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ESTIMATION OF A PROBABILITY DENSITY FUNCTION AND ITS DERIVATIVES[1]

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1. Introduction and summary. Let $X_1, X_2, \cdots$ be independent identically distributed random variables having a common probability density function f. After a so-called kernel class of estimates f_n of f based on $X_1, \cdots, X_n$ was introduced by Rosenblatt [7], various convergence properties of these estimates have been studied. The strongest result in this direction was due to Nadaraya [5] who proved that if f is uniformly continuous then for a large class of kernels the estimates f_n converges uniformly on the real line to f with probability one. For a very general class of kernels, we will show that the above assumptions on f are necessary for this type of convergence. That is, if f_n converges uniformly to a function g with probability one, then g must be uniformly continuous and the distribution F from which we are sampling must be absolutely continuous with $F'(x) = g(x)$ everywhere.

When in addition to the conditions mentioned above, it is assumed that f and its first $r + 1$ derivatives are bounded, we are able to show how to construct estimates f_n such that $f_n^{(s)}$ converges uniformly to $f^{(s)}$ at a given rate with probability one for $s = 0, 1, \cdots, r$.

2. Uniform convergence of $f_n^{(r)}$. Let $X_1, \cdots, X_n$ be independent identically distributed random variables with a common distribution function F. Let F_n be the empirical distribution function based on $X_1, \cdots, X_n$; i.e., $nF_n(x)$ is the number of X_i with $X_i \leqq x$ where $1 \leqq i \leqq n$.

LEMMA 2.1. *There exists a universal constant C such that for any $n > 0$, $\epsilon_n > 0$ and distribution function F,*

$$P_F\{\sup_x |F_n(x) - F(x)| > \epsilon_n\} \leqq C \exp(-2n\epsilon_n^2). \tag{1}$$

PROOF. For the case when F is continuous, see Dvoretzky, Kiefer and Wolfowitz [2]. If F is discontinuous at some point then there exists a continuous distribution function $\bar{F}$ for which

$$P_F\{\sup_x |F_n(x) - F(x)| > \epsilon_n\} \leqq P_{\bar{F}}\{\sup_x |F_n(x) - \bar{F}(x)| > \epsilon_n\}$$

(see [3] and [4]). Thus the lemma is true for all univariate F.

Let $f_n(x)$ be a kernel estimate based on $X_1, \cdots, X_n$ from F as given in [7], that is,

$$f_n(x) = (na_n)^{-1} \sum_{i=1}^n k((x - X_i)/a_n)$$

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where $\{a_n\}$ is a sequence of positive numbers converging to zero and k is a probability density function. In this section we will assume that the kernel k is chosen such that $\int |u|\, k(u)du$ (whenever the integration extends over $(-\infty, \infty)$ no limits of integration will be given) is finite, and such that $k^{(s)}$ is a continuous function of bounded variation for $s = 0, 1, \cdots, r$. The density function of the standard normal, for example, satisfies all these conditions. The variation of $k^{(s)}$ will be denoted μ_s. The continuity assumption on $k^{(r)}$ was made solely to ensure that $\sup_x |f_n^{(r)}(x) - Ef_n^{(r)}(x)|$ is a random variable. With the deletion of this assumption the following lemma remains true when we replace the probability P_F by the outer probability P_F^* of P_F. Our proof remains valid in this case.

LEMMA 2.2. *There exists a universal constant C such that for any $n > 0$, $\epsilon_n > 0$, and distribution function F,*

$$P_F\{\sup_x |f_n^{(r)}(x) - Ef_n^{(r)}(x)| > \epsilon_n\} \leqq C \exp(-2n\epsilon_n^2 a_n^{2r+2}/\mu_r^2)$$

where $\{a_n\}$ is a sequence of positive numbers converging to zero and

$$f_n^{(r)}(x) = (na_n^{r+1})^{-1} \textstyle\sum_{i=1}^n k^{(r)}((x - X_i)/a_n) .$$

PROOF. Since $k^{(r)}$ is of bounded variation on $(-\infty, \infty)$ we know (see [6], page 239) that $k^{(r)}$ is bounded and that $\lim_{x\to\infty} k^{(r)}(x)$ and $\lim_{x\to-\infty} k^{(r)}(x)$ both exist. If $r = 0$ then k is non-negative and $\int k(u)\, du = 1$, so that since $\lim_{x\to\infty} k(x)$ and $\lim_{x\to-\infty} k(x)$ both exist, these limits must be zero. If $r \geqq 1$ then the function $k^{(r-1)}$ has a bounded derivative on $[-a, a]$ for any a, and hence (see [6], page 133) $k^{(r)}$ is Lebesgue integrable on $[-a, a]$. Thus (see [6], page 259)

$$\mathsf{V}_{-a}^{a}\, [k^{(r-1)}] = \textstyle\int_{-a}^{a} |k^{(r)}(u)|\, du.$$

Now

$$\mathsf{V}_{-\infty}^{\infty}\, [k^{(r-1)}] = \lim_{a\to\infty} \mathsf{V}_{-a}^{a}[k^{(r-1)}] = \lim_{a\to\infty} \textstyle\int_{-a}^{a} |k^{(r)}(u)|\, du = \int_{-\infty}^{\infty} |k^{(r)}(u)|\, du$$

so that $\int |k^{(r)}(u)|\, du$ is finite. This fact together with the existence of $\lim_{x\to\infty} k^{(r)}(x)$ and $\lim_{x\to-\infty} k^{(r)}(x)$ imply that these limits must be zero.

Upon integrating by parts and remembering that $\lim_{x\to\infty} k^{(r)}(x) = \lim_{x\to-\infty} k^{(r)}(x) = 0$, we find that

$$\begin{aligned}
&\sup_x |f_n^{(r)}(x) - Ef_n^{(r)}(x)| \\
&\quad = \sup_x a_n^{-(r+1)} |\textstyle\int k^{(r)}((x-u)/a_n)dF_n(u) - \int k^{(r)}((x-u)/a_n)\, dF(u)| \\
&\quad = a_n^{-(r+1)} \sup_x |[\{F_n(u) - F(u)\}k^{(r)}((x-u)/a_n)]_{-\infty}^{\infty} \\
&\qquad - \textstyle\int \{F_n(u) - F(u)\}\, dk^{(r)}((x-u)/a_n)| \\
&\quad = a_n^{-(r+1)} \sup_x |\textstyle\int \{F_n(u) - F(u)\}\, dk^{(r)}((x-u)/a_n)| \\
&\quad \leqq a_n^{-(r+1)} \sup_x |F_n(x) - F(x)|\mu_r .
\end{aligned}$$

Therefore by an application of Lemma 2.1 we have

$$P_F\{\sup_x |f_n^{(r)}(x) - Ef_n^{(r)}(x)| > \epsilon_n\}$$
$$\leqq P_F\{\sup_x |F_n(x) - F(x)| > \epsilon_n a_n^{r+1}/\mu_r\} \leqq C \exp(-2n\epsilon_n^2 a_n^{2r+2}/\mu_r^2)$$

and the proof is complete.

Lemma 2.3 below is found in [1]; however, we note here that the symmetry condition imposed on k in [1] is not needed and that in the proof given there the absolute integrability of the $k^{(s)}$, $s = 1, 2, \cdots, r$, has been tacitly assumed. From the proof of Lemma 2.2, the $k^{(s)}$ tend to zero as $x \to +\infty$ or $-\infty$ and $\int |k^{(s)}(u)|\, du$ is finite for $s = 0, 1, \cdots, r$, so that the proof of Lemma 2.3 can be completed exactly as in [1].

LEMMA 2.3. *Let X be an absolutely continuous random variable with probability density function f and let a be any positive real number. If f and its first $r + 1$ derivatives are bounded then there exists a constant C, not depending on a, such that*

$$\sup_x |E_f[a^{-(r+1)}k^{(r)}((x - X)/a)] - f^{(r)}(x)| \leqq Ca.$$

LEMMA 2.4. *If f and its first $r + 1$ derivatives are bounded and if $\{\epsilon_n\}$ is a sequence of positive numbers such that $a_n = o(\epsilon_n)$, then there exist positive constants C_1 and C_2 such that*

$$P_f\{\sup_x |f_n^{(r)}(x) - f^{(r)}(x)| > \epsilon_n\} \leqq C_1 \exp(-C_2 n\epsilon_n^2 a_n^{2r+2})$$

for n sufficiently large.

PROOF. We have with the aid of Lemma 2.3

$$\begin{aligned}\sup_x |f_n^{(r)}(x) - f^{(r)}(x)| &\leqq \sup_x |f_n^{(r)}(x) - Ef_n^{(r)}(x)| \\ &\quad + \sup_x |Ef_n^{(r)}(x) - f^{(r)}(x)| \\ &\leqq \sup_x |f_n^{(r)}(x) - Ef_n^{(r)}(x)| + Ca_n .\end{aligned}$$

Since $a_n = o(\epsilon_n)$ it follows that for n sufficiently large

$$P\{\sup_x |f_n^{(r)}(x) - f^{(r)}(x)| > \epsilon_n\} \leqq P\{\sup_x |f_n^{(r)}(x) - Ef_n^{(r)}(x)| > \epsilon_n/2\}.$$

An application of Lemma 2.2 yields the desired result.

The theorem below tells us that for special sequences $\{a_n\}$,

$$\sup_x |f_n^{(r)}(x) - f^{(r)}(x)|$$

converges to zero with probability one. A sequence $\{b_n\}$ with b_n going to infinity is introduced to indicate the rate at which the above convergence takes place.

THEOREM 2.5. *If f and its first $r + 1$ derivatives are bounded and if the sequences $\{a_n\}$ and $\{b_n\}$ are such that $a_n b_n = o(1)$ and $\sum_{n=1}^{\infty} \exp(-cna_n^{2r+2}/b_n^2)$ is finite for all positive c, then*

$$\lim_{n\to\infty} \sup_x b_n |f_n^{(r)}(x) - f^{(r)}(x)| = 0$$

with probability one.

PROOF. For any $\epsilon > 0$, we obtain by Lemma 2.4 that

$$P_F\{\sup_x |f_n^{(r)}(x) - f^{(r)}(x)| > \epsilon/b_n\} \leqq C_1 \exp(-C_2\epsilon^2 n a_n^{2r+2}/b_n^2)$$

for n sufficiently large. Since $\sum_{n=1}^{\infty} \exp(-cna_n^{2r+2}/b_n^2)$ is finite for all positive c, it follows that

$$\sum_{n=1}^{\infty} P\{\sup_x |f_n^{(r)}(x) - f^{(r)}(x)| > \epsilon/b_n\}$$

is finite for all positive ϵ. Consequently, with the aid of the Borel-Cantelli Lemma we see that $\lim_{n\to\infty} \sup_x b_n |f_n^{(r)}(x) - f^{(r)}(x)| = 0$ with probability one.

It can be seen that the assumption that $f^{(r+1)}$ be bounded could be relaxed somewhat and the conclusion would still hold. The fact that $f^{(r+1)}$ is bounded was used in Lemma 2.3 in [1] to ensure that $\sup_x |Ef_n^{(r)}(x) - f^{(r)}(x)| = O(a_n)$. To establish $\lim_{n\to\infty} \sup_x |f_n^{(r)}(x) - f^{(r)}(x)| = 0$ with probability one following the lines of our argument we would only need $\sup_x |Ef_n^{(r)}(x) - f^{(r)}(x)| = o(1)$ which would be true, for instance, if $f^{(r)}$ were uniformly continuous.

A corollary follows which will indicate the rate of convergence for a particular choice of a_n.

COROLLARY 2.6. *If f and its first $r + 1$ derivatives are bounded, $a_n = n^{-1/(2r+4)}$ and $0 < c < 1/(2r + 4)$, then*

$$\lim_{n\to\infty} \sup_x n^c |f_n^{(r)}(x) - f^{(r)}(x)| = 0$$

with probability one.

3. A necessary and sufficient condition for the uniform convergence of f_n. Let $f_n(x)$ be a kernel estimate based on a random sample $X_1, X_2, \cdots, X_n$ from F as given in Section 2.

We shall assume that the sequence a_n is such that $\sum_{n=1}^{\infty} \exp(-cna_n^2)$ is finite for all positive c and that k is a probability density function satisfying the following conditions:

(i) k is continuous and of bounded variation on $(-\infty, \infty)$.

(ii) $uk(u) \to 0$ as $u \to +\infty$ or $-\infty$.

(iii) There exists a δ in $(0, 1)$ such that

$$u(\vee_{-\infty}^{-u^\delta}(k) + \vee_{u^\delta}^{\infty}(k)) \to 0 \quad \text{as} \quad u \to \infty.$$

(iv) $\int |u|\, dk(u)$, the integral of $|u|$ with respect to the signed measure determined by k, is finite.

For example, the density function of any normal or Cauchy distribution satisfies these conditions. Lemmas 3.1 through 3.10 below hold for any distribution function F.

LEMMA 3.1. *For any distribution function F,*

$$\lim_{n\to\infty} \sup_x |f_n(x) - Ef_n(x)| = 0$$

with probability one.

PROOF. We note that for $r = 1$ the proof of Lemma 2.2 is valid under the above assumption (i) on k so that by Lemma 2.2 we have

$$P_F\{\sup_x |f_n(x) - Ef_n(x)| > \epsilon\} \leqq C \exp(-\alpha n a_n^2)$$

where $\alpha = 2\epsilon^2/\mu^2$ and $\mu = \vee_{-\infty}^{\infty}(k)$. Since $\sum_{n=1}^{\infty} \exp(-\alpha n a_n^2)$ is finite, it follows that $\sum_{n=1}^{\infty} P_F\{\sup_x |f_n(x) - Ef_n(x)| > \epsilon\}$ is finite, and the proof is complete in view of the Borel-Cantelli Lemma.

We note here that this lemma was proved in [5] for continuous distribution functions F. We have extended this lemma to arbitrary F by using Lemma 2.1 to establish inequality 5 on page 187 of [5].

LEMMA 3.2. *In order for* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some function* g, *it is necessary and sufficient that*

$$\lim_{n\to\infty} \sup_x |Ef_n(x) - g(x)| = 0.$$

PROOF. This result follows directly from Lemma 3.1 in conjunction with the following inequalities:

$$\sup_x |f_n(x) - g(x)| \leqq \sup_x |f_n(x) - Ef_n(x)| + \sup_x |Ef_n(x) - g(x)|$$

and

$$\sup_x |Ef_n(x) - g(x)| \leqq \sup_x |f_n(x) - Ef_n(x)| + \sup_x |f_n(x) - g(x)|.$$

LEMMA 3.3. *If* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some function* g, *then* g *is uniformly continuous.*

PROOF. For any $\epsilon > 0$ there exists by Lemma 3.2 an $M = M(\epsilon)$ such that $\sup_x |Ef_n(x) - g(x)| < \epsilon/4$ for $n \geqq M$. Conditions (i) and (ii) on k imply that k is uniformly continuous, so that given $\epsilon' > 0$ there exists a δ' such that $|k(x) - k(y)| < \epsilon'$ whenever $|x - y| < \delta'$. With $\epsilon' = \frac{1}{2}\epsilon a_m$ we define δ to be $\delta' a_m$ so that whenever $|x - y| < \delta$, we shall have

$$\begin{aligned}
|g(x) - g(y)| &\leqq |g(x) - Ef_M(x)| + |Ef_M(x) - Ef_M(y)| + |Ef_M(y) - g(y)| \\
&\leqq |Ef_M(x) - Ef_M(y)| + 2 \sup_x |Ef_M(x) - g(x)| \\
&\leqq |Ef_M(x) - Ef_M(y)| + \tfrac{1}{2}\epsilon \\
&= |\textstyle\int a_M^{-1}\{k((x - u)/a_M) - k((y - u)/a_M)\}\, dF(u)| + \tfrac{1}{2}\epsilon \\
&< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.
\end{aligned}$$

LEMMA 3.4. *If* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some function* g, *then* $\lambda\{x\varepsilon(-\infty, \infty) \mid F'(x) \neq g(x)\} = 0$ (λ *represents the Lebesgue measure on the real line*).

PROOF. Suppose x is a point where $F'(x)$ exists. Using integration by parts we see that

$$\begin{aligned}
Ef_n(x) &= \textstyle\int a_n^{-1} k((x - u)/a_n)\, dF(u) \\
&= -\textstyle\int a_n^{-1} F(u)\, dk((x - u)/a_n) \\
&= \textstyle\int a_n^{-1} F(x - a_n u)\, dk(u) \\
&= \textstyle\int (F(x - a_n u) - F(x)) a_n^{-1}\, dk(u), \qquad \text{since } \textstyle\int dk(u) = 0.
\end{aligned}$$

Let δ be such that condition (iii) on k holds. Then we may write

$$Ef_n(x) = \int_{|n|>a_n}^{-\delta} (F(x - a_n u) - F(x))a_n^{-1}\, dk(u)$$
$$+ \int I_n(u)(F(x - a_n u) - F(x))(-a_n u)^{-1}(-u)\, dk(u)$$

where I_n is the indicator function of $[-a_n^{-\delta}, a_n^{-\delta}]$. We observe by (iii) that

$$\lim_{n\to\infty}\left|\int_{|n|>a_n}^{-\delta} (F(x - a_n u) - F(x))a_n^{-1}\, dk(u)\right|$$
$$\leqq \lim_{n\to\infty} 2/a_n^{-1}\left(\bigvee_{-\infty}^{-a_n^{-\delta}}(k) + \bigvee_{a_n^{-\delta}}^{\infty}(k)\right) = 0.$$

Also, given $\epsilon > 0$ there exists an $N = N(\epsilon, x)$ such that for $n > N$

$$|I_n(u)(F(x - a_n u) - F(x))(-a_n u)^{-1}| \leqq F'(x) + \epsilon.$$

By condition (iv) on k we have that $\int[F'(x) + \epsilon]\,|u|\, dk(u)$ is finite. Thus Lebesgue's dominated convergence theorem for signed measures applies and hence

$$\lim_{n\to\infty} \int I_n(u)(F(x - a_n u) - F(x))(-a_n u)^{-1}(-u)\, dk(u)$$
$$= \int \lim_{n\to\infty} I_n(u)(F(x - a_n u) - F(x))(-a_n u)^{-1}(-u) dk(u)$$
$$= \int F'(x)(-u)\, dk(u)$$
$$= F'(x)$$

since $\int (-u)\, dk(u) = 1$. Therefore $\lim_{n\to\infty} Ef_n(x) = F'(x)$ whenever $F'(x)$ exists. By Lemma 3.2, $\lim_{n\to\infty} Ef_n(x) = g(x)$ everywhere and hence $F'(x) = g(x)$ whenever $F'(x)$ exists. Since it is well known that the derivative of a monotone function exists almost everywhere, this completes the proof.

LEMMA 3.5. *If* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some function* g, *then* $\int g(u)\, du \leqq 1$.

PROOF. Let $F(x) = F_{AC}(x) + F_S(x) + F_D(x)$ where F_{AC}, F_S and F_D denote the absolutely continuous, the singular, and the discrete part of F respectively. Now $F'(x) = F'_{AC}(x)$ almost everywhere, and $F'(x) = g(x)$ almost everywhere by Lemma 3.4, so that $F'_{AC}(x) = g(x)$ almost everywhere. Thus

$$F_{AC}(x) = \int_{-\infty}^{x} F'_{AC}(u)\, du = \int_{-\infty}^{x} g(u)\, du$$

which implies $\int g(u)\, du \leqq 1$ since $\lim_{x\to\infty} F_{AC}(x)$ exists and is less than or equal to one.

LEMMA 3.6. *If* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some function* g, *then* $F'_{AC}(x) = g(x)$ *everywhere.*

PROOF. In the proof of Lemma 3.5 we have shown that $F'_{AC}(x) = g(x)$ almost everywhere. Consequently

$$F_{AC}(x) - F_{AC}(a) = \text{(Lebesgue)} \int_a^x F'_{AC}(u)\, du$$
$$= \text{(Lebesgue)} \int_a^x g(u)\, du$$
$$= \text{(Riemann)} \int_a^x g(u)\, du$$

since g is uniformly continuous on $[a, x]$ by Lemma 3.3. So $F'_{AC}(x) = g(x)$ by the fundamental theorem of calculus for Riemann integrals.

Lemma 3.7. *If* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some function* g, *then*

$$\lim_{n\to\infty} \sup_x \int a_n^{-1} k((x-u)/a_n)\, d[F_S(u) + F_D(u)] = 0.$$

Proof. From $Ef_n(x) = \int a_n^{-1} k((x-u)/a_n)\, dF(u)$ we obtain

$$Ef_n(x) - \int a_n^{-1} k((x-u)/a_n)\, dF_{AC}(u) = \int a_n^{-1} k((x-u)/a_n)\, d[F_S(u) + F_D(u)].$$

So for $\delta > 0$ we have with the aid of Lemma 3.6

$$\begin{aligned}
0 &\leqq \int a_n^{-1} k((x-u)/a_n)\, d[F_S(u) + F_D(u)] \\
&\leqq |Ef_n(x) - g(x)| + |g(x) - \int a_n^{-1} k((x-u)/a_n)\, dF_{AC}(u)| \\
&= |Ef_n(x) - g(x)| + |g(x) - \int a_n^{-1} k((x-u)/a_n) g(u)\, du| \\
&= |Ef_n(x) - g(x)| + |\int \{g(x) - g(x-u)\} a_n^{-1} k(u/a_n)\, du| \\
&\leqq |Ef_n(x) - g(x)| + \int_{|u|<\delta} |g(x) - g(x-u)|\, |a_n^{-1} k(u/a_n)\, du \\
&\quad + \int_{|u|\geqq\delta} |g(x) - g(x-u)| a_n^{-1} k(u/a_n)\, du \\
&\leqq |Ef_n(x) - g(x)| + \sup_{|u|<\delta} |g(x) - g(x-u)| \\
&\quad + 2 \sup_x g(x) \int_{|u|>\delta/a_n} k(u)\, du.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1)\quad \sup_x \int a_n^{-1} k((x-u)/a_n)\, d[F_S(u) + F_D(u)] &\leqq \sup_x |Ef_n(x) - g(x)| \\
&+ \sup_x \sup_{|u|<\delta} |g(x) - g(x-u)| + 2 \sup_x g(x) \int_{|u|>\delta/a_n} k(u)\, du.
\end{aligned}$$

In view of Lemmas 3.3, 3.5 and 3.6, g is uniformly continuous and non-negative and $\int g(u)\, du$ is finite, whence g is bounded.

Let $\epsilon > 0$ be given. Since g is uniformly continuous we can choose δ so small that the second term on the right side of (1) is less than $\epsilon/3$. Having so chosen δ we can now choose N so large that if $n \geqq N$, then the remaining terms on the right side of (1) will each be less than $\epsilon/3$, since the first term tends to 0 by Lemma 3.2 and the last term goes to zero for any fixed $\delta > 0$. The desired conclusion now follows.

Lemma 3.8. *If* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some function* g, *then* $F_D(x) = 0$ *for all* x.

Proof. Suppose there exists an x_0 such that $F_D(x_0) - F_D(x_0 - 0) > 0$. Then

$$\int a_n^{-1} k((x-u)/a_n)\, dF_D(u) \geqq a_n^{-1} k((x-x_0)/a_n)\{F_D(x_0) - F_D(x_0 - 0)\}.$$

If c is such that $k(c) > 0$ and $x_n = ca_n + x_0$ then

$$\begin{aligned}
\sup_x \int a_n^{-1} k((x-u)/a_n)\, dF_D(u) &\geqq \int_{-\infty}^{\infty} a_n^{-1} k((x_n - u)/a_n)\, dF_D(u) \\
&\geqq k(c) a_n^{-1} \{F_D(x_0) - F_D(x_0 - 0)\}
\end{aligned}$$

which contradicts Lemma 3.7. (Recall that $a_n \to 0^+$.)

LEMMA 3.9. *If* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some* g, *then* 0 *is a derived number of* F_S *at* x_0 *(as defined on page* 207 *of* [6]) *for any* x_0 *in* $(-\infty, \infty)$.

PROOF. Let a be such that $k(a) > 0$. Since k is continuous there exists a number $b > a$ such that $\inf_{a \leq x \leq b} k(x) \geq \frac{1}{2} k(a)$. Now

$$\int a_n^{-1} k((x - u)/a_n) \, dF_S(u) \geq \int_{x-ba_n}^{x-aa_n} a_n^{-1} k((x - u)/a_n) \, dF_S(u)$$
$$\geq \inf_{x-ba_n \leq u \leq x-aa_n} k((x - u)/a_n) \cdot (F_S(x - aa_n) - F_S(x - ba_n))a_n^{-1}$$
$$\geq \tfrac{1}{2} (b - a) k(a) \cdot (F_S(x - aa_n) - F_S(x - ba_n))((b - a)a_n)^{-1} \geq 0.$$

Let x_0 be an arbitrary but fixed real number and $x_n = x_0 + aa_n$. It then follows that

$$\sup_x \int a_n^{-1} k((x - u)/a_n) \, dF_S(u)$$
$$\geq \tfrac{1}{2}(b - a)k(a) \cdot (F_S(x_n - aa_n) - F_S(x_n - ba_n))((b - a)a_n)^{-1}$$
$$= \tfrac{1}{2}(b - a)k(a) \cdot (F_S(x_0) - F_S(x_0 - (b - a)a_n))((b - a)a_n)^{-1}.$$

From Lemma 3.7 we can easily deduce that

$$\lim_{n\to\infty} (F_S(x_0) - F_S(x_0 - (b - a)a_n))((b - a)a_n)^{-1} = 0.$$

Since x_0 was arbitrary the proof is complete.

LEMMA 3.10. *If* $\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$ *with probability one for some function* g, *then* $F_S(x) = 0$ *for all* x.

PROOF. Let a and b be real numbers with $a < b$, and put $h(x) = F_S(x) + x$. Then h is strictly increasing on $[a, b]$ and by Lemma 3.9 it has a derived number equal to one at every point. Thus if we take $E = [a, b]$ in Lemma 2 on page 208 of [6] then we have

$$0 \leq \lambda^*(h[a, b]) \leq 1 \cdot \lambda^*([a, b]) \tag{2}$$

where $\lambda^*(E)$ denotes the Lebesgue outer measure of E and $h[a, b]$ is the image of $[a, b]$ under h. Since $h[a, b] = [a + F_S(a), b + F_S(b)]$ we can rewrite (2) as

$$0 \leq b + F_S(b) - a - F_S(a) \leq b - a$$

which means $F_S(b) = F_S(a)$. Since a and b were arbitrary, F_S must be constant and hence F_S must be identically zero since $\lim_{x\to-\infty} F_S(x) = 0$.

We are now ready to obtain the main theorem of this section.

THEOREM 3.11. *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \sup_x |f_n(x) - g(x)| = 0$$

with probability one for a function g *is that* g *be the uniformly continuous derivative of* F.

PROOF. The sufficiency of this condition has been established by Nadaraya [5] for a larger class of kernels than that considered here.

Conversely, Lemmas 3.8 and 3.10 show that $F = F_{AC}$. Lemma 3.6 states that

$F'_{AC}(x) = g(x)$ everywhere and hence $F'(x) = g(x)$ everywhere. Finally Lemma 3.3 yields the uniform continuity of g and the necessity of the condition is established.

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