The Annals of Mathematical Statistics
1969, Vol. 40, No. 4, 1167–1176

STARSHAPED TRANSFORMATIONS AND THE POWER OF RANK TESTS[1]

By Kjell Doksum

University of California, Berkeley

1. Introduction and summary. It is known (Lehmann (1959), p. 187, and Bell, Moser and Thompson (1966), p. 134) that for the two-sample problem, the closer the two samples are together stochastically, the smaller is the power of monotone rank tests. Here it is shown that if one uses the ideas of van Zwet (1964) to define "skewness" and "heavy tails," then the more skew the distributions of the two samples are, the smaller is the power of monotone rank tests; and heavy tails similarly leads to smaller power of monotone rank tests.

Skewness and heavy tails are defined using convex and star-shaped transformations of random variables. These are the same transformations used in reliability theory (Barlow and Proschan (1965), Birnbaum, Esary and Marshall (1966), and others) to describe the concept of "wear-out." Thus if X is a random variable that represents "time to failure," and if failure is caused by wear-out or by the environment, then there exists a convex or starshaped function g such that $Z = g(X)$ is an exponential $(1 - \exp[-\lambda z])$ random variable. The distributions of these variables X are called increasing failure rate (IFR) distributions when g is convex and (IFRA) distributions when g is starshaped. It turns out that if one restricts attention to such distributions, then the results of this paper can be used to construct a simple optimality theory for rank tests. This is done in a later paper [6].

The power inequalities related to skewness and heavy tails readily extends to sequential rank tests. It is shown (Example 5.1) that the sequential probability ratio test based on ranks for exponential scale alternatives (e.g. [11] and [12]) also is valid for the class of IFRA scale alternatives.

2. Starshaped transformations of random variables. Weakening the convexity condition $g(\lambda x_0 + (1-\lambda)x_1) \leqq \lambda g(x_0) + (1-\lambda)g(x_1)$, $0 \leqq \lambda \leqq 1$, we call a function g defined on the interval $I \subset [0, \infty)$ *starshaped* on I if $g(\lambda x) \leqq \lambda g(x)$ whenever $x \varepsilon I$, $\lambda x \varepsilon I$ and $0 \leqq \lambda \leqq 1$ (see [2]). Thus if $I = (0, \infty)$, then the graph of g initially lies on or below any straight line through the origin, and then lies on or above it. On the class $\mathfrak{F}_0$ of continuous distributions F with $F(0) = 0$, the following ordering (partial) is defined: $F <_* H$ (F is *starshaped* with respect to H) if $H^{-1}F$ is starshaped on $\{x : 0 < F(x) < 1\}$, where $H^{-1}(u) = \inf\{x : H(x) \geqq u\}$. Thus if $F <_* H$ and X has distribution F, then $Z = H^{-1}[F(X)]$ has distribution H and is a starshaped transformation of X; hence one would expect the distribution of Z to be more "skewed to the left" than the distribution

Received 7 August 1968; revised 27 January 1969.

[1] This research has been partially supported by the Office of Naval Research under Contract Nonr-222(83) with the University of California and NSF-GP-7454. Reproduction in whole or in part is permitted for any purpose of the United States Government.

of X (see [13]). Moreover, if X and Z are random variables that represent "time to failure," then $F <_* H$ indicates that X is more subject to wear-out than Z (see [4]).

For the class $\mathfrak{F}$ of continuous distributions F with median 0, i.e., $F(0) = \frac{1}{2}$, the ordering $<_r$ is defined by (see [8]): $F <_r H$ (F is *r-ordered* with respect to H) if $H^{-1}[F(x)]$ is starshaped on $\{x: \frac{1}{2} < F(x) < 1\}$ and $-H^{-1}[F(-x)]$ is starshaped on $\{x: 0 < F(-x) < \frac{1}{2}\}$. If $F <_r H$ and X has distribution F, then $Z = H^{-1}[F(X)]$ has distribution H and one would expect the distribution of Z to have "heavier tails" than the distribution of X. More generally, if F and H are continuous and have medians m_1 and m_2, let $\hat{F}(x) = F(x + m_1)$, $\hat{H}(x) = H(x + m_2)$, and define $<_r$ by: $F <_r H$ if $\hat{H}^{-1}[\hat{F}(x)]$ is starshaped on $\{x: \frac{1}{2} < \hat{F}(x) < 1\}$ and $-\hat{H}^{-1}[\hat{F}(-x)]$ is starshaped on $\{x: 0 < \hat{F}(-x) < \frac{1}{2}\}$. Since all the tests in the next sections are translation invariant, it will be assumed (without loss of generality) that $m_1 = m_2 = 0$. Examples of r-ordered distributions are: uniform $<_r$ normal $<_r$ logistic $<_r$ double exponential $<_r$ Cauchy (van Zwet (1964)).

Note that if F is any continuous distribution with $F(0) = \frac{1}{2}$ and if $H_c(x) = F(cx)$, then $F <_r H_c$ and $H_c <_r F$ for each $c > 0$. Thus the ordering $<_r$ is independent of scale. On the other hand, the power functions of rank tests are not independent of scale. For instance, if the shift parameter in a two-sample problem is fixed, then the power of the Wilcoxon test tends to one as the scale parameter tends to zero, while the power tends to the significance level α as the scale parameter tends to ∞. Thus, in addition to $F <_r H$, a condition is needed on the "dispersion" of the distributions before one can hope to obtain bounds for power functions. If F and H have densities f and h, then it is reasonable to require that the density with the heaviest tail have the smallest value at 0, i.e., $h(0) \leqq f(0)$. The following lemma gives the connection between the condition $h(0) \leqq f(0)$ and two more familiar conditions on the dispersion of f and h. The condition in (ii) has been considered by Lawton (1968). Part of the lemma (as indicated) was proved by van Zwet in a personal communication.

LEMMA 2.1. *Let X and Z have distributions F and H in $\mathfrak{F}$ and let F and H have continuous densities f and h satisfying $f(0) > 0$, $h(0) > 0$, then*

(i) *if $F <_r H$, if F and H are symmetric, and if X and Z have finite variances, then $h(0) \leqq f(0)$ implies* Var $(X) \leqq$ Var (Z), *and* (*van Zwet*) Var $(X) \geqq$ Var (Z) *implies* $h(0)/f(0) \geqq [\text{Var}(X)/\text{Var}(Z)]^{\frac{1}{2}} \geqq 1$.

(ii) *if $F <_r H$, then $h(0) \leqq f(0)$ is equivalent to $P(v \leqq Z \leqq t) \leqq P(v \leqq X \leqq t)$ for all $v < 0 < t$.*

PROOF. Suppose $h(0) \leqq f(0)$. Let $g(x) = H^{-1}[F(x)]$, then $g'(0) = f(0)/h(0) \geqq 1$ which together with $F <_r H$ implies $H^{-1}[F(x)] \geqq x$ for $x > 0$ and $H^{-1}[F(x)] \leqq x$ for $x < 0$. (ii) and the first part of (i) follow easily from this. Suppose Var $(X) \geqq$ Var (Z) and set $c = [\text{Var}(Z)/\text{Var}(X)]^{\frac{1}{2}} \leqq 1$, then

$$(2.1) \qquad E[(g(X) - cX)(g(X) + cX)] = \text{Var}(Z) - \text{Var}(cX) = 0.$$

If $g(x) = cx$, then the second part of (i) clearly holds. Suppose $g(x) \neq cx$, and suppose $g(x)$ does not cross the line cx (except at 0), then either $(g(x) - cx) \cdot$

$(g(x) + cx) \geqq 0$ for all x or $(g(x) - cx)(g(x) + cx) \leqq 0$ for all x, and since $g(x) \neq cx$, this implies either $E[(g(X) - cX)(g(X) + cX)] > 0$ or $E[(g(X) - cX)(g(X) + cX)] < 0$. In either case, (2.1) is contradicted and one can conclude that $g(x)$ crosses cx for $x \neq 0$. This can only happen when $g'(0) = f(0)/h(0) < c$, and the proof is complete.

Hájek (1969) defines the density f to have shorter tails than the density h if $F^{-1}(u) = a(u)H^{-1}(u)$ for some non-decreasing function $a(u)$ on $(0, 1)$. Under the conditions of Lemma 2.1, this is equivalent to $F <_r H$. To see this, make the substitution $u = F(x)$ in the quantity $H^{-1}[F(x)]x^{-1}$ and the substitution $u = F(-x)$ in $-H^{-1}[F(-x)]x^{-1}$.

It turns out (Section 4) that one can obtain the desired inequalities for the power functions of rank tests under slightly weaker conditions. For F and H in $\mathfrak{F}$ one defines: $F <_t H$ (F is *tail ordered* with respect to H) if $H^{-1}[F(x)] - x$ is non-decreasing on $\{x:0 < F(x) < 1\}$. It is clear from the proof of Lemma 2.1 that:

LEMMA 2.2. *Let X and Z have distributions F and H in $\mathfrak{F}$, then*

(i) *if F is tail ordered with respect to H, then* $P(v \leqq Z \leqq t) \leqq P(v \leqq X \leqq t)$ *for all $v < 0 < t$, and* $\text{Var}(X) \leqq \text{Var}(Z)$.

(ii) *if F and H have densities f and h continuous at 0, if F is r-ordered with respect to H, and if $h(0) \leqq f(0)$, then F is tail ordered with respect to H.*

3. Skewness, the scale model and the power of monotone tests. Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent random samples from populations with continuous distributions F and G, and let $r_1 < \cdots < r_m$ denote the ordered ranks of the X's in the combined sample. A test φ is said to be monotone if

$$(3.1) \qquad y_j' \geqq y_j \quad \text{for } j = 1, \cdots, n \quad \text{implies}$$
$$\varphi(x_1, \cdots, x_m, y_1', \cdots, y_n') \leqq \varphi(x_1, \cdots, x_m, y_1, \cdots, y_n).$$

Thus monotone tests are used for one-sided alternatives under which the Y's are stochastically smaller than the X's. All the usual one-sided rank tests such as the Wilcoxon, normal scores, Savage, etc., are monotone. In fact, all one-sided tests based on statistics of the form $\sum J(r_i)$ with J non-decreasing, are monotone. The power of a test φ for the scale alternative $G(y) = F(\Delta y)$ will be denoted by $\beta_s(\varphi; F; \Delta)$, i.e.,

$$(3.2) \quad \beta_s(\varphi; F; \Delta) = E_{F,G}(\varphi) \quad \text{with} \quad G(y) = F(\Delta y), \quad -\infty < y < \infty; \quad \Delta > 0.$$

The null hypothesis considered is $H_0: \Delta \leqq 1$ and the alternative is $H_1: \Delta > 1$. It will be shown that the starshaped ordering (skewness) on the class of distributions induces orderings on the probabilities of type I and type II errors of monotone rank tests. More precisely, increased skewness to the left leads to larger probabilities of these errors. Note that $\beta_s(\varphi; F; \Delta)$, $0 < \Delta \leqq 1$, is the probability of a type I error, while $1 - \beta_s(\varphi; F; \Delta)$, $\Delta > 1$, is the probability of a type II error.

THEOREM 3.1. *If φ is a monotone rank test, if F, $H \varepsilon \mathfrak{F}_0$, and if F is starshaped with respect to H, then*

$$\beta_s(\varphi; H; \Delta) \leqq \beta_s(\varphi; F; \Delta) \quad \textit{for each} \quad \Delta > 1. \tag{3.3}$$

If in addition, $H(x) < 1$ for each $x < \infty$, or $F(x) < 1$ for each $x < \infty$, then

$$\beta_s(\varphi; H; \Delta) \geqq \beta_s(\varphi; F; \Delta) \quad \textit{for each} \quad 0 < \Delta \leqq 1. \tag{3.4}$$

PROOF. The proof essentially consists of showing that the starshaped ordering is equivalent to some stochastic ordering. Let $X_i' = H^{-1}[F(X_i)]$ and $Y_j' = H^{-1}[F(Y_j)]$, $i = 1, \cdots, m$; $j = 1, \cdots, n$. Since $H^{-1}F$ is increasing and φ is a rank test, then

$$\varphi(X_1, \cdots, X_m, Y_1, \cdots, Y_n) = \varphi(X_1', \cdots, X_m', Y_1', \cdots, Y_n'). \tag{3.5}$$

Next note that if one lets $Y_j'' = \{H^{-1}[F(\Delta Y_j)]\}/\Delta$, then the definition of $F <_* H$ implies that for each $\Delta > 1$,

$$Y_j' = H^{-1}[F(Y_j)] = H^{-1}[F(\Delta^{-1}(\Delta Y_j))] \leqq \Delta^{-1}H^{-1}[F(\Delta Y_j)] = Y_j'' \tag{3.6}$$

provided Y_j and ΔY_j are in $I_F = \{x:0 < F(x) < 1\}$. Note that ΔY_j has the same distribution as X_i, thus ΔY_j is in I_F with probability one. If Y_j is not in I_F, then $F(Y_j) = 0$ and $Y_j' = H^{-1}(0) \leqq Y_j''$. Hence in all cases, $Y_j' \leqq Y_j''$ a.s. (almost surely), and since φ is monotone, then

$$\begin{aligned} &\varphi(X_1', \cdots, X_m', Y_1'', \cdots, Y_n'') \\ &\qquad \leqq \varphi(X_1', \cdots, X_m', Y_1', \cdots, Y_n') \quad \text{a.s.} \quad (\Delta > 1). \end{aligned} \tag{3.7}$$

Note that X_i' has distribution $H(x)$, Y_j'' has distribution $H(\Delta y)$, and that (3.5) and (3.7) imply that

$$\begin{aligned} &\varphi(X_1', \cdots, X_m', Y_1'', \cdots, Y_n'') \\ &\qquad \leqq \varphi(X_1, \cdots, X_m, Y_1, \cdots, Y_n) \quad \text{a.s.} \quad (\Delta > 1). \end{aligned} \tag{3.8}$$

Thus (3.3) follows by taking expectations in (3.8). Next suppose that $0 < \Delta \leqq 1$, then the definition of the starshaped ordering implies that

$$Y_j'' = \Delta^{-1}H^{-1}[F(\Delta Y_j)] \leqq H^{-1}[F(Y_j)] = Y_j' \tag{3.9}$$

provided that Y_j and ΔY_j are in I_F. Again ΔY_j is in I_F a.s. If Y_j is not in I_F, then $F(Y_j) = 1$, and $Y_j' = H^{-1}[1] = \infty$ under the additional condition: $H(x) < 1$ for each $x < \infty$. On the other hand, if $F(x) < 1$ for each $x < \infty$, then Y_j is in I_F a.s. In either case, $Y_j'' \leqq Y_j'$ a.s. (3.4) now follows upon noting that for $0 < \Delta \leqq 1$, (3.7) and (3.8) hold with the inequality signs reversed.

The following example shows that the rank condition can not be omitted from Theorem 3.1. Whether or not it can be replaced by the condition "φ is a Pitman permutation test" is not known.

EXAMPLE 3.1. The uniformly most powerful level α test for one-sided scale alternatives when $F(x)$ is of the exponential form $1 - \exp[-\lambda x]$, $x \geqq 0$, $\lambda > 0$, is given by (e.g. [12])

$$\varphi^* = 1 \quad \text{if} \quad \bar{x}/\bar{y} \geqq k_\alpha \qquad (3.10)$$
$$= 0 \quad \text{otherwise}$$

where k_α is the $100(1 - \alpha)$th percentile of the F distribution with $2m$ and $2n$ degrees of freedom. It is clear that this test is monotone, but it is not a rank test. It does not satisfy (3.3) for $0 < \alpha < \frac{1}{2}$. To see this, let $K(x) = 1 - \exp(-x)$ and $F_0(x) = 1 - \exp(-x^2)$, $x \geqq 0$, then F_0 is starshaped with respect to K. Let $N = m + n$, Φ stand for the standard normal distribution function, and let $c_\alpha = \Phi^{-1}(\alpha) + (\Delta - 1)[mnN^{-1}]^{\frac{1}{2}}$. It is shown in [5] that if $\Delta = 1 + cN^{-\frac{1}{2}}$ for some constant $c > 0$, then for each $\epsilon > 0$ there exist m_1 and n_1 such that

$$|\beta_s(\varphi^*; K; \Delta) - \Phi(c_\alpha)| < \epsilon, \quad \text{and} \qquad (3.11)$$

$$|\beta_s(\varphi^*; F_0; \Delta) - \Phi(c_\alpha[\pi(4 - \pi)^{-1}]^{\frac{1}{2}})| < \epsilon \qquad (3.12)$$

for $m \geqq m_1$ and $n \geqq n_1$. Since $\alpha < \frac{1}{2}$, $\Phi^{-1}(\alpha) < 0$, and there exists c such that $c_\alpha < 0$. Upon computing $[\pi(4 - \pi)^{-1}]^{\frac{1}{2}} = 1.91 > 1$, one obtains from (3.11) and (3.12) that there exist m, n and $\Delta > 1$ for which $\beta_s(\varphi^*; K; \Delta) > \beta_s(\varphi^*; F_0; \Delta)$, thus (3.3) does not hold. Using the arguments of van Zwet (1964, p. 10), one can generalize the above example and show that if F_1 and F_2 are any two distinct distributions in $\mathfrak{F}_0$ with finite variances such that F_1 is starshaped with respect to F_2, then there exist m, n and $\Delta > 1$ such that (3.3) does not hold for the test φ^* (provided $0 < \alpha < \frac{1}{2}$).

To obtain a counterexample to (3.4) for the test φ^*, take $\Delta = 1$, $\frac{1}{2} < \alpha < 1$, and note that the above computations and [5,], p. 1734, yield $\beta_s(\varphi^*; K; 1) = \alpha$ while $\beta_s(\varphi^*; F_0; 1) \to \Phi(\Phi^{-1}(\alpha)[\pi(4 - \pi)^{-1}]^{\frac{1}{2}}) > \alpha$.

For a simpler counterexample illustrating the same inequalities but using a less reasonable test, consider

EXAMPLE 3.2. Let F and H in $\mathfrak{F}_0$ be such that $H^{-1}[F(x)] = x^2$, e.g., $H(x) = 1 - e^{-x}$, $F(x) = 1 - e^{-x^2}$. Consider the monotone test

$$\varphi_1 = 1 \quad \text{if} \quad y_1 \leqq \tfrac{1}{2}x_1$$
$$= 0 \quad \text{otherwise.}$$

Suppose $0 < \frac{1}{2}\Delta < 1$, then $\beta_s(\varphi_1; H; \Delta) = P_H(Y_1 \leqq \frac{1}{2}X_1) = P_H((Y_1/X_1) \leqq \frac{1}{2}\Delta \mid \Delta = 1) > P_H((Y_1/X_1)^{\frac{1}{2}} \leqq \frac{1}{2}\Delta \mid \Delta = 1) = P_F(Y_1 \leqq \frac{1}{2}X) = \beta_s(\varphi_1; F; \Delta)$. Thus (3.3) is contradicted for $1 < \Delta < 2$.

REMARK. If alternatives with the Y's stochastically larger than the X's were of interest, then monotone tests would be defined to be non-increasing functions of the x's rather than the y's (see (3.1)). In that case, the results of this paper would continue to hold if one lets the F's and H's denote distributions of the Y's and the G's distributions of the X's.

4. Heavy tails, the translation model, and the power of monotone tests. Consider the two samples and the monotone tests of Section 3. For the translation alternative $G(y) = F(y + \theta)$, the power of each test φ will be denoted by $\beta_t(\varphi; F; \theta)$, i.e.,

$$\beta_t(\varphi; F; \theta) = E_{F,G}(\varphi) \quad \text{with} \quad G(y) = G(y + \theta), \qquad -\infty < y < \infty, \quad -\infty < \theta < \infty. \tag{4.1}$$

The null hypothesis $H_0': \theta \leqq 0$ is to be tested against the alternative $H_1': \theta > 0$. The next result in conjunction with Lemma 2.2 (ii) shows that for monotone rank tests, the probabilities of type I and type II errors increase as one makes the tails of the distributions heavier (in the sense of Section 2).

Theorem 4.1. *If φ is a monotone rank test, if $F, H \in \mathfrak{F}$, and if F is tail ordered with respect to H, then*

$$\beta_t(\varphi; H; \theta) \leqq \beta_t(\varphi; F; \theta) \qquad \textit{for each} \quad \theta > 0. \tag{4.2}$$

If in addition $F(x) < 1$ for each $x < \infty$, or $H(x) < 1$ for each $x < \infty$, then

$$\beta_t(\varphi; H; \theta) \geqq \beta_t(\varphi; F; \theta) \qquad \textit{for each} \quad \theta \leqq 0. \tag{4.3}$$

Proof. Let $X_i' = H^{-1}[F(X_i)]$ and $Y_j' = H^{-1}[F(Y_j)]$, $i = 1, \cdots, m; j = 1, \cdots, n$. Since $H^{-1}F$ is increasing and φ is a rank test, then

$$\varphi(X_1, \cdots, X_m, Y_1, \cdots, Y_n) = \varphi(X_i', \cdots, X_m', Y_1', \cdots, Y_n'). \tag{4.4}$$

Set $\hat{Y}_j = H^{-1}[F(Y_j + \theta)] - \theta$, then the definition of tail ordering implies that for each $\theta \geqq 0$,

$$Y_j' - Y_j = H^{-1}[F(Y_j)] - Y_j \leqq H^{-1}[F(Y_j + \theta)] - (Y_j + \theta) = \hat{Y}_j - Y_j \tag{4.5}$$

provided Y_j and $Y_j + \theta$ are in $I_F = \{x: 0 < F(x) < 1\}$. $Y_j + \theta$ has the same distribution as X, thus $Y_j + \theta \in I_F$ a.s. If Y_j is not in I_F, then $F(Y_j) = 0$ and $Y_j' = H^{-1}(0) \leqq \hat{Y}_j$. Thus in all cases, the inequality (4.5) holds a.s., and we obtain $Y_j' \leqq \hat{Y}_j$ a.s., $j = 1, \cdots, n$. Since φ is monotone, this yields

$$\varphi(X_1', \cdots, X_m', \hat{Y}_1, \cdots, \hat{Y}_n) \leqq \varphi(X_1', \cdots, X_m', Y_1', \cdots, Y_n') \qquad \text{a.s.} \quad (\theta > 0). \tag{4.6}$$

Finally note that X_i' has distribution $H(x)$, $\hat{Y}_j$ has distribution $H(y + \theta)$, and that (4.4) and (4.6) imply that

$$\varphi(X_i', \cdots, X_m', \hat{Y}_1, \cdots, \hat{Y}_n) \leqq \varphi(X_1, \cdots, X_m, Y_1, \cdots, Y_n) \qquad \text{a.s.} \quad (\theta > 0). \tag{4.7}$$

Thus (4.2) follows upon taking expectations in (4.7). Next suppose that $\theta \leqq 0$, then

$$\hat{Y}_j - Y_j = H^{-1}[F(Y_j + \theta)] - (Y_j + \theta) \leqq H^{-1}[F(Y_j)] - Y_j = Y_j' - Y_j \tag{4.8}$$

provided Y_j, $Y_j + \theta \varepsilon I_F$. $Y_j + \theta \varepsilon I_F$ a.s. as before. If $F(x) < 1$ for each $x < \infty$, then $Y_j \varepsilon I_F$ a.s. If $Y_j \not\varepsilon I_F$, but $H(x) < 1$ for each $x < \infty$, then $Y_j' = H^{-1}(1) = \infty$. In all cases, $\hat{Y}_j \leqq Y_j'$ a.s. when $\theta \leqq 0$. Finally, (4.3) follows upon noting that for $\theta \leqq 0$, (4.7) holds with the inequality sign reversed.

For each distribution function F, let $\sigma^2(F)$ denote the variance $\int x^2\, dF(x) - (\int x\, dF(x))^2$. The next example shows that the condition of tail ordering in Theorem 4.1 cannot be replaced by the condition $\sigma^2(F) \leqq \sigma^2(H)$.

Example 4.1. Let φ be any non-constant, non-randomized, monotone rank test, let F_1 denote the standard normal distribution function, and let F_2 denote the uniform distribution on $(-2, 2)$. Then $\sigma^2(F_1) = 1 < \frac{4}{3} = \sigma^2(F_2)$, but for $\theta > 4$, $\beta_t(\varphi; F_2; \theta) = 1 > \beta_t(\varphi; F_1; \theta)$. $\beta_t(\varphi; F_2; \theta) = 1$ for $\theta > 4$ since the test must reject when all the x's are larger than all the y's (which happens a.s. under F_2), while $\beta_t(\varphi; F_1; \theta) < 1$ for all θ since under F_1, no rank ordering has probability one.

Next, consider replacing the tail ordering of Theorem 4.1 by $P_H(v \leqq Z \leqq t) \leqq P_F(v \leqq X \leqq t)$ for all $v < 0 < t$, or $|H^{-1}[F(x)]| \geqq x$ for all x. That this is not possible (even in the presence of symmetry) is shown in

Example 4.2. Let φ be any non-constant, non-randomized, monotone rank test, let H_1 denote the uniform distribution on $(-1, 1)$ and let H_2 denote the distribution uniform on $(-2 - \delta, -2) \cup (2, 2 + \delta)$, where $0 < \delta < 1$. Then $P_{H_2}(v \leqq Z \leqq t) \leqq P_{H_1}(v \leqq X \leqq t)$ for all $v < 0 < t$, and $|H_2^{-1}[H_1(x)]| \geqq x$ for all x. However, if φ has level $\alpha = \binom{N}{m}^{-1}$, then there exist $\theta > 0$ and $0 < \delta < 1$ such that

$$(4.9) \qquad \beta_t(\varphi; H_1; \theta) < \beta_t(\varphi; H_2; \theta).$$

To see this, note that $\varphi = 1$ if and only if all the x's are larger than all the y's. Thus,

$$(4.10) \qquad \beta_t(\varphi; H_1; \theta) \to \binom{N}{m}^{-1} \quad \text{as} \quad \theta \to 0^+.$$

Next assume (without loss of generality) that $m \leqq n$ and note that for $0 < \delta < \theta < 1$,

$$(4.11) \qquad \beta_t(\varphi; H_2; \theta) \geqq P_{H_2}(X_1 \geqq 0, \cdots, X_m \geqq 0) = 2^{-m}.$$

Since $2^{-m} > \binom{N}{m}^{-1}$, (4.10) and (4.11) imply that there exists $0 < \theta_0 < 1$ such that $\beta_t(\varphi; H_2; \theta) > \beta_t(\varphi; H_1; \theta)$ for all $0 < \delta < \theta < \theta_0$.

It is sometimes easier to show that $H^{-1}[F(x)]$ is starshaped and that $h(0) \leqq f(0)$ than it is to show that $H^{-1}[F(x)] - x$ is increasing. Thus the following result which follows from Lemma 2.2 (ii) and Theorem 4.1 would be used.

Corollary 4.1. *If F and H in $\mathfrak{F}$ have densities f and h continuous at zero, if F is r-ordered with respect H, and if $h(0) \leqq f(0)$, then the conclusions of Theorem 4.1 hold*

Example 4.3. Let $F_a^{(1)}(x)$, $F_b^{(2)}(x)$, $F_c^{(3)}(x)$, $F_d^{(4)}(x)$ and $F_k^{(5)}(x)$ denote the uniform, normal, logistic, double exponential and Cauchy distributions with densities

$$f_a^{(1)}(x) = \tfrac{1}{2}a, \qquad -a \leqq x \leqq a,\ a > 0,$$
$$f_b^{(1)}(x) = (1/(2\pi)^{\frac{1}{2}}b) \exp(-x^2/b^2), \qquad -\infty < x < \infty,\ b > 0,$$
$$f_c^{(3)}(x) = c^{-1} \exp(-x/c)[1 + \exp(-x/c)]^{-2}, \qquad -\infty < x < \infty,\ c > 0,$$
$$f_d^{(4)}(x) = (\tfrac{1}{2}/d) \exp|-x/d|, \qquad -\infty < x < \infty,\ d > 0, \text{ and}$$
$$f_k^{(5)}(k) = 1/k\pi(1 + x^2/k^2), \qquad -\infty < x < \infty,\ k > 0$$

respectively. $F^{-1(j)}F^{(i)}$ is starshaped for each $i < j$, thus according to Corollary 4.1,

$$(4.12) \quad \beta_t(\varphi; F_k^{(5)}; \theta) \leqq \beta_t(\varphi; F_d^{(4)}; \theta) \leqq \beta_t(\varphi; F_c^{(3)}; \theta) \leqq \beta_t(\varphi; F_b^{(2)}; \theta) \leqq \beta_t(\varphi; F_a^{(1)}; \theta)$$

for all monotone rank tests φ and all $\theta > 0$ provided $f_k^{(5)}(0) \leqq f_d^{(4)}(0) \leqq f_c^{(3)}(0) \leqq f_b^{(2)}(0) \leqq f_a^{(1)}(0)$, i.e., the inequalities (4.12) hold for all a, b, c, d, and k such that

$$(4.13) \qquad 0 \leqq 2a \leqq (2\pi)^{\frac{1}{2}}b \leqq 4c \leqq 2d \leqq \pi k.$$

When $\theta < 0$, (4.12) holds with the inequalities reversed (provided a, b, c, d, and k satisfies (4.13)).

Next we give an example which shows that the rank condition cannot be omitted from the assumptions of Theorem 4.1 and Corollary 4.1.

EXAMPLE 4.4. Let $Z_i = 2^{X_i}$, $W_j = 2^{Y_j}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$ and define the monotone test

$$\varphi_1^* = 1 \quad \text{if } \bar{z}/\bar{w} \geqq k_\alpha$$
$$= 0 \quad \text{otherwise}$$

where $F_{2m,2n}(k_\alpha) = 1 - \alpha$ when $F_{2m,2n}$ is the F distribution with $2m$ and $2n$ degrees of freedom. Let F_1 and F_2 be the uniform distributions on $(-1, 1)$ and $(-2, 2)$ respectively. Then equation (3.5) of [5] and few computations show that if Φ stands for the standard normal distribution,

$$|\beta_t(\varphi_1^*; F_1; \theta) - \Phi(c_\alpha' \cdot (\tfrac{3}{2})^{\frac{1}{2}})| \rightarrow 0$$

and $$|\beta_t(\varphi_1^*; F_2; \theta) - \Phi(c_\alpha' \cdot (\tfrac{15}{13})^{\frac{1}{2}})| \rightarrow 0 \quad \text{as } m, n \rightarrow \infty,$$

where $$c_\alpha' = \Phi^{-1}(\alpha) + (e^\theta - 1)[mnN^{-1}]^{\frac{1}{2}}.$$

This implies that for $0 < \alpha < \frac{1}{2}$, there exist m, n, and $\theta > 0$ such that $\beta_t(\varphi_1^*; F_1; \theta) < \beta_t(\varphi_1^*; F_2; \theta)$. See Example 3.1.

5. Sequential tests. The situation considered here is the one in which the observations at the nth stage form two independent random samples $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ from populations with continuous distributions F and G. A sequential test φ is *monotone* if

$$y_j \geqq y_j' \text{ for } j = 1, 2, \cdots \quad \text{implies}$$

$$\varphi(x_1, x_2, \cdots; y_1', y_2', \cdots) \leqq \varphi(x_1, x_2, \cdots; y_1, y_2, \cdots). \tag{5.1}$$

Rank tests are tests which at the nth stage depend on the ordered ranks $R_1, \cdots, R_n$ of the X's in the combined sample $X_1, \cdots, X_n; Y_1, \cdots, Y_n$. Let $\beta_s(\varphi; F; \Delta)$ and $\beta_t(\varphi; F; \theta)$ denote the power of φ when the Y's have distributions $F(\Delta y)$ and $F(y + \theta)$ respectively, then it is immediate that the results of Theorem 3.1, 4.1 and Corollary 4.1 hold for these quantities.

The following example is typical of the kind of applications the results of this and the preceding sections have. It shows that a test which is appropriate for exponential alternatives actually is valid (in terms of bounds on the probabilities of type I and II errors) for the more general class of IFRA scale alternatives. Further applications to the IFRA scale problem are given in [6].

EXAMPLE 5.1. The sequential probability ratio test $\hat{\varphi} = \hat{\varphi}(a, b)$ based on ranks for the exponential alternative where $F(x) = 1 - \exp(-\lambda x) = K(x)$ (say) and $G(x) = K(\Delta x)$ has been studied by Parent (1965), Savage and Sethuraman (1966), and others. For testing $\Delta = 1$ against $\Delta = \Delta_1 > 1$, it can be written

$$\begin{aligned} &\text{Take two more observations } (X_{n+1}, Y_{n+1}) \text{ if } a \leqq L_n \leqq b, \\ &\text{accept } H_0 \text{ if } L_n < a, \\ &\text{reject } H_0 \text{ if } L_n > b, \\ &n = 1, 2, \cdots, \end{aligned} \tag{5.2}$$

where $0 < a < 1 < b$ are constants independent of n,

$$L_n = (2n)!\,\Delta_1^n \prod_{k=1}^{2n} [k + (\Delta_1 - 1)V_k]^{-1}, \tag{5.3}$$

and V_k is the number of Y's among the k largest observations in the combined sample. Note that $(V_1, \cdots, V_{2n})$ is equivalent to the set of ranks $(R_1, \cdots, R_n)$.

It is clear that this test is monotone; thus the bounds on the error probabilities given in Theorem 3.1 apply. Let α_0 and α_1 be desired error probabilities for $\Delta = 1$ and $\Delta = \Delta_1$ respectively, and suppose that a and b are such that the test $\hat{\varphi}(a, b)$ achieve these bounds for the exponential distribution, i.e., $\beta_s(\hat{\varphi}; K; 1) = \alpha_0$ and $1 - \beta_s(\hat{\varphi}; K; \Delta_1) = \alpha_1$. Then the sequential version of Theorem 3.1 implies that α_0 and α_1 are bounds on the error probabilities for all IFRA distributions for $\Delta \leqq 1$ and $\Delta \geqq \Delta_1$ respectively; i.e.,

$$\begin{aligned} &\beta_s(\hat{\varphi}; F; \Delta) \leqq \alpha_0 \quad \text{for} \quad \Delta \leqq 1, \qquad \text{and} \\ &1 - \beta_s(\hat{\varphi}; F; \Delta) \leqq \alpha_1 \quad \text{for} \quad \Delta \geqq \Delta_1 \\ &\text{for all continuous IFRA distributions } F. \end{aligned} \tag{5.4}$$

Note that the test $\hat{\varphi}$ is easy to carry out as a and b can be closely approximated as follows:

$$a \doteq \alpha_1(1-\alpha_0)^{-1} \quad \text{and} \quad b \doteq (1-\alpha_1)\alpha_0^{-1}. \tag{5.5}$$

REFERENCES

[1] BARLOW, R. E. and PROSCHAN, F. (1965). *Mathematical Theory of Reliability*. Wiley, New York.

[2] BARLOW, R. E., MARSHALL, A. W. and PROSCHAN, F. (1967). Some inequalities for star-shaped and convex functions. Boing Scientific Research Laboratories, Document D1-82-0643, to appear in the *Pacific J. Math.*

[3] BELL, C. B., MOSER, J. M. and THOMPSON, R. (1966). Goodness criteria for two-sample distribution-free tests. *Ann. Math. Statist.* **37** 133–142.

[4] BIRNBAUM, Z. W., ESARY, J. D. and MARSHALL, A. W. (1966). A stochastic characterization of wear-out for components and systems. *Ann. Math. Statist.* **37** 816–826.

[5] DOKSUM, K. D. (1967). Asymptotically optimal statistics in some models with increasing failure rate averages. *Ann. Math. Statist.* **38** 1731–1739.

[6] DOKSUM, K. D. (1969). Minimax results for IFRA scale alternatives. Submitted.

[7] HÁJEK, J. (1969). *A Course in Nonparametric Statistics*. Holden-Day, San Francisco.

[8] LAWRENCE, M. J. (1966). Inequalities and tolerance limits for s-ordered distributions. Technical Report ORC-66-37, Operations Research Center, Univ. of California, Berkeley.

[9] LAWTON, W. H. (1968). Concentration of random quotients. *Ann. Math. Statist.* **39** 466–480.

[10] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.

[11] PARENT, E. A. (1965). Sequential ranking procedures. Stanford University, Department of Statistics, Tech. Rept. No. 80.

[12] SAVAGE, I. R. and SETHURAMAN, J. (1966). Stopping time of a rank-order sequential probability ratio test on Lehmann alternatives. *Ann. Math. Statist.* **37** 1154–1160.

[13] VAN ZWET, W. R. (1964). *Convex Transformations of Random Variables*. Math. Centre, Amsterdam.