nu j}) \to 0 \ \text{as}\ \nu \to \infty,$$

then $\mathbf{b}_\nu$ *has asymptotically the multinormal distribution with mean vector* $E\mathbf{b}_\nu$ *and a dispersion matrix* $\mathbf{A}_\nu$.

Thus it follows from (4.12) and Lemmas 4.2 and 4.3 that for proving the asymptotic normality of $\mathbf{S}_\nu$, it is sufficient to show that $\{Z_{\nu, jk}, j = 1, \cdots, p; k = 1, \cdots, q\}$ has asymptotically a multinormal distribution. This will be accomplished by showing that any arbitrary linear combination of $Z_{\nu, jk}$'s has asymptotically a normal distribution. With this end in view, we define

$$(4.15) \quad Z_\nu = \sum_{j=1}^{p} \sum_{k=1}^{q} \ell_{jk} Z_{\nu, jk},$$

where ℓ_{jk}'s are real and finite and not all of them are zero. From (4.9), (4.10) and (4.15), we obtain that

$$(4.16) \quad Z_\nu = \sum_{i=1}^{N_\nu} g_i(\mathbf{X}_{\nu i}),$$

where

$$(4.17) \quad g_i(\mathbf{X}_{\nu i}) = N^{-1} \sum_{r=1}^{N_\nu} \sum_{j=1}^{p} \sum_{k=1}^{q} \ell_{jk} (c_{\nu r}^{(k)} - c_{\nu i}^{(k)}) B_{j(r,i)}(X_{\nu i}^{(j)}),$$

for $i = 1, \cdots, N_\nu$. By definition, $g_i(\mathbf{X}_{\nu i})$ are independent random variables. Further

$$(4.18) \quad \sum_{i=1}^{N_\nu} \text{Var}\{g_i(\mathbf{X}_{\nu i})\} = \sum_{j=1}^{p} \sum_{k=1}^{q} \sum_{j'=1}^{p} \sum_{k'=1}^{q} \ell_{jk} \ell_{j'k'} \sigma_{\nu, jk; j'k'}$$

is finite and positive by (4.4). Proceeding then precisely on the same line as in the proof of Theorem 2.1 of Hájek (1968), it follows that $\{g_i(\mathbf{X}_{\nu i}), i = 1, \cdots, N_\nu\}$ satisfy the Lindeberg condition of the central limit theorem when the assumptions of Section 2 hold. Here we note that when φ_j's do not have bounded second derivative, we consider the same modifications as in the proof of Theorem 2.3

of Hájek (1968), whereby we approximate the φ_j by some polynomials up to any preassigned level of accuracy. Hence, we arrive at the following.

THEOREM 4.1. *Under* (4.4) *and the Assumption* I, III–VII *of Section* 2, *the elements of* $\mathbf{S}_\nu$ *have (jointly) asymptotically a multinormal distribution with mean* $E\mathbf{S}_\nu$ *and dispersion matrix* $\mathbf{E}_\nu$, *defined in* (4.1)–(4.3).

Let us now confine ourselves to the model (1.1) and impose the restrictions (2.14)–(2.20). Then, by Lemma 4.1, the covariance matrix $\mathbf{\Sigma}_\nu$ is asymptotically equivalent to $\mathbf{\Lambda}(F) \otimes \mathbf{C}_\nu$. So, in order to find the limiting distribution of $\mathcal{L}_\nu$, defined by (2.26), we require to simplify the expression for $E\mathbf{S}_\nu$ when (1.1) holds. Now, it is not known whether under the assumptions made by Hájek (1968), the expression for $ES_{\nu,jk}$ can be simplified as a linear function of the expected scores [i.e. $\int_{-\infty}^{\infty} \varphi_j(H_{\nu[j]}(x))\, dF_{\nu i[j]}(x)$]. However, it has been shown by Hoeffding (1968) that if Assumption II of Section 2 is superimposed on I, then $ES_{\nu,jk}$ can as well be replaced by

$$\mu_{\nu,jk} = \sum_{i=1}^{N_\nu} c_{\nu i}^{(k)} \int_{-\infty}^{\infty} \varphi_j(H_{\nu[j]}(x)\, dF_{\nu i[j]}(x)), \tag{4.19}$$

for $j = 1, \cdots, p$; $k = 1, \cdots, q$. In addition to $F(\mathbf{x})$ in (1.1) being assumed to be absolutely continuous, we impose either of the following two assumptions: either (a) for each $j(= 1, \cdots, p)$, $f_{[j]}(x) = (d/dx)F_{[j]}(x)$ is absolutely continuous and

$$I(f_{[j]}) = \int_{-\infty}^{\infty} [f'_{[j]}(x)/f_{[j]}(x)]^2\, dF_{[j]}(x) < \infty, \tag{4.20}$$

or, (b) for each $j(= 1, \cdots, p)$, $\varphi_j(u)$ has a continuous derivative $\varphi_j'(u)$ such that

$$\lim_{u\to 0 \text{ or } 1} \{\varphi_j'(u) f_{[j]}(F_{[j]}^{-1}(u))\} \quad \text{is bounded.} \tag{4.21}$$

If (4.20) holds, we let $\Psi_j(u) = -f'_{[j]}(F_{[j]}^{-1}(u))/f_{[j]}(F_{[j]}^{-1}(u))$, $0 < u < 1$, $j = 1, \cdots, p$ and define

$$B_1(\varphi_j, F_{[j]}) = \int_0^1 \varphi_j(u)\Psi_j(u)\, du, \quad j = 1, \cdots, p. \tag{4.22}$$

On the other hand, if (4.21) holds, we let

$$B_2(\varphi_j, F_{[j]}) = \int_{-\infty}^{\infty} \varphi_j'(F_{[j]}(x)) f_{[j]}^2(x)\, dx, \quad j = 1, \cdots p. \tag{4.23}$$

Now, upon making use of Assumption IV, we may write

$$\mu_{\nu,jk} = \sum_{i=1}^{N_\nu} c_{\nu i}^{(k)} \int_{-\infty}^{\infty} \varphi_j(H_{\nu[j]}(x))\, d[F_{\nu i[j]}(x) - H_{\nu[j]}(x)], \tag{4.24}$$

$j = 1, \cdots, p$; $k = 1, \cdots, p$. Thus, if (4.20) holds, we obtain under (1.1), and Assumptions V-VII that

$$|\mu_{\nu,jk} + (\textstyle\sum_{t=1}^{q} \beta_{jt} C_{\nu,jt}) B_1(\varphi_j, F_{[j]})| = o(1), \tag{4.25}$$

$j = 1, \cdots, p$, $k = 1, \cdots, q$. On the other hand, if (4.21) holds, we obtain from (4.24) by partial integration and using (1.1), (2.15)–(2.20) that

$$|\mu_{\nu,jk} + (\textstyle\sum_{t=1}^{q} \beta_{jt} C_{\nu,jt}) B_2(\varphi_j, F_{[j]})| = o(1), \tag{4.26}$$

$j = 1, \cdots, p$, $k = 1, \cdots, q$. Of course, if both (4.20) and (4.21) hold, then it is easy to verify that

$$\lim_{u\to 0 \text{ or } 1} \phi_j(u) f_{[j]}(F_{[j]}^{-1}(u)) = 0, \quad j = 1, \cdots, p, \tag{4.27}$$

and hence, using the results of section 5.2 of Hájek and Sidák (1967, page 216), it is easy to verify that

$$B_1(\varphi_j, F_{[j]}) = B_2(\varphi_j, F_{[j]}) = B(\varphi_j, F_{[j]}), \tag{4.28}$$

$j = 1, \cdots, p$. Hence, for later convenience, we shall replace $\mu_{\nu,jk}$ by

$$-(\textstyle\sum_{t=1}^{q} \beta_{jt} C_{\nu,jt}) B(\varphi_j, F_{[j]}), \; j = 1, \cdots, p, \; k = 1, \cdots, p \tag{4.29}$$

where it is understood that $B(\varphi_j, F_{[j]})$ stands for $B_1(\varphi_j, F_{[j]})$ or $B_2(\varphi_j, F_{[j]})$ or any of them when (4.20) or (4.21) or both hold.

In the next section, we shall make use of this result for the study of the asymptotic properties of the test based on $\mathfrak{L}_\nu$ in (2.26).

5. Asymptotic properties of the $\mathfrak{L}_\nu$-test. Now, using the results in Lemma 4.1, Theorem 4.1 and the discussion following the latter, it follows from some well-known results on the limiting distributions of quadratic forms in asymptotically normally distributed random variables that

$$\mathfrak{L}_\nu^* = \textstyle\sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} S_{\nu,jk} S_{\nu,j'k'} \lambda^{jj'}(F) C_\nu^{kk'} \tag{5.1}$$

[where $((\lambda^{jj'}(F))) = \mathbf{\Lambda}^{-1}(F)$] has asymptotically [under (1.1), the Assumptions I–VII of Section 2] and (4.20) or (4.21) a non-central χ^2 distribution with pq df and the non-centrality parameter

$$\begin{aligned}\Delta_{\mathfrak{L}_\nu} &= \textstyle\sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} \beta_{jk} \beta_{j'k'} C_{\nu,kk'} \lambda^{jj'}(F) B(\varphi_j, F_{[j]}) \\ &\quad \cdot B(\varphi_{j'}, F_{[j']}) \\ &= \textstyle\sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} \beta_{jk} \beta_{j'k'} C_{\nu,kk'} \nu^{jj'}(F),\end{aligned} \tag{5.2}$$

where

$$((\nu^{jj'}(F))) = ((\nu_{jj'}(F)))^{-1}; \tag{5.3}$$

$$\nu_{jj'}(F) = \lambda_{jj'}(F)/[B(\varphi_j, F_{[j]}) B(\varphi_{j'}, F_{[j']})], \quad j, j' = 1, \cdots, p. \tag{5.4}$$

Now, using Corollary 3.1 and Theorem 4.1, it is straightforward to show that under (1.1) and the assumptions of Section 2,

$$\mathfrak{L}_\nu - \mathfrak{L}_\nu^* \to_p 0, \quad \text{as} \quad \nu \to \infty. \tag{5.5}$$

This leads to the following theorem

THEOREM 5.1. *Under* (1.1), *the Assumptions* I–VII *and* (4.20) *or* (4.21), *the statistic* $\mathfrak{L}_\nu$ *has asymptotically a non-central chi-square distribution with pq df and the non-centrality parameter* $\Delta_{\mathfrak{L}_\nu}$, *defined by* (5.2).

Theorem 5.1 may be used to study the asymptotic power properties of the test in (3.12) and to compare its (asymptotic) efficiency relative to standard parametric tests. Unlike the univariate situation, in the multivariate general linear hypothesis testing problem, more than one test is available in the literature, and each one is admissible against certain specific alternatives. Thus, the different test criteria do not have uniformly (in the range of the parameters) good or bad performances. The standard parametric tests are based on the assumption that $F(\mathbf{x})$ in (1.1) is a multivariate normal distribution. The notable test criteria include the so called normal-theory likelihood ratio test criterion (which happens to be based actually on the least squares estimators of $\boldsymbol{\beta}$), Roy's largest characteristic root of certain determinantal equation, and others [cf. Adnerson (1958, Chapter 8)]. The performance characteristics of these tests when $F(\mathbf{x})$ ceases to be a multinormal cdf are not precisely known, even when N_ν is large. The question arises: how does a normal theory test behave when $F(\mathbf{x})$ is not necessarily normal and how does it compare with our $\mathcal{L}_\nu$ test in (3.12)? This will be the subject matter of the current section. In this study, we have specifically considered the normal-theory likelihood ratio test based on the conventional least squares estimators, though we believe that a similar study can be made with that of Roy's largest characteristic root or some other criterion. Second, when $F(\mathbf{x})$ is not necessarily normal but is specified (and satisfies certain regularity conditions to be stated in Section 6), it is possible to derive the so-called likelihood-ratio test which possesses some asymptotically optimum properties in the sense of Wald (1943). The comparison of $\mathcal{L}_\nu$ with such a likelihood ratio test is itself worthy of investigation, and is considered in the next section.

5.1. Comparison with the normal-theory likelihood ratio test. Let us write

$$\text{(5.6)} \qquad \mathbf{Z}_\nu^{p\times q} = (Z_{\nu,jk}); \qquad Z_{\nu,jk} = \textstyle\sum_{i=1}^{N_\nu} X_{\nu i}^{(j)} C_{\nu i}^{(k)};$$

$$\text{(5.7)} \qquad \mathbf{V}_\nu = N_\nu^{-1}\{\textstyle\sum_{i=1}^{N_\nu} \mathbf{X}_{\nu i} - \bar{\mathbf{X}}_\nu)(\mathbf{X}_{\nu i} - \bar{\mathbf{X}}_\nu)'\}; \qquad \bar{\mathbf{X}}_\nu = N_\nu^{-1}\textstyle\sum_{i=1}^{N_\nu}\mathbf{X}_{\nu i}.$$

Also, we define $\mathbf{C}_\nu$ as in (2.17), and denote the covariance matrix of $F(\mathbf{x})$ by $\mathbf{H} = ((\eta_{jj'}))$ which we assume to be positive-definite and finite. The least squares estimator of $\boldsymbol{\beta}$ in (1.1) is given by

$$\text{(5.8)} \qquad \hat{\boldsymbol{\beta}}_\nu = \mathbf{Z}_\nu\mathbf{C}_\nu^{-1},$$

so that the covariance matrix of $\hat{\boldsymbol{\beta}}$ is $\mathbf{H} \otimes \mathbf{C}_\nu^{-1}$, which by Assumption VI is positive definite and finite.

The normal-theory likelihood ratio criterion is

$$\text{(5.9)} \qquad \lambda_\nu = \{\|N_\nu\mathbf{V}_\nu - \hat{\boldsymbol{\beta}}_\nu\mathbf{C}_\nu\hat{\boldsymbol{\beta}}_\nu'\|/\|N\mathbf{V}_\nu\|\}^{\frac{1}{2}N_\nu}$$

[cf. Anderson (1958, page 188)]. To simplify the expression for λ_ν (actually $-2\log\lambda_\nu$) for large values of ν, we first consider the following.

LEMMA 5.1. *Under* (1.1) *and the Assumptions* IV–VII *of Section* 2, *when* $\mathbf{H}$ *is finite*, $\mathbf{V}_\nu \to_p \mathbf{H}$ *as* $\nu \to \infty$.

PROOF. Let us write

$$\text{(5.10)} \qquad \mathbf{Y}_{\nu i} = \mathbf{X}_{\nu i} - \boldsymbol{\alpha} - \boldsymbol{\beta}\mathbf{C}_{\nu i}, \qquad i = 1, \cdots, N_\nu, \qquad \bar{\mathbf{Y}}_\nu = N_\nu^{-1}\textstyle\sum_{i=1}^{N_\nu}\mathbf{Y}_{\nu i};$$

$$\text{(5.11)} \qquad \mathbf{V}_\nu^* = N_\nu^{-1}\textstyle\sum_{i=1}^{N_\nu}\mathbf{Y}_{\nu i}\mathbf{Y}_{\nu i}'.$$

Since $\mathbf{Y}_{\nu i}$'s are independent and identically distributed random variables (vectors) and $\mathbf{H}$ is finite, by Kintchine's law of large numbers,

$$(5.12) \qquad \bar{\mathbf{Y}}_\nu \rightarrow_p 0 \quad \text{and} \quad \mathbf{V}_\nu^* \rightarrow_p \mathbf{H} \quad \text{as} \quad \nu \rightarrow \infty .$$

From (5.7), (5.10), (5.11) and some simple algebraic manipulations, we obtain that

$$(5.13) \quad \mathbf{V}_\nu - \mathbf{V}_\nu^* = 2\boldsymbol{\beta}[(N_\nu)^{-1} \textstyle\sum_{i=1}^{N_\nu} \mathbf{C}_{\nu i}\mathbf{Y}'_{\nu i}] - 2(\bar{\mathbf{Y}}_\nu - \boldsymbol{\alpha})(\bar{\mathbf{X}}_\nu - \boldsymbol{\alpha})' + N_\nu^{-1}\boldsymbol{\beta}\mathbf{C}_\nu\boldsymbol{\beta}' + (\bar{\mathbf{X}}_\nu - \boldsymbol{\alpha})'.$$

Now, by (5.10), (5.12) and (2.15), $\bar{\mathbf{X}}_\nu - \boldsymbol{\alpha} \rightarrow_p \mathbf{0}$ as $\nu \rightarrow \infty$. Also, by (2.16), $N_\nu^{-1}\boldsymbol{\beta}\mathbf{C}_\nu\boldsymbol{\beta}' \rightarrow_p \mathbf{0}^{p\times p}$ as $\nu \rightarrow \infty$. Finally, using the Cauchy-Schwarz inequality and (2.16) as well as (5.12), we get that $N_\nu^{-1} \sum_{i=1}^{N_\nu} \mathbf{C}_{\nu i}\mathbf{Y}'_{\nu i} \rightarrow_p \mathbf{0}^{q\times p}$ as $\nu \rightarrow \infty$. Hence, from (5.12) and (5.13), we obtain that as $\nu \rightarrow \infty$

$$(5.14) \qquad \mathbf{V}_\nu \sim_p \mathbf{V}_\nu^* \rightarrow_p \mathbf{H}.$$

Hence the lemma.

By virtue of this lemma, $\mathbf{V}_\nu$ is positive definite, in probability, (as $\nu \rightarrow \infty$) whenever $\mathbf{H}$ is positive definite (as has been assumed). Hence, by elementary expansion of the determinant in the numerator on the right hand side of (5.9) and followed by some simple algebraic manipulations we get that for any fixed $\boldsymbol{\beta}$:

$$(5.15) \quad -2 \log \lambda_\nu = \textstyle\sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} \hat{\beta}_{\nu,jk}\, \hat{\beta}_{\nu,j'k'}\, C_{\nu,kk'}\, v_\nu^{jj'}[1 + o_p(1)],$$

where $(v_\nu^{jj'}) = \mathbf{V}_\nu^{-1}$. Thus, by virtue of (5.15) and Lemma 5.1, we have for the model (1.1) (for any fixed $\boldsymbol{\beta}$) and under the assumptions of Section 2,

$$(5.16) \quad -2 \log \lambda_\nu \sim p \textstyle\sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} \hat{\beta}_{\nu,jk}\, \hat{\beta}_{\nu,j'k'}\, C_{\nu,kk'}\, \eta^{jj'},$$

where $(\eta^{jj'}) = \mathbf{H}^{-1}$. Since $\hat{\boldsymbol{\beta}}_\nu$ in (5.8) is a linear estimator it readily follows by generalizing the results of Eicker (1963) to the multivariate case that under the assumptions of Section 2, $\hat{\boldsymbol{\beta}}$ has asymptotically a multinormal distribution with mean $\boldsymbol{\beta}$ and dispersion matrix $\mathbf{H} \otimes \mathbf{C}_\nu^{-1}$. Hence, the statistic on the right hand side of (5.16) has asymptotically a non-central chi-square distribution with pq degrees of freedom and the non-centrality parameter

$$(5.17) \qquad \Delta_{\lambda_\nu} = \textstyle\sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} \eta^{jj'} C_{\nu,kk'}\, \beta_{jk}\, \beta_{j'k'}\, .$$

Thus, under $H_0\colon \boldsymbol{\beta} = \mathbf{0}$, $-2 \log \lambda_\nu$ has asymptotically a chi-square distribution with pq df, provided $F(\mathbf{x})$ in (1.1) has finite second order moments. Hence, the same (asymptotic) test procedure as in (3.12) applies to the normal-theory likelihood ratio test.

It is seen that in multivariate situations, the asymptotic relative efficiency (A.R.E.) of the test based on $\mathcal{L}_\nu$ with respect to the one based on λ_ν, as measured by $\Delta_{\mathcal{L}_\nu}/\Delta_{\lambda_\nu}$, depends not only on $\mathbf{V}$ and $\mathbf{H}$, but also on $\boldsymbol{\beta}$ and $\mathbf{C}_\nu$, even when it is assumed that $\lim_{\nu\to\infty} \mathbf{C}_\nu = \mathbf{C}$ exists (which is required to justify the A.R.E.). The following points are worth noting in this context.

(i) If $p = 1$, no matter whatever be $\boldsymbol{\beta}$ and $\mathbf{C}_\nu$

$$\Delta_{\mathcal{L}_\nu}/\Delta_{\lambda_\nu} = \eta_{11}/\nu_{11} \tag{5.18}$$

which depends only on the parent cdf F and happens to coincide with the usual A.R.E. expression for the two-sample location problem. A few important cases may be mentioned here. First, if for $\mathcal{L}_\nu$, we use the Wilcoxon scores (i.e., $\varphi_1(u) = u: 0 < u < 1$), (5.18) reduces to $12\nu_{11}(\int_{-\infty}^{\infty} f_{[1]}^2(x)\, dx)^2$, which has known values (as well as lower bounds) for various $F_{[1]}(x)$. Secondly, if $\varphi_1(u)$ is taken to be the inverse of the standard normal cumulative distribution function (i.e., the $a_\nu(i)$ are the normal scores), (5.18) is bounded below by 1 where the lower bound is obtained only when $F_{[1]}$ itself is normal. This clearly illustrates the asymptotic efficiency of the proposed tests for $p = 1$.

(ii) If $F(\mathbf{x})$ in (1.1) consists of totally independent coordinates, i.e.,

$$F(\mathbf{x}) = \prod_{j=1}^{p} F_{[j]}(x_j), \qquad \mathbf{x} \in R^p. \tag{5.19}$$

Then, both $\boldsymbol{\nu}$ and $\mathbf{H}$ are diagonal matrices and as such

$$\frac{\Delta_{\mathcal{L}_\nu}}{\Delta_{\lambda_\nu}} = \frac{\{\sum_{j=1}^{p} (1/\nu_{jj}) \sum_{k=1}^{q} \sum_{k'=1}^{q} \beta_{jk}\, \beta_{jk'}\, C_{\nu,kk'}\}}{\{\sum_{j=1}^{p} (1/\eta_{jj}) \sum_{k=1}^{q} \sum_{k'=1}^{q} \beta_{jk}\, \beta_{jk'}\, C_{\nu,kk'}\}}. \tag{5.20}$$

Thus, if we write

$$e = \min_{j=1,\cdots,p} (\eta_{jj}/\nu_{jj}), \tag{5.21}$$

we obtain from (5.20) and (5.21) that

$$\Delta_{\mathcal{L}_\nu}/\Delta_{\lambda_\nu} \geqq e, \quad \text{uniformly in} \quad \boldsymbol{\beta} \quad \text{and} \quad \mathbf{C}_\nu. \tag{5.22}$$

Hence, if we use the normal scores for φ_j, e is bounded below by 1, and hence, the same bound applies to (5.20). Incidentally, here also the equality sign in (5.22) holds only when $F_{[j]}$ is normal for all $j = 1, \cdots, p$. Similarly, for Wilcoxon scores, e is bounded below by 0.864 (for all continuous $F_{[j]}$'s), and hence, (5.20) is also bounded below by 0.864.

(iii) If $F(\mathbf{x})$ in (1.1) is itself a multinormal cdf, then if we use the normal scores, it readily follows that $\boldsymbol{\nu} = \mathbf{H}$, and hence, $\Delta_{\mathcal{L}_\nu} = \Delta_{\lambda_\nu}$ for all $\boldsymbol{\beta}$ and $\mathbf{C}_\nu$. Thus, for parent normal distribution, the normal scores test and the normal-theory likelihood ratio tests are asymptotically power equivalent.

(iv) In general, for arbitrary $F(\mathbf{x})$, $\Delta_{\mathcal{L}_\nu}/\Delta_{\lambda_\nu}$ is bounded below and above by the minimum and maximum characteristic roots of $\mathbf{H}\boldsymbol{\nu}^{-1}$, (the proof follows by a straightforward application of a theorem by Courant on the bounds of the ratio of two quadratic forms (cf. [13])). Now, the bounds of $\mathbf{H}\boldsymbol{\nu}^{-1}$ have been studied in detail by the authors [Sen and Puri (1967)] in connection with the multivariate one-sample location problem. As such, the details are omitted here.

6. Asymptotic optimality of $\mathcal{L}_\nu$ for certain $F(\mathbf{x})$. For this we assume that $F(\mathbf{x})$ in (1.1) has the absolutely continuous density function $f(\mathbf{x})$, and define

$$f(x_j) \mid x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_p) = f(\mathbf{x})/\int_{-\infty}^{\infty} f(\mathbf{x})\, dx_j, \quad j = 1, \cdots, p; \tag{6.1}$$

$$f_{[j]}(x) = (d/dx)F_{[j]}(x), \quad j = 1, \cdots, p. \tag{6.2}$$

Let then

$$(6.3)\qquad f_j'(x_j \mid \mathbf{x}) = (\partial/\partial x_j) f(x_j \mid \mathbf{x}),$$

$$(6.4)\qquad f_{[j]}'(x) = (d/dx) f_{[j]}(x), \quad j = 1, \cdots, p;$$

$$(6.5)\qquad g_j(x_j \mid \mathbf{x}) = f_j'(x_j \mid \mathbf{x})/f(x_j \mid \mathbf{x}), \quad j = 1, \cdots, p.$$

We define the statistics

$$(6.6)\quad U_{\nu,jk} = \sum_{i=1}^{N_\nu} c_{\nu i}^{(k)} g_j(X_{\nu i}^{(j)} \mid X_{\nu i}^{(1)}, \cdots, X_{\nu i}^{(j-1)}, X_{\nu i}^{(j+1)}, \cdots, X_{\nu i}^{(p)});$$

for $j = 1, \cdots, p$, $k = 1, \cdots, q$.
We also define $\mathbf{\Xi} = ((\xi_{jj'}))$ by

$$(6.7)\qquad \xi_{jj'} = E_0\{g_j(X_{\nu i}^{(j)} \mid \mathbf{X}_{\nu i}) g_{j'}(X_{\nu i}^{(j')} \mid \mathbf{X}_{\nu i})\}, \qquad j, j' = 1, \cdots, p,$$

where E_0 denotes the expectation over the random vector $\mathbf{X}_{\nu i}$, computed under the null hypothesis $\boldsymbol{\beta} = \mathbf{0}$. Finally, let $\mathbf{T}_\nu = ((\tau_{\nu, jk, j'k}))= \mathbf{\Xi} \otimes \mathbf{C}_\nu$, where $\mathbf{C}_\nu$ is defined by (2.17), and let

$$(6.8)\qquad U_\nu^* = \sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} \tau_\nu^{jk,j'k'} U_{\nu,jk} U_{\nu,j'k'},$$

where $((\tau_\nu^{jk,j'k'})) = \mathbf{T}_\nu^{-1} = \mathbf{\Xi}^{-1} \otimes \mathbf{C}_\nu^{-1}$, and it is of course assumed that $\mathbf{\Xi}$ is positive definite.

Let now $p(\mathbf{E}_{\nu i}; \boldsymbol{\alpha\beta})$ be the joint density function of $\mathbf{E}_\nu = (\mathbf{X}_{\nu i}, \cdots, X_{\nu N_\nu})$ and let $\hat{\boldsymbol{\alpha}}_\nu$, $\hat{\boldsymbol{\beta}}_\nu$ be the maximum likelihood estimates of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively. Also let $\hat{\boldsymbol{\alpha}}_\nu^*$ be the maximum likelihood estimate of $\boldsymbol{\alpha}$ under the assumption that $\boldsymbol{\beta}_\nu = \mathbf{0}$. We then define the likelihood function by

$$(6.9)\qquad L_\nu = p(\mathbf{E}_\nu, \hat{\boldsymbol{\alpha}}_\nu^*, \mathbf{0})/p(\mathbf{E}_\nu, \hat{\boldsymbol{\alpha}}_\nu, \hat{\boldsymbol{\beta}}_\nu).$$

If now (i) the range of $\mathbf{x}$ in $F(\mathbf{x})$ [in (1.1)] does not depend on $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, (ii) $\partial^2 f(\mathbf{x})/(\partial x_i \partial x_j)$ exists for all $i, j = 1, \cdots, p$ and these are continuous functions of $\mathbf{x}$, and (iii) the maximum likelihood estimates of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are uniformly (in $\boldsymbol{\alpha}, \boldsymbol{\beta}$) consistent, then it follows from the result of Wald (1943) that

$$(6.10)\qquad -2 \log L_\nu \sim_p U_\nu^*.$$

Using his results, it also follows that under $H_0: \boldsymbol{\beta} = \mathbf{0}$, $-2 \log L_\nu$ has asymptotically a chi-square distribution with pq df. Hence, a large sample test for $\boldsymbol{\beta} = \mathbf{0}$ can be constructed as in (3.12). Also, using Theorem X of Wald (1943, p. 480), it follows that under (1.1) and the assumptions of Section 2 [namely IV–VIII] $-2 \log L_\nu$ has asymptotically the non-central χ^2 distribution with pq df and the non-centrality parameter

$$(6.12)\qquad \Delta_{U_\nu^*} = \sum_{j=1}^{p} \sum_{j'=1}^{p} \sum_{k=1}^{q} \sum_{k'=1}^{q} \xi^{jj'} C_{\nu,kk'} \beta_{jk} \beta_{j'k'}.$$

Now, for each $\boldsymbol{\beta} \varepsilon R^{pq}$, we define a surface by the equation

$$(6.12)\qquad \Delta_{U_\nu^*} = a^2, \qquad a > 0.$$

Consider now a transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}_\nu^*$, such that (6.12) in terms of $\boldsymbol{\beta}_\nu^*$

reduces to

$$\sum_{j=1}^{p} \sum_{k=1}^{q} (\beta^*_{\nu,jk})^2 = a^2, \qquad a > 0. \tag{6.13}$$

We denote by $S_a(\boldsymbol{\beta})$ and $S_a'(\boldsymbol{\beta}_\nu^*)$ the surfaces in (6.12) and (6.13) respectively. Now, for any point $\boldsymbol{\beta}_0$ and any positive ρ, consider the set $\omega(\boldsymbol{\beta}_0, \rho)$ consisting of all points $\boldsymbol{\beta}$, which lie on the same $S_a(\boldsymbol{\beta})$ as $\boldsymbol{\beta}_0$ and satisfy the condition that $|\boldsymbol{\beta} - \boldsymbol{\beta}_0| < \rho$. We denote by $\omega'(\boldsymbol{\beta}, \rho)$ the image of $\omega(\boldsymbol{\beta}, \rho)$ by the transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\beta}_\nu^*$, and by $A(\omega)$ the area of the set ω. Then, from Theorem 8 of Wald (1943, p. 478), we arrive at the following.

THEOREM 6.1. *For testing H_0: $\boldsymbol{\beta} = \mathbf{0}$ against $\boldsymbol{\beta} \neq \mathbf{0}$ the likelihood ratio test considered above has* (i) *asymptotically best average power with respect to the surfaces $S_a(\boldsymbol{\beta})$ and weight functions $\gamma(\boldsymbol{\beta}) = \lim_{\rho\to 0} \{A[\omega'(\boldsymbol{\beta}, \rho)]/A[\omega(\boldsymbol{\beta}, \rho)]\}$,* (ii) *asymptotically best constant power on the surfaces $S_a(\boldsymbol{\beta})$, and* (iii) *it is an asymptotically most stringent test.*

Thus, when we speak of the asymptotic optimality of the likelihood ratio test, we keep in mind the regularity condition of Wald (1943) and regard the optimality in the light of Theorem 6.1.

We shall show that under certain conditions on $F(\mathbf{x})$, the test based on $\mathcal{L}_\nu$ has also asymptotically the best average power on the same family of ellipsoids, provided φ_j's are chosen suitably. Our treatment deals with a set of sufficient conditions for this asymptotic optimality and the authors are not aware of necessary conditions for the same.

Regarding $F(\mathbf{x})$ we assume that

$$g_j(X_{\nu i}^{(j)} \mid \mathbf{X}_{\nu i}) = \sum_{j'=1}^{p} h_{jj'} f'_{j'[j']}(x_j)/f_{[j']}(x_j) \tag{6.14}$$

where $h_{jj'}$'s are real constants, not all zero, for $j = 1, \cdots, p$. (6.14) holds for (i) the multivariate normal distribution, (ii) any coordinate-wise independent distribution, and may also hold for other distributions.

Let us now define

$$W_{\nu,jk} = \sum_{i=1}^{N_\nu} c_{\nu i}^{(k)} f'_{j[j]}(X_{\nu i}^{(j)})/f_{[j]}(X_{\nu i}^{(j)}), \tag{6.15}$$

for $j = 1, \cdots, p$, $k = 1, \cdots, q$. If (6.14) holds, it follows that $U_{\nu,jk}$'s are linear functions of $W_{\nu,jk}$'s, and consequently, the quadratic form based on $W_{\nu,jk}$'s (analogous to U_ν^*), will also have the same properties as that of U_ν^*.

We now define

$$\varphi_j(u) = f'_{j[j]}(F_{[j]}^{-1}(u))/f_{[j]}(F_{[j]}^{-1}(u)), \quad 0 < u < 1, \quad j = 1, \cdots, p. \tag{6.16}$$

and define $S_{\nu,jk}$'s as in (2.6). Following then Hájek's (1962) elegant approach it is seen that $S_{\nu,jk} - W_{\nu,jk}$ converges in quadratic mean to zero for all $j = 1, \cdots, p$, $k = 1, \cdots, q$. Consequently, on using Lemma 4.2 and some routine computations, we obtain that for φ_j's given by (6.16), $\mathcal{L}_\nu$ in (2.26) is stochastically equivalent to the quadratic form in $W_{\nu,jk}$'s, under (1.1) and the assumptions of Section 2. Since, under (6.14), this quadratic form in $V_{\nu,jk}$'s is also equal to U_ν^* in (6.8), it follows that under (1.1), the assumptions made in

Section 2 and (6.14), (6.16),

$$\mathfrak{L}_\nu \sim_p U_\nu^*. \tag{6.17}$$

Consequently, we arrive at the following.

THEOREM 6.2. *Under* (i) *the model* (1.1), (ii) *the assumptions made in Section* 2, *and* (iii) (6.14) *and* (6.15), $\mathfrak{L}_\nu$ *is asymptotically the best test in accordance with the optimality criteria in Theorem* 6.1.

In particular, if $F(\mathbf{x})$ is normal, the condition (6.14) is satisfied and (6.16) leads to the coordinate-wise normal scores. Hence the asymptotic optimality of the normal scores test for normal cdf's.

A special case of Theorem 6.2 (that is, for $p = 1$) is dealt with in an interesting paper by Matthes and Truax (1965).

7. A characterization of the multivariate multisample location problem. Let $\mathbf{X}_{k1}, \cdots, \mathbf{X}_{kn_k}$ be n_k independent and identically distributed p-variate random variables having a continuous p-variate cumulative distribution function $F_k(\mathbf{x})$ for $k = 1, \cdots, c(\geqq 2)$. In the multivariate multisample location problem [cf. Puri and Sen (1966) and the other references cited therein], we may let

$$F_k(\mathbf{x}) = F(\mathbf{x} - \boldsymbol{\theta}_k), \quad k = 1, \cdots, c. \tag{7.1}$$

We define $N = n_1 + \cdots + n_c$, and consider a sequence of N p-vectors $\mathbf{Z}_1, \cdots, \mathbf{Z}_N$, of which the first n_1 observations are from the first sample, the next n_2 from the second sample and so on. Now, we may write $\boldsymbol{\theta}_k$ as $\boldsymbol{\theta} + \boldsymbol{\beta}_k$, where $\sum_{k=1}^{c} (n_k/N)\boldsymbol{\beta}_k = \mathbf{0}$. Thus, only $c - 1$ of the $\boldsymbol{\beta}_k$ are linearly independent and the null hypothesis: $F_1 = \cdots = F_c$ implies that $\boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_c = 0$. Hence, by reparameterization, we may write

$$F_k(\mathbf{x}) = F(\mathbf{x} - \boldsymbol{\theta} - c_{\nu k}^{(1)}\boldsymbol{\beta}_1 - \cdots - c_{\nu k}^{(c-1)}\boldsymbol{\beta}_{c-1}), \quad k = 1, \cdots, c, \tag{7.2}$$

where the constants $c_{\nu k}^{(r)}$ satisfy the condition:

$$\sum_{k=1}^{c} n_k c_{\nu k}^{(r)} = 0 \quad \text{for} \quad r = 1, \cdots, c - 1. \tag{7.3}$$

Moreover, if the Assumptions V and VI of Section 2 are to be satisfied, we require that (i)

$$C_{\nu k}^{(r)} = O(N^{-\frac{1}{2}}), \tag{7.4}$$

and as $N \to \infty$

$$n_{k/N} = \lambda_N^{(k)}: 0 < \lambda_N^{(k)} < 1 \quad \text{for all} \quad k = 1, 2, \cdots c. \tag{7.5}$$

The resulting statistic $\mathfrak{L}_\nu$ in (2.26) can be shown to be identical with the corresponding $\mathfrak{L}_\nu$ in Puri and Sen (1966). Thus, the results derived in this paper also generalize the results of Puri and Sen (1966) in the sense that the conditions on φ_j's in this paper are much less stringent than in Puri and Sen (1966) and the regression constants $\{\mathbf{c}_{\nu i}, i = 1, \cdots, N_\nu\}$, contain the latter as a particular case.

REMARK. The model (1.1) could be made a little more general as follows.

$$(7.6) \quad X_{\nu i}^{(j)} = \alpha_j + \textstyle\sum_{k=1}^{q} \beta_{jk}\, c_{\nu j i}^{(k)} + \gamma_{\nu i}^{(j)}, \quad j = 1, \cdots, p, \quad i = 1, \cdots, N_\nu$$

where the $c_{\nu j i}^{(k)}$ are known constants. The theory can be developed along the same line as in Sections 3–6. However, this model will lead to certain complications. First, $\mathbf{D}_\nu$ and $\mathbf{\Gamma}_\nu$ cannot be expressed as the Kronecker product of two matrices, and as a result the proof of Corollary 3.1 will have to be changed. The same problem also arises in (4.7) and for $\mathbf{T}_\nu$ in Section 6. Secondly, in actual practice (1.1) is a more natural model which arises in many situations; for example, in the c-sample problem (cf. Section 7), or when we want to fit a regression of $\mathbf{X}_{\nu i}$ on $\mathbf{c}_{\nu i}$, $i = 1, \cdots, p$. Hence, the model (7.6) is not considered in detail.

8. Acknowledgment. The authors wish to express their gratitude to the referee for his useful comments on this paper.

REFERENCES

[1] ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[2] CHATTERJEE, S. K , and SEN, P. K. (1966). Nonparametric tests for the multivariate multisample location problem. *Essays in Probability and Statistics.* (R. C. Bose *et al.*, eds.). Univ. North Carolina Press. To appear.

[3] CHERNOFF, H., and SAVAGE, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.* **29** 972–994.

[4] EICKER, F. (1963). Asymptotic normality and consistency of the least squares estimators for families of linear regressions. *Ann. Math. Statist.* **34** 447–456.

[5] HÁJEK, J. (1961). Some extensions of the Wald-Wolfowitz-Noether theorem. *Ann. Math. Statist.* **32** 506–523.

[6] HÁJEK, J. (1962). Asymptotically most powerful rank order tests. *Ann. Math. Statist.* **33** 1124–1147.

[7] HÁJEK, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. *Ann. Math. Statist.* **39** 325–346.

[8] HÁJEK, J., and ŠIDÁK, Z. (1967). *Theory of Rank Tests.* Academic Press, New York.

[9] HOEFFDING, W. (1968). On the centering of a simple linear rank statistic. Institute of Statistics Mimeo Series No. 585, Univ. North Carolina.

[10] MATTHES, T. K., and TRUAX, D. R. (1965). Optimal invariant rank tests for the k-sample problem. *Ann. Math. Statist.* **36** 1207–1222.

[11] PURI, M. L., and SEN, P. K. (1966). On a class of multivariate multisample rank-order tests. *Sankhyā Ser. A* **28** 353–376.

[12] RAO, C. R. (1965). *Linear Statistical Inference and its Applications.* Wiley, New York.

[13] SEN, P. K., and PURI, M. L. (1967). On the theory of rank order tests for location in the multivariate one-sample problem. *Ann. Math. Statist.* **38** 1216–1228.

[14] WALD A. (1943). Tests of statistical hypotheses concerning several parameters when the number of observations is large. *Trans. Amer. Math. Soc.* **45** 426–482.