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SOME MULTIPLE PRODUCTS OF POLYKAYS

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Dwyer and Tracy [1] presented a combinatorial method to obtain general expressions for products of two polykays as a linear function of polykays. They gave formulae for double products $k(P)k(Q)$ where $k(P) = k_P = k_{p_1 \ldots p_\pi}$ and Q has weight 2, 3 or 4.

TABLE 1
Formulae for multiplication of $k(P)$ by polykay products of weight 3

Term	$k(P)k(2)k(1)$	$k(P)k(11)k(1)$	$k(P)k^3(1)$	Divisor
$k(P3)$	1		1	n^2
$k(P21)$	1	2	3	n
$k(P111)$		1	1	1
$k(P \oplus 3)$	1		1	n^3
$k(P \oplus 2, 1)$	1	2	3	n^2
$k(P \oplus 1, 2)$	1	2	3	n^2
$k(P \oplus 1, 11)$		3	3	n
$k(P \oplus 21)$	$n - 3$	$2n$	$3(n - 1)$	$n^2n^{(2)}$
$k(P \oplus 11, 1)$	-1	$3n - 2$	$3(n - 1)$	$nn^{(2)}$
$k(P \oplus 111)$	-1	n	$n - 1$	$n^2n^{(2)}$
$\sum C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 21)$	2	-2		$nn^{(2)}$
$\sum C(P_i \mid 2)k(P1{:}P_i \mid 2 \oplus 11)$	1	-1		$n^{(2)}$
$\sum C(P_i \mid 2)k(P \oplus 1{:}P_i \mid 2 \oplus 11)$	1	-1		$nn^{(2)}$
Multiplier	n	1	1	

Tracy [2] gave rules extending the combinatorial method in [1] to obtain multiple products of polykays. These rules have been used to obtain formulae for multiplying $k(P)$ by products of polykays up to weight 4. Since double products have already appeared in [1], only formulae for multiple products are listed in this paper. Notation used is the same as in [1].

The only multiple product when $k(P)$ is multiplied by a product of weight 2 is $k(P)k^2(1)$. This expression [1] is

$$k(P)k^2(1) = k(P11) + k(P2)/n + 2k(P \oplus 1, 1)/n + k(P \oplus 2)/n^2 + k(P \oplus 11)/n^2.$$

Formulae for multiplying $k(P)$ by products of weights 3 and 4 are presented in Tables 1 and 2 respectively. Each entry is to be multiplied by the multiplier in the

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TABLE 2

Formulae for multiplication of $k(P)$ by polykay products of weight 4

Term	$k(P)k(3)k(1)$	$k(P)k(21)k(1)$	$k(P)k(111)k(1)$	$k(P)k(2)k^2(1)$	$k(P)k(11)k^2(1)$	$k(P)k^2(2)$	$k(P)k(2)k(11)$	$k(P)k^2(11)$	$k(P)k^4(1)$	Divisor
$k(P4)$	1			1		1			1	n^3
$k(P31)$	1	1		2	2		2		4	n^2
$k(P22)$		$n-1$		$n-1$	$2(n-1)$	$n+1$	-2	$2n$	$3(n-1)$	$nn^{(2)}$
$k(P211)$		1	3	1	5		1	4	6	n
$k(P1111)$			1		1			1	1	1
$k(P \oplus 4)$	1			1		1			1	n^4
$k(P \oplus 3, 1)$	1	1		2	2		2		4	n^3
$k(P \oplus 2, 2)$		$2(n-1)$		$2(n-1)$	$4(n-1)$	$2(n+1)$	-4	$4n$	$6(n-1)$	$n^2n^{(2)}$
$k(P \oplus 1, 3)$	1	1		2	2		2		4	n^3
$k(P \oplus 2, 11)$		1	3	1	5		1	4	6	n^2
$k(P \oplus 1, 21)$		2	6	2	10		2	8	12	n^2
$k(P \oplus 1, 111)$			4		4			4	4	n
$k(P \oplus 31)$	$n-4$	n		$2(n-2)$	$2n$	-4	$2n$		$4(n-1)$	$n^3n^{(2)}$
$k(P \oplus 22)$	$-3(n-1)$	$n^{(2)}$		$(n-1)(n-3)$	$2n^{(2)}$	n^2-2n+3	$-2n$	$2n^2$	$3(n-1)^2$	$n^2n^{(2)}n^{(2)}$
$k(P \oplus 21, 1)$	-3	$2n-3$	$6n$	$2(n-3)$	$2(5n-3)$		$2(n-3)$	$8n$	$12(n-1)$	$n^2n^{(2)}$
$k(P \oplus 11, 2)$		$(n-1)^{(2)}$	$3n^{(2)}$	$(n-1)^{(2)}$	$(n-1)(5n-4)$	$-2(n+1)$	n^2-n+4	$4n(n-2)$	$6(n-1)^2$	$nn^{(2)}n^{(2)}$
$k(P \oplus 11, 11)$		-1	$3(2n-1)$	-1	$6n-5$		-1	$2(3n-2)$	$6(n-1)$	$nn^{(2)}$
$k(P \oplus 211)$	$-3(n-1)(n-4)$	$n^{(2)}(n-5)$	$3nn^{(2)}$	$(n-1)^{(2)}(n-6)$	$5n^{(3)}$	$-2(n-2)^{(2)}$	$n(n-2)(n-5)$	$4n^2(n-2)$	$6(n-1)(n-1)^{(2)}$	$n^2n^{(2)}n^{(3)}$
$k(P \oplus 111, 1)$	2	$-2(n-1)$	$2n(2n-3)$	$-2(n-2)$	$2(n-2)(2n-1)$		$-2(n-2)$	$4n(n-2)$	$4(n-1)^{(2)}$	$n^2n^{(3)}$
$k(P \oplus 1111)$	$2(n-1)$	$-n^{(2)}$	$nn^{(2)}$	$-(n-1)^{(2)}$	$n^{(3)}$	$n-2$	$-n(n-2)$	$n^2(n-2)$	$(n-1)(n-1)^{(2)}$	$n^2n^{(2)}n^{(3)}$
Multiplier	n^2	n	1	n	1	n^2	n	1	1	
$\Sigma C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 31)$	3	-1		2	-2	4	-2			$n^2n^{(2)}$
$\Sigma C(P_i \mid 2)k(P{:}P_i \mid 2 \oplus 22)$	$3(n-1)$	$-(n-1)$		$2(n-1)$	$-2(n-1)$	$2(n-2)$	2	$-2n$		$nn^{(2)}n^{(2)}$
$\Sigma C(P_i \mid 2)k(P1{:}P_i \mid 2 \oplus 21)$	3	1	-6	4	-4		4	-8		$nn^{(2)}$
$\Sigma C(P_i \mid 2)k(P2{:}P_i \mid 2 \oplus 11)$		$n-1$	$-3(n-1)$	$n-1$	$-(n-1)$	$2(n+1)$	$-(n+3)$	4		$n^{(2)}n^{(2)}$
$\Sigma C(P_i \mid 2)k(P11{:}P_i \mid 2 \oplus 11)$		1	-3	1	-1		1	-2		$n^{(2)}$
$\Sigma C(P_i \mid 3)k(P{:}P_i \mid 3 \oplus 211)$	$3(n-1)$	$-3(n-1)$	$6(n-1)$			$4(n-2)$	$4(n-1)^{(2)}$	$4(n-2)$		$n^{(2)}n^{(3)}$
$\Sigma C(P_i \mid 2)k(P \oplus 1{:}P_i \mid 2 \oplus 21)$	$3(n-1)(n-4)$	$(n-1)(n+4)$	$-6n^{(2)}$	$4(n-1)^{(2)}$	$-4(n-1)^{(2)}$	$-8(n-2)$	$4(n+1)(n-2)$	$-8n(n-2)$		$nn^{(2)}n^{(3)}$
$\Sigma C(P_i \mid 2)k(P \oplus 2{:}P_i \mid 2 \oplus 11)$	-3	$n+1$	$-3n$	$n-2$	$-(n-2)$	$2(n-2)$	$-(n-2)$			$n^2n^{(3)}$
$\Sigma C(P_i \mid 3)k(P1{:}P_i \mid 3 \oplus 111)$	1	-1	2							$n^{(3)}$
$\Sigma C(P_i \mid 2) \cdot k(P \oplus 1, 1{:}P_i \mid 2 \oplus 11)$	-3	$2n-1$	$-6(n-1)$	$2(n-2)$	$-2(n-2)$		$2(n-2)$	$-4(n-2)$		$nn^{(3)}$
$\Sigma C(P_i \mid 3) \cdot k(P \oplus 1{:}P_i \mid 3 \oplus 111)$	1	-1	2							$nn^{(3)}$
$\Sigma C(P_i \mid 2) \cdot k(P \oplus 11{:}P_i \mid 2 \oplus 11)$	$-3(n-1)$	$(n+1)(n-1)$	$-3n^{(2)}$	$(n-1)^{(2)}$	$-(n-1)^{(2)}$	$-2(n-2)$	$(n+1)(n-2)$	$-2n(n-2)$		$nn^{(2)}n^{(3)}$
$\Sigma C(P_i \mid 2)C(P_j \mid 2) \cdot k(P{:}P_i \mid 2 \oplus 11{:}P_j \mid 2 \oplus 11)$						1	1	1		$3n^{(2)}n^{(2)}$
Multiplier	n^2	n	1	n	1	n^2	n	1	1	

last row and divided by the divisor in the last column. It then indicates the coefficient of the term, appearing in the row, in the expression for the product at the head of the column.

Using these tables, specific formulae for products of polykays beyond those given by Wishart [3] are being evolved.

Checks, using general formulae given for double products in [1] and specific formulae for products of polykays in [3], have been utilized to verify the correctness of the expressions. Thanks are due Miss P. N. Nagambal for assistance in checking the results.

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