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RANK ORDER TESTS FOR MULTIVARIATE PAIRED COMPARISONS[1]

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1. Summary and introduction. The only non–parametric *multivariate* paired comparison tests presently available for testing the hypothesis of no difference among several treatments are (i) the Sen-David (1968) test and (ii) the Davidson-Bradley (1969) test. Both these tests are applicable to situations which involve the preferences of each individual comparison. Both these tests are the generalizations of the one–sample multivariate sign tests [1]. As such their A.R.E.'s (Asymptotic Relative Efficiencies) with respect to the normal theory $\mathcal{F}$-test are not expected to be high. In fact the A.R.E. of the Sen-David (1968) test with respect to the normal theory $\mathcal{F}$-test can be as low as zero (under normality).

The purpose of this paper is to develop test procedures which could be considered as competitors to the Sen-David (1968) and to the Davidson-Bradley (1968) tests. The proposed procedures are based on the ranks of the observed comparison differences, and include as special cases the multivariate normal scores and the multivariate rank sum paired comparison tests. For convenience of presentation we develop the theory when the paired comparisons involve paired characteristics. Under suitable regularity conditions the limiting distributions of the proposed test statistics are derived under the null as well as non-null hypotheses, and their large sample properties are studied. It is shown that for various situations of interest the proposed procedures have considerable efficiency improvements over the Sen–David (1968) and the normal theory procedures.

2. Mathematical model and the proposed tests. Let us consider t treatments in an experiment involving paired comparisons, and suppose that for the pair (i, j) of treatments $(1 \leqq i < j \leqq t)$, the N_{ij} encounters yield the random variables $\mathbf{Z}_{ij,l} = (X_{ij,l}, Y_{ij,l})$, $l = 1, \cdots, N_{ij}$ which are independent and identically distributed according to an absolutely continuous cdf (cumulative distribution function) $\prod_{ij}(\mathbf{z}) = \prod_{ij}(x, y)$, $1 \leqq i < j \leqq t$. The null hypothesis states that

$$\prod\nolimits_{ij}(\mathbf{z}) = \prod(\mathbf{z}) \tag{2.1}$$

where $\prod(\mathbf{z})$ is unknown and $\prod(\mathbf{z})$ is diagonally symmetric about $\mathbf{0}$; that is, its density $\pi(\mathbf{z})$ is invariant under the simultaneous changes of signs of all the coordinate variates.

Let $N = \frac{1}{2} \sum_{i=1, i \neq j}^{t} \sum_{j=1}^{t} N_{ij}$ be the total number of observations. For each variate separately, arrange the absolute values of the observations in increasing

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order of magnitude. Let

$$a_{XN,r}^{(i,j)} = +1 \ (-1) \tag{2.2}$$

if the rth smallest of the $|X_{ab,c}|$, $c = 1, \cdots, N_{ab}$, $1 \leqq a < b \leqq t$ is from the pair (i, j) and the corresponding observation is positive (negative) and otherwise let $a_{XN,r}^{(i,j)} = 0$.

$$a_{YN,r}^{(i,j)} = +1 \ (-1) \tag{2.3}$$

if the rth smallest of the $|Y_{ab,c}|$, $c = 1, \cdots, N_{ab}$, $1 \leqq a < b \leqq t$ is from the pair (i, j) and the corresponding observation is positive (negative) and otherwise let $a_{YN,r}^{(i,j)} = 0$. Denote

$$T_{N1(i,j)} = N_{ij}^{-1} \sum_{r=1}^{N} E_{N1,r} a_{XN,r}^{(i,j)}, \qquad T_{N2(i,j)} = N_{ij}^{-1} \sum_{r=1}^{N} E_{N2,r} a_{YN,r}^{(i,j)} \tag{2.4}$$

where $E_{Ni,r} = J_{Ni}(r/(N+1))$, $i = 1, 2$, $r = 1, \cdots, N$ is the expected value of the rth order statistic of a sample of size N from a distribution

$$\begin{aligned} \psi_i(x) &= \psi_i^*(x) - \psi_i^*(-x) && \text{if } x \geqq 0, \\ &= 0 && \text{otherwise}, \qquad i = 1, 2. \end{aligned} \tag{2.5}$$

We assume that $\psi_i^*(x)$ satisfies the following assumptions (cf. [12]):

Assumption I. $\psi_i^*(x)$ is symmetric about $x = 0$, $i = 1, 2$.

Assumption II. $N^{-1} \sum_{r=1}^{N} [E_{Ni,r} - \psi_i^{*-1}(r/(N+1))]\lambda_{N,r} = o_p(N^{-\frac{1}{2}})$ where $\lambda_{N,r} = a_{XN,r}^{(\alpha,\beta)}, a_{YN,r}^{(\alpha,\beta)}$; $1 \leqq \alpha < \beta \leqq t$; $i = 1, 2$.

Assumption III. $J_i^*(u) = \psi_i^{*-1}(u)$ is absolutely continuous, and $|J_i^{*(j)}(u)| = |dJ_i^{*(j)}(u)/du^j| \leqq K[u(1-u)]^{\delta - j - \frac{1}{2}}$, $j = 0, 1$, for some K and some $\delta > 0$; $i = 1, 2$. Further, let

$$U_{N,i} = \sum_{j=1, j\neq i}^{t} N_{ij}^{\frac{1}{2}} T_{N1(i,j)}, \qquad V_{N,i} = \sum_{j=1, j\neq i}^{t} N_{ij}^{\frac{1}{2}} T_{N2(i,j)}. \tag{2.6}$$

Then the proposed test statistic is

$$\mathfrak{L}_N = t^{-1} \sum_{i=1}^{t} (U_{n,i}, V_{N,i}) \hat{A}^{-1} (U_{N,i}, V_{N,i})' \tag{2.7}$$

where the matrix $(t-1)\hat{A}$ is an estimator of the covariance matrix of $(U_{N,i}, V_{N,i})$ under H_0, to be specified later. The test consists in rejecting H_0 at level α if $\mathfrak{L}_N \geqq C_\alpha$, where C_α is determined by $P_{H_0}(\mathfrak{L}_N \geqq C_\alpha) = \alpha$. It shall be established that when H_0 is true, if $(t-1)\hat{A}$ is a consistent estimator of the covariance matrix of $(U_{N,i}, V_{N,i})$, then $\mathfrak{L}_N$ has asymptotically a central chi-square distribution with $2(t-1)$ degrees of freedom. This provides a large sample approximation to the critical point C_α.

3. Joint asymptotic normality. Before proving the main theorem of this section, we introduce a few notations and assumptions.

Let $c = \binom{t}{2}$ denote the number of all possible pairs and label the pair (i, j) by $\alpha = (i-1)t + j - \binom{i+1}{2}$, $1 \leqq i < j \leqq t$. Then $\mathbf{Z}_{\alpha 1}, \cdots, \mathbf{Z}_{\alpha N_\alpha}$ are the observations corresponding to the αth pair. They are independently distributed according to the continuous cdf $\prod_\alpha(\mathbf{z}) = \prod_\alpha(x, y)$, $\alpha = 1, \cdots, c$.

Let $N = \sum_{\alpha=1}^{c} N_\alpha$ and $\rho_N^{(\alpha)} = N_\alpha/N$ and assume that for all N, the inequalities $0 < \rho_0 \leqq \rho_N^{(\alpha)} \leqq 1 - \rho_0 < 1$ hold for some fixed $\rho_0 \leqq 1/c$ and $\alpha = 1, \cdots, c$.

Let $F_1^{(\alpha)}(x)$, $F_2^{(\alpha)}(y)$, $H_1^{(\alpha)}(x)$ and $H_2^{(\alpha)}(y)$ be the marginal cdf's of $X_{\alpha r}$, $Y_{\alpha r}$, $|X_{\alpha r}|$ and $|Y_{\alpha r}|$ respectively for $\alpha = 1, \cdots, c$.

$F_{1,N}^{(\alpha)}(x)$, $F_{2,N}^{(\alpha)}(y)$, $H_{1,N}^{(\alpha)}(x)$ and $H_{2,N}^{(\alpha)}(y)$ will denote the sample cdf's of $X_{\alpha r}$, $Y_{\alpha r}$, $|X_{\alpha r}|$ and $|Y_{\alpha r}|$, $r = 1, \cdots, N_\alpha$ respectively, for $\alpha = 1, \cdots, c$.

Denote

$$(3.1) \quad F_{j,N}(x) = \sum_{\alpha=1}^{c} \rho_N^{(\alpha)} F_{j,N}^{(\alpha)}(x), \qquad H_{j,N}(x) = \sum_{\alpha=1}^{c} \rho_N^{(\alpha)} H_{j,N}^{(\alpha)}(x),$$
$$\Pi_j(x) = \sum_{\alpha=1}^{c} \rho_N^{(\alpha)} F_j^{(\alpha)}(x) \qquad H_j(x) = \sum_{\alpha=1}^{c} \rho_N^{(\alpha)} H_j^{(\alpha)}(x), \; j = 1, 2.$$

Thus $F_{1,N}(x)$, $H_{1,N}(x)$, $F_{2,N}(x)$ and $H_{2,N}(x)$ are the combined sample cdf's of X's, $|X|$'s, Y's and $|Y|$'s respectively whose population cdf's are $\Pi_1(x)$, $H_1(x)$, $\Pi_2(x)$ and $H_2(x)$ respectively. The distribution function $\Pi_\alpha(\mathbf{z})$, and hence $F_j^{(2)}(x)$, $j = 1, 2$ may depend upon N, (as for example in Section 4), but for the sake of convenience this notation is suppressed.

As in [12], we write $E_{Ni,r} = J_{Ni}(r/(N + 1))$, $i = 1, 2$, $r = 1, \cdots, N$ and extend the domain of definition of J_{Ni} to $(0, 1)$ by letting it have constant value over $[r/(N + 1), (r + 1)/(N + 1))$, $r = 1. \cdots, N$. Finally, let

$$(3.2) \qquad J_i(u) = \lim_{N\to\infty} J_{Ni}(u) \quad \text{for} \quad 0 < u < 1, \qquad i = 1, 2.$$

With these notations, we may represent $T_{Nj(\alpha)}$, $j = 1, 2$ defined by (2.4) equivalently as

$$(3.3) \quad \begin{aligned} T_{Nj(\alpha)} &= \int_{x=0}^{\infty} J_{Nj}(N(N + 1)^{-1}H_{j,N}(x))\, dF_{j,N}^{(\alpha)}(x) \\ &\quad - \int_{x=-\infty}^{0} J_{Nj}(N(N + 1)^{-1}H_{j,N}(-x))\, dF_{j,N}^{(\alpha)}(x) \\ &= \int_{x=0}^{\infty} J_{Nj}(N(N + 1)^{-1}H_{j,N}(x))\, d[F_{j,N}^{(\alpha)}(x) + F_{j,N}^{(\alpha)}(-x)], \end{aligned}$$
$$j = 1, 2; \alpha = 1, \cdots, c.$$

Finally, set

$$(3.4) \qquad \mu_{Nj(\alpha)} = \int_{x=0}^{\infty} J_j(H_j(x))\, d[F_j^{(\alpha)}(x) + F_j^{(\alpha)}(-x)],$$
$$j = 1, 2; \alpha = 1, \cdots, c.$$

THEOREM 3.1. *Under the assumptions* I, II *and* III, *the random variables* $[N_\alpha^{\frac{1}{2}}(T_{N1(\alpha)} - \mu_{N1(\alpha)}, T_{N2(\alpha)} - \mu_{N2(\alpha)}), \alpha = 1, \cdots, c]$ *have asymptotically a multivariate normal distribution with zero means and covariance matrix* $\mathbf{\Sigma} = ((\sigma_{N,ij,\alpha\beta}))$ *where* $\sigma_{N,ij,\alpha\beta}$, $i, j = 1, 2$; $\alpha, \beta = 1, \cdots, c$ *are given by* (6.15), (6.16), (6.17) *and* (6.18).

PROOF. We can express $T_{Nj(\alpha)}$, $j = 1, 2$; $\alpha = 1, \cdots, c$ as

$$(3.5) \qquad T_{Nj(\alpha)} = \mu_{Nj}(\alpha) + B_{1N,j}^{(\alpha)} + B_{2N,j}^{(\alpha)} + \sum_{k=1}^{c} C_{kN,j}^{(\alpha)},$$
$$j = 1, 2; \alpha = 1, \cdots, c$$

where $\mu_{Nj(\alpha)}$ is given by (3.4),

$$(3.6)\quad B_{1N,j}^{(\alpha)} = \int_{x=0}^{\infty} J_j(H_j(x))\, d[F_{j,N}^{(\alpha)}(x) + F_{j,N}^{(\alpha)}(-x) - F_j^{(\alpha)}(x) - F_j^{(\alpha)}(-x)]$$

$$(3.7)\quad B_{2N,j}^{(\alpha)} = \int_{x=0}^{\infty} J_j'(H_j(x))[H_{j,N}(x) - H_j(x)]\, d[F_j^{(\alpha)}(x) + F_j^{(\alpha)}(-x)]$$

$$(3.8)\quad C_{1N,j}^{(\alpha)} = -(N+1)^{-1} \int_{x=0}^{\infty} J_j'(H_j(x))H_{j,N}(x)\, d[F_{j,N}^{(\alpha)}(x) + F_{j,N}^{(\alpha)}(-x)]$$

$$(3.9)\quad C_{2N,j}^{(\alpha)} = \int_{x=0}^{\infty} J_j'(H_j(x))[H_{j,N}(x) - H_j(x)] \cdot d[F_{j,N}^{(\alpha)}(x) + F_{j,N}^{(\alpha)}(-x) - F_j^{(\alpha)}(x) - F_j^{(\alpha)}(-x)]$$

$$(3.10)\quad C_{3N,j}^{(\alpha)} = \int_{x=0}^{\infty} \{J_j(N(N+1)^{-1}H_{j,N}(x)) - J_j(H_j(x)) - J_j'(H_j(x)) \cdot [N(N+1)^{-1}H_{j,N}(x) - H_j(x)]\}\, d[F_{j,N}^{(\alpha)}(x) + F_{j,N}^{(\alpha)}(-x)]$$

$$(3.11)\quad C_{4N,j}^{(\alpha)} = \int_{x=0}^{\infty} \{J_{N,j}(N(N+1)^{-1}H_{j,N}(x)) - J_j(N(N+1)^{-1}H_{j,N}(x))\} \cdot d[F_{j,N}^{(\alpha)}(x) + F_{j,N}^{(\alpha)}(-x)]$$

By virtue of the assumption III and the fact that $d[F_j^{(\alpha)}(x) + F_j^{(\alpha)}(-x] \leqq \rho_0^{-1}\, dH_j(x)$, it follows that $\mu_{Nj(\alpha)}$ is finite. Furthermore, proceeding precisely as in [12], it can be shown that $C_{kN,j}^{(\alpha)} = o_p(N^{-\frac{1}{2}})$ for $k = 1, 2, 3, 4$ and $j = 1, 2$. Thus the difference $[N^{\frac{1}{2}}(T_{N1(\alpha)} - \mu_{N1(\alpha)}, T_{N2(\alpha)} - \mu_{N2(\alpha)}) - N^{\frac{1}{2}}(B_{1N,1}^{(\alpha)} + B_{2N,1}^{(\alpha)}, B_{1N,2}^{(\alpha)} + B_{2N,2}^{(\alpha)})]$ converges in probability to zero, as $N \to \infty$, for all $\alpha = 1, \cdots, c$. Hence to prove the theorem, it suffices to show

LEMMA 3.1. $B_N = \sum_{\alpha=1}^{c} \sum_{j=1}^{2} N_\alpha^{\frac{1}{2}}(B_{1N,j}^{(\alpha)} + B_{2N,j}^{(\alpha)})\lambda_{j\alpha}$ *(where* $\lambda_{j\alpha}$, $j = 1, 2$; $\alpha = 1, \cdots, c$ *are real constants, not all zero) has the limiting normal distribution as* $N \to \infty$.

Integrating $B_{1N,j}^{(\alpha)}$ by parts and making some routine computations, we can express $B_{1N,1}^{(\alpha)}$ and $B_{1N,2}^{(\alpha)}$ as

$$(3.12)\quad B_{1N,1}^{(\alpha)} = N\alpha^{-1} \sum_{r=1}^{N_\alpha} B_1(X_{\alpha r}), \qquad B_{1N,2}^{(\alpha)} = N_\alpha^{-1} \sum_{r=1}^{N_\alpha} B_2(Y_{\alpha r})$$

where

$$(3.13)\quad B_1(X_{\alpha r}) = -\int_{x=0}^{\infty} [c(x - X_{\alpha r}) + c(-x - X_{\alpha r}) - F_1^{(\alpha)}(x) - F_1^{(\alpha)}(-x)] \cdot dJ_1(H_1(x))$$

$$(3.14)\quad B_2(Y_{\alpha r}) = -\int_{x=0}^{\infty} [c(x - Y_{\alpha r}) + c(-x - Y_{\alpha r}) - F_2^{(\alpha)}(x) - F_2^{(\alpha)}(-x)] \cdot dJ_2(H_2(x))$$

and where

$$(3.15)\quad c(u) = 1 \text{ if } u \geqq 0 \text{ and } c(u) = 0 \text{ otherwise.}$$

Let us now turn our attention to $B_{2N,j}$, $j = 1, 2$. Noting (cf. (3.1)) that $H_{j,N}(x) - H_j(x) = \sum_{\beta=1}^{c} \rho_n^{(\beta)}[H_{j,N}^{(\beta)}(x) - H_j^{(\beta)}(x)]$, we can rewrite $B_{2N,j}^{(\alpha)}$ as

$$(3.16)\quad \begin{aligned} B_{2N,j}^{(\alpha)} &= \sum_{\beta=1}^{c} \rho_N^{(\beta)} \int_{x=0}^{\infty} [H_{j,N}^{(\beta)}(x) - H_j^{(\beta)}(x)]J_j'(H_j(x)) \cdot d[F_j^{(\alpha)}(x) + F_j^{(\alpha)}(-x)] \\ &= N^{-1} \sum_{\beta=1}^{c} \sum_{r=1}^{N_\beta} C_{1\alpha}(X_{\beta r}) \quad \text{if } j = 1 \\ &= N^{-1} \sum_{\beta=1}^{c} \sum_{r=1}^{N_\beta} C_{2\alpha}(Y_{\beta r}) \quad \text{if } j = 2, \end{aligned}$$

where

$$(3.17)\quad C_{1\alpha}(X_{\beta r}) = \int_{x=0}^{\infty} [c(x - |X_{\beta r}|) - H_1^{(\beta)}(x)] J_1'(H_1(x)) \cdot d[F_1^{(\alpha)}(x) + F_1^{(\alpha)}(-x)]$$

and

$$(3.18)\quad C_{2\alpha}(Y_{\beta r}) = \int_{x=0}^{\infty} [c(x - |Y_{\beta r}|) - H_2^{(\beta)}(x)] \cdot J_2'(H_2(x))\, d[F_2^{(\alpha)}(x) + F_2^{(\alpha)}(-x)]$$

and $c(\cdot)$ is defined in (3.15).

Now making use of (3.12) and (3.16), we obtain

$$(3.19)\quad \begin{aligned} \textstyle\sum_{j=1}^{2} \lambda_{j\alpha}(B_{1N,j}^{(\alpha)} + B_{2N,j}^{(\alpha)}) &= \lambda_{12}\{N_\alpha^{-1} \textstyle\sum_{r=1}^{N_\alpha} B_1(X_{\alpha r}) + N^{-1} \sum_{\beta=1}^{c} \sum_{r=1}^{N_\beta} C_{1\alpha}(X_{\beta r})\} \\ &\quad + \lambda_{2\alpha}\{N_\alpha^{-1} \textstyle\sum_{r=1}^{N_\alpha} B_2(Y_{\alpha r}) + N^{-1} \sum_{\beta=1}^{c} \sum_{r=1}^{N_\beta} C_{2\alpha}(Y_{\beta r})\} \end{aligned}$$

Finally, using (3.19), we can express B_N as

$$(3.20)\quad B_N = \textstyle\sum_{\alpha=1}^{c} N_\alpha^{\frac{1}{2}}[N_\alpha^{-1} \sum_{r=1}^{N_\alpha} \{\lambda_{1\alpha} B_1(X_{\alpha r}) + \lambda_{2\alpha} B_2(Y_{\alpha r})\} + N^{-1} \sum_{\beta=1}^{c} \sum_{r=1}^{N_\beta} \{\lambda_{1\alpha} C_{1\alpha}(X_{\beta r}) + \lambda_{2\alpha} C_{2\alpha}(Y_{\beta r})\}]$$

Now denote

$$(3.21)\quad \eta_\alpha(\mathbf{Z}_\alpha) = \textstyle\sum_{r=1}^{N_\alpha} \{\lambda_{1\alpha} B_1(X_{\alpha r}) + \lambda_{2\alpha} B_{2\alpha}(Y_{\alpha r})\}/N_\alpha$$

and

$$(3.22)\quad \xi_{\beta,\alpha}(\mathbf{Z}_\beta) = \textstyle\sum_{r=1}^{N_\beta} \{\lambda_{1\alpha} C_{1\alpha}(X_{\beta r}) + \lambda_{2\alpha} C_{2\alpha}(Y_{\beta r})\}/N_\beta$$

(note that $\eta_\alpha(\mathbf{Z}_\alpha)$ is a function only of $\mathbf{Z}_{\alpha 1}, \cdots, \mathbf{Z}_{\alpha N\alpha}$ and $\xi_{\beta,\alpha}(\mathbf{Z}_\beta)$ is a function only of $\mathbf{Z}_{\beta 1}, \cdots, \mathbf{Z}_{\beta N\beta}$ where $\mathbf{Z}_{\gamma r} = (X_{\gamma r}, Y_{\gamma r})$). Then from (3.20), (3.21) and (3.22), we obtain

$$(3.23)\quad \begin{aligned} B_N &= \textstyle\sum_{\alpha=1}^{c} N_\alpha^{\frac{1}{2}} \eta_\alpha(\mathbf{Z}_\alpha) + \sum_{\alpha=1}^{c} N_\alpha^{\frac{1}{2}} \sum_{\beta=1}^{c} \rho_N^{(\beta)} \xi_{\beta,\alpha}(\mathbf{Z}_\beta) \\ &= \textstyle\sum_{\alpha=1}^{c} N_\alpha^{\frac{1}{2}} \eta_\alpha(\mathbf{Z}_\alpha) + \sum_{r=1}^{c} \sum_{\beta=1}^{c} N_r^{\frac{1}{2}} \rho_N^{(\beta)} \xi_{\beta,r}(\mathbf{Z}_\beta) \\ &= \textstyle\sum_{\alpha=1}^{c} N_\alpha^{\frac{1}{2}} [\eta_\alpha(\mathbf{Z}_\alpha) + \sum_{r=1}^{c} (\rho_N^{(\alpha)} \rho_N^{(r)})^{\frac{1}{2}} \xi_{\alpha,r}(\mathbf{Z}_\alpha)] \\ &= \textstyle\sum_{\alpha=1}^{c} N_\alpha^{\frac{1}{2}} [N_\alpha^{-1} \sum_{r=1}^{N_\alpha} \{B(X_{\alpha r}) + C(Y_{\alpha r})\}] \end{aligned}$$

where $B(X_{\alpha r}) = \lambda_{1\alpha} B_{1\alpha}(X_{\alpha r}) + (\rho_N^{(\alpha)})^{\frac{1}{2}} \sum_{\nu=1}^{c} (\rho_N^{(\nu)})^{\frac{1}{2}} \lambda_{1\nu} C_{1\nu}(X_{\alpha r})$ and $C(Y_{\alpha r}) = \lambda_{2\alpha} B_{2\alpha}(Y_{\alpha r}) + (\rho_N^{(\alpha)})^{\frac{1}{2}} \sum_{\nu=1}^{c} (\rho_N^{(\nu)})^{\frac{1}{2}} \lambda_{2\nu} C_{2\nu}(Y_{\alpha r})$.

The right hand side of (3.23) represents c-summations. They involve independent samples of identically distributed random variables, and it can easily be shown by the use of c_r-inequality that each of random variables $B(X_{\alpha r}) + C(Y_{\alpha r})$ has finite absolute moment of order $2 + \delta'$, $0 < \delta' \leqq 1$. Hence by the central limit theorem [cf. Esseen (1965), page 43, Puri (1964), page 109], each sum properly normalized has the normal distribution in the limit with the result that the sum of c-summations will have the normal distribution in the limit. The proof follows. The computation of the covariance matrix is deferred to the appendix.

4. The limiting distribution of the proposed test under Pitman shift alternatives. From this section onward, we shall be concerned with a sequence of admissable alternative hypotheses H_N, which specify that for each $\alpha = 1, \cdots, c$, $\Pi_\alpha(\mathbf{z}) = \Pi(\mathbf{z} + \boldsymbol{\mu}_\alpha N^{-\frac{1}{2}})$, where $\mathbf{z} = (x, y)$, $\boldsymbol{\mu}_\alpha = (\mu_{1\alpha}, \mu_{2\alpha})$, $\Pi(\mathbf{z})$ is a fixed absolutely continuous bivariate cdf diagonally symmetric about 0 and for some pair (α, β), $\boldsymbol{\mu}_\alpha \neq \boldsymbol{\mu}_\beta$. Note that with this representation $F_1^{(\alpha)}(x) = F_1(x + \mu_{1\alpha}N^{-\frac{1}{2}})$, $F_2^{(\alpha)}(y) = F_2(y + \mu_{2\alpha}N^{-\frac{1}{2}})$, and $F_j(x) + F_j(-x) = 1, j = 1, 2$. The following theorem (the proof of which follows from Theorem 3.1) is used in deriving the limiting distribution of the statistic $\mathcal{L}_N$ and studying its efficiency properties.

THEOREM 4.1. *If* (i) $\rho_N^{(\alpha)} \to \rho_\alpha$ *as* $N \to \infty$ *and* $0 < \rho_\alpha < 1$, $\alpha = 1, \cdots, c$. (ii) *The conditions of Theorem* 4.1 *are satisfied.* (iii) *For each fixed* N, *the hypothesis* H_N *is true. Then, the random variables* $[N_\alpha^{\frac{1}{2}}(T_{N1(\alpha)} - \mu_{N1(\alpha)}, T_{N2(\alpha)} - \mu_{N2(\alpha)}),$ $\alpha = 1, \cdots, c]$ *have a limiting multivariate normal distribution as* $N \to \infty$ *with means zero and covariance matrix* $\boldsymbol{\tau} = (\boldsymbol{\tau}_{ij,\alpha\beta})$, $i, j = 1, 2; \alpha, \beta = 1, \cdots, c$ *where*

$$\tau_{jj,\alpha\beta} = \delta_{\alpha\beta}A_j^2, \qquad j = 1, 2; \alpha, \beta = 1, \cdots, c \tag{4.1}$$

$$\tau_{12,\alpha\beta} = \tau_{21,\alpha\beta} = \delta_{\alpha\beta}\mathcal{S}_{12}, \tag{4.2}$$

$$A_j^2 = \int_0^1 J_j^2(u)\,du = \int_0^1 [J_j^*(u)]^2\,du, \qquad j = 1, 2 \tag{4.3}$$

$$\mathcal{S}_{12} = \int_{x=-\infty}^{+\infty}\int_{y=-\infty}^{+\infty} J_1^*(F_1(x))J_2^*(F_2(y))\,d\Pi(x, y), \tag{4.4}$$

where $J_j(u) = \psi_j^{-1}(u)$, $J_j^*(u) = \psi_j^{*-1}(u)$ *and* $\delta_{\alpha\beta}$ *is the Kronecker delta.*

PROOF. The asymptotic normality follows from Theorem 3.1. The only thing we have to establish is that $\lim_{N\to\infty}\sigma_{N,ij,\alpha\beta} = \tau_{ij,\alpha\beta}$, $i, j = 1, 2; \alpha, \beta = 1, \cdots, c$. To obtain this result, we note the following:

(a) $$J_j^*(u) + J_j^*(1 - u) = 0, \qquad J_j(2u - 1) = J_j^*(u);$$

(b) $$\lim_{N\to\infty} A_j^{(\alpha)}(x, y) = F_j(x)(1 - F_j(y)),$$

$$\lim_{N\to\infty} B_{jk}(x, y) = dJ_j^*(F_j(x))\,dJ_k^*(F_k(y)),$$

$$\lim_{N\to\infty} C_{jk}^{(\alpha)}(x, y) = 0, \qquad \lim_{N\to\infty} D_{jk}^{(\alpha)}(x, y) = 0,$$

$$\lim_{N\to\infty} U_{jk}(x, y) = J_j'(2F_j(x) - 1)J_k'(2F_k(y) - 1) = \tfrac{1}{4}J_j^{*\prime}(F_j(x)J_k^{*\prime}(F_k(y))),$$

(where $A_j^{(\alpha)}$, B_{jk}, $C_{jk}^{(\alpha)}$, $D_{jk}^{(\alpha)}$ and U_{jk} are defined in the Appendix), and

(c) the application of the Lebesgue dominated convergence theorems permits the interchange of the limit and integration sign.

Then from (6.15)

$$\begin{aligned}
\tau_{jj,\alpha\alpha} &= \lim_{N\to\infty} \sigma_{N,jj,\alpha\alpha} \\
&= 4 \int\int_{0<x<y<\infty} F_j(x)(1 - F_j(y)\, dJ_j^*(F_j(x))\, dJ_j^*(F_j(y)) \\
(4.5) \qquad &\quad + 4 \int\int_{0<x<y<\infty} (1 - F_j(x))(\mathrm{P} - F_j(y))\, dJ_j^*(F_j(x))\, dJ_j^*(F_j(y)) \\
&= 4 \int\int_{\frac{1}{2}<u<v<1} (1 - v)\, dJ_j^*(u)\, dJ_j^*(v) = \int_0^1 J_j^{*2}(u)\, du \\
&= \int_0^1 J_j^2(u)\, du.
\end{aligned}$$

From (6.16)

$$\begin{aligned}
\tau_{12,\alpha\alpha} &= \tau_{21,\alpha\alpha} = \lim_{N\to\infty} \sigma_{N,12,\alpha\alpha} \\
(4.6) \qquad &= \int_{x=0}^{\infty} \int_{y=0}^{\infty} [\{\Pi(x, y) - F_1(x)F_2(y)\} + \{\Pi(x, -y) - F_1(x)F_2(-y)\} \\
&\quad + \{\Pi(-x, y) - F_1(-x)F_2(y)\} \\
&\quad + \{\Pi(-x, -y) - F_1(-x)F_2(-y)\}]\, dF_1^{\alpha}(F_1(x)\, dJ_2^*(F_2(y)) \\
(4.7) \qquad &= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} [\Pi(x, y) - F_1(x)F_2(y)]\, dJ_1^*(F_1(x))\, dJ_2^*(F_2(y)).
\end{aligned}$$

We now establish the equivalence of (4.7) and $\mathcal{S}_{12}$.

Let (X, Y) be a random variable whose joint distribution is $\Pi(x, y)$ and whose marginal distributions are $F_1(x)$ and $F_2(y)$ respectively. Then since $E[c(x - X) - F_1(x)][c(y - Y) - F_2(y)] = \Pi(x, y) - F_1(x)F_2(y)$, we can write (4.7) as

$$\begin{aligned}
&= \mathrm{Cov}\, (\textstyle\int_{-\infty}^{\infty} [c(x - X) - F_1(x)\, dJ_1^*[F_1(x)], \\
&\qquad \textstyle\int_{-\infty}^{\infty} [c(y - Y) - F_2(y)]\, dJ_2^*(F_2(y)) \\
&= \mathrm{Cov}\, (\textstyle\int_X^{\infty} dJ_1^*[F_1(x)], \int_Y^{\infty} dJ_2^*(F_2(y))) \\
&= \mathrm{Cov}\, (J_1^*(F_1(x)), J_2^*(F_2(Y))) \text{ which is the same as } \mathcal{S}_{12} \text{ as defined in (4.4).}
\end{aligned}$$

The other expressions can be evaluated in a similar manner.

COROLLARY 4.1.1. *Suppose that the hypothesis H_0 is true. Then under the conditions of Theorem* 4.1, *the random variables* $[N_\alpha^{\frac{1}{2}}(T_{N1(\alpha)}, T_{N2(\alpha)}), \alpha = 1, \cdots, c]$ *have a limiting multivariate normal distribution with means zero and covariance matrix* $\boldsymbol{\tau} = (\tau_{ij,\alpha\beta}), i, j = 1, 2; \alpha, \beta = 1, \cdots, c$ *given by* (4.1) *and* (4.2).

We now revert back to our original notation. Then we have

COROLLARY 4.1.2. *Under the assumptions of Theorem* 4.1, *the random variables* $[N_{ij}^{\frac{1}{2}}(T_{N1(ij)} - \mu_{N1(ij)}, T_{N2(ij)} - \mu_{N2(ij)}, 1 \leqq i < j \leqq t]$ *where*

$$\mu_{Nr(ij)} = \int_{x=0}^{\infty} J_r(H_r(x))\, d[F_r^{(ij)}(x) + F_r^{(ij)}(-x)], \qquad (4.8)$$

$$r = 1, 2; 1 \leqq i < j \leqq t,$$

have, in the limit as $N \to \infty$, *the multivariate normal distribution with zero mean and covariance matrix* $\boldsymbol{\tau} = \tau_{kl(ij)(i'j')}), k, l = 1, 2; 1 \leqq i < j \leqq t; 1 \leqq i' < j' \leqq t;$

where,

$$\tau_{kk(ij)(i'j')} = \delta_{(ij)(i'j')} A_k^2, \qquad k = 1, 2 \tag{4.9}$$

$$\tau_{kl(ij)(i'j')} = \tau_{lk(ij)(i'j')} = \delta_{(ij)(i'j')} S_{12}, \qquad k \neq l$$

where A_k^2 *and* S_{12} *are given by* (4.3) *and* (4.4) *respectively and*

$$\begin{aligned} \delta_{(ij)(i'j')} &= +1 \quad \text{if } i = i', \quad j = j'; \\ &= -1 \quad \text{if } i = j', \quad j = i'; \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{4.10}$$

Now let

$$U_{N,i} = \textstyle\sum_{j=1, j\neq i}^{t} N_{ij}^{\frac{1}{2}} T_{N1(ij)}, \qquad V_{N,i} = \textstyle\sum_{j=1, j\neq i}^{t} N_{ij}^{\frac{1}{2}} T_{N2(ij)} \tag{4.11}$$

$$\mu_{Nk(i\cdot)} = \textstyle\sum_{j=1, j\neq i}^{t} N_{ij}^{\frac{1}{2}} \mu_{Nk(ij)}, \qquad k = 1, 2; i = 1, \cdots, t \tag{4.12}$$

$$\eta_{k(ij)} = \lim_{N\to\infty} N_{ij}^{\frac{1}{2}} \mu_{Nk(ij)}, \qquad \eta_{k(i\cdot)} = \textstyle\sum_{j=1, j\neq i}^{t} \eta_{k(ij)}, \qquad k = 1, 2; i = 1, \cdots, t \tag{4.13}$$

and assume that $\eta_{k(ij)}$ exists and is finite for $k = 1, 2$ and $1 \leqq i < j \leqq t$. Then we have the following.

THEOREM 4.2. *Under the assumptions of Theorem* 4.1, *the random vector* $(U_{N,1}, \cdots, U_{N,t}, V_{N,1}, \cdots, V_{N,t})$ *has asymptotically a multivariate normal distribution with mean vector* $(\eta_{1(1\cdot)}, \cdots, \eta_{1(t\cdot)}, \eta_{2(1\cdot)}, \cdots, \eta_{2}(t\cdot))$ and covariance matrix $\mathbf{M} = (M_{ij})$ given by

$$\begin{aligned} M_{ij} &= (t\delta_{ij} - 1)A_1^2 && \text{if } 1 \leqq i, j \leqq t; \\ &= (t\delta_{t+i,j} - 1)S_{12} && \text{if } 1 \leqq i \leqq t, t < j \leqq 2t; \\ &= (t\delta_{i,t+j} - 1)S_{12} && \text{if } t < i \leqq 2t, 1 \leqq j \leqq t; \\ &= (t\delta_{ij} - 1)A_2^2 && \text{if } t < i, j \leqq 2t; \end{aligned} \tag{4.14}$$

and the rank of $\mathbf{M}$ *is* $2(t-1)$ *if and only if* $A_1^2A_2^2 - S_{12}^2 \neq 0$. The asymptotic normality follows directly from Theorem 4.1 and Corollary 4.1.2. The covariance terms are obtained by using

$$\begin{aligned} \tau_{kl(ij)(i'j')} &= A_k^2 && \text{if } (i, j) = (i', j'), k = l = 1, 2; \\ &= -A_k^2 && \text{if } (i, j) = (j', i'), k = l = 1, 2; \\ &= S_{12} && \text{if } (i, j) = (i', j'), k \neq l; \\ &= -S_{12} && \text{if } (i, j) = (j', i'), k \neq l; \\ &= 0 && \text{otherwise;} \end{aligned}$$

and routine computations.

To obtain the rank of the matrix $\mathbf{M}$, we proceed as follows. Denote

$$\mathbf{C} = (C_{ij}) = ((t\delta_{ij} - 1)), \qquad i, j = 1, \cdots, t. \tag{4.15}$$

Then we can rewrite $\mathbf{M}$ as

$$\mathbf{M} = \begin{pmatrix} A_1^2\mathbf{C} & \mathcal{S}_{12}\mathbf{C} \\ \mathcal{S}_{12}\mathbf{C} & A_2^2\mathbf{C} \end{pmatrix}.$$

Let $\|\mathbf{M}\|$ = Determinant $\mathbf{M}$. In $\|\mathbf{M}\|$, subtracting row 1 from each of the rows number $2, \cdots, t$ and the $(t+1)$st row from each of the rows number $t+2, \cdots 2t$ leaves $\|\mathbf{M}\|$ unchanged. Then add columns $2, \cdots, t$ to column 1 and column $t+2, \cdots, 2t$ to column $(t+1)$. The resulting determinant has columns 1 and $t+1$ all zeros. Hence $\|\mathbf{M}\| = 0$ and since it has two columns of zero, we have that the rank of $\mathbf{M}$ is at most $2(t-1)$.

Striking the 1st and $(t+1)$st rows and columns of the new determinant leaves the minor

$$\|\mathbf{M}^*\| = \begin{Vmatrix} tA_1^2\mathbf{I} & t\mathcal{S}_{12}\mathbf{I} \\ t\mathcal{S}_{12}\mathbf{I} & tA_2^2\mathbf{I} \end{Vmatrix}$$

where $\mathbf{I}$ is $(t-1) \times (t-1)$ identity matrix. Since $A_1^2 \neq 0$ by assumption, we may rewrite $\|\mathbf{M}\|^*$ as

$$\|\mathbf{M}^*\| = \begin{Vmatrix} tA_1^2\mathbf{I} & \mathbf{0} \\ t\mathcal{S}_{12}\mathbf{I} & t(A_2^2 - \mathcal{S}_{12}^2/A_1^2)\mathbf{I} \end{Vmatrix} = \prod_{i=1}^{t-1} t^2(A_1^2A_2^2 - \mathcal{S}_{12}^2)$$
$$= t^{2(t-1)}(A_1^2A_2^2 - \mathcal{S}_{12}^2)^{t-1}$$

and so $\|\mathbf{M}^*\| = 0$ if and only if $A_1^2A_2^2 - \mathcal{S}_{12}^2 = 0$.

Remark. We have proved above that the rank of the matrix $\mathbf{M}$ is $2(t-1)$ if and only if the matrix

$$(4.16) \qquad A = \begin{pmatrix} A_1^2 & \mathcal{S}_{12} \\ \mathcal{S}_{12} & A_2^2 \end{pmatrix}$$

is non-singular, that is, the dispersion matrix of $[J_1^*(F_1(X)), J_2^*(F_2(Y))]$ is non-singular. In what follows we make the assumption that the distribution function $\Pi(x, y)$ and the score functions J_1 and J_2 are such that the moment matrix $\mathbf{A}$ is non-singular. The moment matrix will be singular if and only if $J_1^*(F_1(X)) = aJ_2^*(F_2(Y)) + b$ a.s. Π.

Let us now define

$$(4.17) \qquad \mathcal{L}_N^* = t^{-1} \sum_{i=1}^{t} (U_{Ni}, V_{Ni})\mathbf{A}^{-1}(U_{Ni}, V_{Ni})'$$

and find the asymptotic distribution of $\mathcal{L}_N^*$. We first state the following theorem due to Sverdrup (1952).

Theorem 4.3 [Sverdrup]. *Let* $\mathbf{X}^{(n)} = (X_1^{(n)} \cdots X_p^{(n)})$, $n = 1, 2, \cdots$ *be an infinite sequence of random vectors and* $g(x_1 \cdots x_p)$ *be a real-valued continuous function for all* $x_1 \cdots x_p$. *Assume that the limit of the probability distribution of the random vector* $\mathbf{X}^{(n)}$ *is the probability distribution of the random vector* $\mathbf{X} = (X_1 \cdots$

X_p). *Then the limit of the probability distribution of* $g(X_1^{(n)} \cdots X_p^{(n)})$ *is the probability distribution of* $g(X_1 \cdots X_p)$.

We may now establish

THEOREM 4.4. *Under the assumptions of Theorem* 4.2, $\mathcal{L}_N^* = t^{-1} \sum_{i=1}^t (U_{N,i}, V_{N,i})\mathbf{A}^{-1}(U_{N,i}, V_{N,i})'$ *has asymptotically a non-central* χ^2*-distribution with* $2(t-1)$ *degrees of freedom and noncentrality parameter*

$$\Delta_{\mathcal{L}}^2 = t^{-1} \sum_{i=1}^t (\eta_{1(i\cdot)}, \eta_{2(i\cdot)})\mathbf{A}^{-1}(\eta_{1(i\cdot)}, \eta_{2(i\cdot)})'. \tag{4.18}$$

PROOF. Let $(\xi_1, \cdots, \xi_t, \xi_{t+1}, \cdots, \xi_{2t}) = \boldsymbol{\xi}$ be the normal vector to which $(U_{N,1}, \cdots, U_{N,t}, V_{N,1}, \cdots, V_{N,t})$ converges in law. Then $\boldsymbol{\xi}$ has a normal distribution with mean

$$\mathbf{n}. = (\eta_{1(1\cdot)}, \cdots, \eta_{1(t\cdot)}, \eta_{2(1\cdot)}, \cdots, \eta_{2(t\cdot)}) \tag{4.19}$$

and covariance matrix $\mathbf{M}$ defined by (4.14).

Let

$$\mathbf{A}^{-1} = ((a^{ij})), \qquad i, j = 1, 2, \tag{4.20}$$

and we may rewrite $\mathcal{L}_N^*$ as

$$\begin{aligned}\mathcal{L}_N^* &= t^{-1} \sum_{i=1}^t [a^{11}U_{N,i}^2 + 2a^{12}U_{N,i}V_{N,i} + a^{22}V_{N,i}^2] \\ &= t^{-1}[a^{11}U_N U_N' + a^{12}U_N V_N' + a^{22}V_N V_N' + a^{12}V_N U_N']\end{aligned}$$

where

$$\mathbf{U}_N = (U_{N,1}, \cdots, U_{N,t}), \qquad \mathbf{V}_N = (V_{N,1}, \cdots, V_{N,t}). \tag{4.21}$$

Letting

$$\boldsymbol{\Gamma} = t^{-1}\begin{pmatrix} a^{11}\mathbf{I} & a^{12}\mathbf{I} \\ a^{12}\mathbf{I} & a^{22}\mathbf{I}\end{pmatrix}, \quad \mathbf{I} = t \times t \text{ identity matrix}, \tag{4.22}$$

we have

$$\mathcal{L}_N^* = (\mathbf{U}_N, \mathbf{V}_N)\boldsymbol{\Gamma}(\mathbf{U}_N, \mathbf{V}_N)'. \tag{4.23}$$

By virtue of Theorem 4.3, $\mathcal{L}_N^*$ has asymptotically the same distribution as that of $\mathcal{L}_N^{**} = \boldsymbol{\xi}\boldsymbol{\Gamma}\boldsymbol{\xi}'$. Next, it is easy to check after routine computations that (i) $\mathbf{M}(\boldsymbol{\Gamma}\mathbf{M}\boldsymbol{\Gamma} - \boldsymbol{\Gamma})\mathbf{M} = \mathbf{0}$ and (ii) $2(t-1) = \text{trace } (\mathbf{M}\boldsymbol{\Gamma})$. The result now follows as an application of the well-known property (cf. Rao (1965), page 443 (viii)) of the multivariate normal distribution.

COROLLARY 4.4. *Suppose that the hypothesis* H_0 *is true. Then under the assumptions of Theorem* 4.2, $\mathcal{L}_N^*$ *has limiting central* χ^2*-distribution with* $2(t-1)$ *degrees of freedom.*

Now, let $\hat{\mathbf{A}}$ be a consistent estimator of $\mathbf{A}$, then it follows that $\mathcal{L}_N - \mathcal{L}_N^*$ converges to zero in probability as $N \to \infty$. Hence $\mathcal{L}_N$, too, has the limiting central χ^2-distribution with $2(t-1)$ degrees of freedom and so the critical function

$$\phi(\mathcal{L}_N) = 1 \quad \text{if } \mathcal{L}_N \geqq \chi^2_{2(t-1),\alpha}, \qquad = 0 \quad \text{if } \mathcal{L}_N < \chi^2_{2(t-1),\alpha}, \tag{4.24}$$

where $\chi^2_{r,\alpha}$ is the $(1 - \alpha)$ percent point of the chi-square distribution with r degrees of freedom provides an asymptotically level α test of H_0.

From Theorem 4.4, it is clear that any consistent estimator of $\mathbf{A}^{-1}$ will preserve the asymptotic distribution of the test statistic. In what follows we propose one such estimator. Looking at the matrix $\mathbf{A}$ defined by (4.16), we find that we require only to obtain a consistent estimator of $\mathcal{S}_{12}$ defined in (4.4) and this is done as follows.

Define

$$H_N(x, y) = \Pi_N(x, y) + \Pi_N(-x-, y) + \Pi_N(x, -y-) + \Pi_N(-x-, -y-) \tag{4.25}$$

and

$$H(x, y) = \Pi(x, y) + \Pi(-x, y) + \Pi(x, -y) + \Pi(-x, -y), \tag{4.26}$$

and assume that

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} [J_{N1}(N(N + 1)^{-1}H_{1,N}(x))J_{N2}(N(N + 1)^{-1}H_{2,N}(y)) - J_1(N(N+1)^{-1}H_{1,N}(x))J_2(N(N+1)^{-1}H_{2,N}(y))\, dH_N(x, y) = o_p(1), \tag{4.27}$$

where $H_{j,N}(x)$ is defined by (3.1).

Consider the statistic

$$\hat{\mathcal{S}}_{12,N} = N^{-1} \sum_{\alpha=1}^{c} \sum_{r=1}^{N_\alpha} E_{N1,R(X_{\alpha r})}E_{N2,R(Y_{\alpha r})} \tag{4.28}$$

where $R(X_{\alpha r})$ and $R(Y_{\alpha r})$ are the ranks of $X_{\alpha r}$ and $Y_{\alpha r}$ among $(X_{\alpha r}, r = 1, \cdots, N_\alpha, \alpha = 1, \cdots, c)$ and $(Y_{\alpha r}, r = 1, \cdots, N_\alpha, \alpha = 1, \cdots, c)$ respectively. Then

THEOREM 4.5. *$\hat{\mathcal{S}}_{12,N}$ is a translation invariant, consistent estimator of $\mathcal{S}_{12}$.*

PROOF. Since the ranks of the observations themselves are invariant under change of origin, it follows that $\hat{\mathcal{S}}_{12,N}$ remains translation invariant. Now, writing $\hat{\mathcal{S}}_{12,N}$ as

$$\hat{\mathcal{S}}_{12,N} = \int_{x=0}^{\infty} \int_{y=0}^{\infty} J_{N1}(N(N + 1)^{-1}H_{1,N}(x)) \cdot J_{N2}(N(N + 1)^{-1}H_{2,N}(y))\, dH_N(x, y) \tag{4.29}$$

and proceeding as in Theorem 3.1 of Sen and Puri (1967) it follows that $\hat{\mathcal{S}}_{12,N}$ converges in probability to

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} J_1(H_1(x))J_2(H_2(y))\, dH(x, y)$$

which equals $\mathcal{S}_{12}$. The proof follows.

In most cases, the quantities $\eta_{j,\alpha} = \lim_{N\to\infty} N_\alpha^{\frac{1}{2}}\mu_{Nj(\alpha)}$ take on simple forms through the help of the following lemma similar to Lemma 7.2 of Puri (1964).

LEMMA 4.1. *If* (i) $F_j(x), j = 1, 2$ *is continuous* cdf *differentiable in each of the* *open intervals* $(-\infty, a_1^{(j)})$, $(a_i^{(j)}, a_2^{(j)})$, $\cdots$, (a_{s-1}, a_s), (a_s, ∞), *and the derivative of* $F_j(x)$ *is bounded in each of these intervals,* (ii) *the function* $dJ_j(F_j(x) - F_j(-x))/dx$ *is bounded as* $x \to \pm\infty$ *and* (iii) $\psi_j^*(x)$ *is symmetric and unimodel with density* $\psi_j'^*(x)$, $j = 1, 2$, *then*

$$\begin{aligned} \eta_{j(\alpha)} &= \lim_{N\to\infty} N_\alpha^{\frac{1}{2}} \int_{x=0}^{\infty} \\ &\quad \cdot J[\textstyle\sum_{\beta=1}^{c} \rho_N^{(\beta)}\{F_j(x + \mu_{j\beta}N^{-\frac{1}{2}}) - F_j(-x + \mu_{j\beta}N^{-\frac{1}{2}})\}] \\ &\quad \cdot d[F_j(x + \mu_{j\alpha}N^{-\frac{1}{2}}) + F_j(-x + \mu_{j\alpha}N^{-\frac{1}{2}})] \\ &= -2\rho_\alpha^{\frac{1}{2}}\mu_{j\alpha} \int_{x=0}^{\infty} (d/dx)J_j[2F_j(x) - 1]\, dF_j(x). \end{aligned} \tag{4.30}$$

In case the conditions of Lemma 4.1 are satisfied, then

$$\begin{aligned} \Delta_{\mathcal{L}}^2 &= t^{-1} \textstyle\sum_{i=1}^{t} (a_1^*\vartheta_{1,i}, a_2^*\vartheta_{2,i})\mathbf{A}^{-1}(a_1^*\vartheta_{1,i}, a_2^*\vartheta_{2,i})' \\ &= t^{-1} \textstyle\sum_{i=1}^{t} (\vartheta_{1,i}, \vartheta_{2,i})\mathbf{A}^{-1*}(\vartheta_{1,i}, \vartheta_{2,i})' \end{aligned} \tag{4.31}$$

where

$$a_k^* = -2 \int_{x=0}^{\infty} (d/dx)J_k[F_k(x) - F_k(-x)]\, dF_k(x), \qquad k = 1, 2 \tag{4.32}$$

$$\vartheta_{k,i} = \textstyle\sum_{j=1, j\neq 1}^{t} \rho_{ij}^{\frac{1}{2}}\mu_{kij}, \qquad k = 1, 2;\ i = 1, \cdots, t \tag{4.33}$$

$$A^* = \begin{pmatrix} A_1^2/(a_1^*)^2 & \mathcal{S}_{12}/a_1^*a_2^* \\ \mathcal{S}_{12}/a_1^*a_2^* & A_2^2/(a_2^*)^2 \end{pmatrix} \tag{4.34}$$

and A_j^2, $j = 1, 2$ and $\mathcal{S}_{12}$ are defined in (4.3) and (4.4) respectively.

Special cases. (a) Let J_j be the inverse of the chi distribution with one degree of freedom. Then the $\mathcal{L}_N$ test reduces to the bivariate paired comparison normal scores, $\mathcal{L}_N(\Phi)$ test. [For $t = 2$, this reduces to the one sample bivariate normal scores test [15].] In this case the noncentrality parameter (4.28) reduces to

$$\Delta_{\mathcal{L}(\Phi)} = t^{-1} \textstyle\sum_{i=1}^{t} (1 - \rho_\Phi^2)^{-1}[\vartheta_{1,i}^2 a_\Phi^2 - 2\vartheta_{1,i}\vartheta_{2,i}\rho_\Phi a_\Phi b_\Phi + \vartheta_{2,i}^2 b_\Phi^2] \tag{4.35}$$

where

$$\rho_\Phi = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} \Phi^{-1}(F_1(x))\Phi^{-1}(F_2(y))\, d\Pi(x, y) \tag{4.36}$$

and

$$\begin{aligned} a_\Phi &= \int_{x=-\infty}^{+\infty} \{\phi[\Phi^{-1}(F_1(x))]\}^{-1} f_1^2(x)\, dx, \\ b_\Phi &= \int_{x=-\infty}^{+\infty} \{\phi[\Phi^{-1}(F_2(x))]\}^{-1} f_2^2(x)\, dx \qquad \phi(x) = \Phi'(x). \end{aligned} \tag{4.37}$$

(b) Let $J_j^*(x) = 2x - 1$, then the $\mathcal{L}_N$ test reduces to the bivariate paired comparison rank sum test. [For $t = 2$ this reduces to the one–sample bivariate rank sum test [13].] In this case the non–centrality parameter (4.28) becomes

$$\Delta_{\mathcal{L}(R)} = t^{-1} \textstyle\sum_{i=1}^{t} 12(1 - \rho_R^2)^{-1}[\vartheta_{1,i}^2 a_R^2 - 2\rho_R\vartheta_{1,i}\vartheta_{2,i}a_R b_R + \vartheta_{2,i}^2 b_R^2] \tag{4.38}$$

where

$$\rho_R = 3 \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} [2F_1(x) - 1][2F_2(y) - 1]\, d\Pi(x, y) \tag{4.39}$$

(4.40) $a_R = \int_{-\infty}^{+\infty} f_1^2(x)\,dx, \qquad b_R = \int_{-\infty}^{+\infty} f_2^2(x)\,dx.$

5. Asymptotic relative efficiency (A.R.E.). In this section we discuss briefly the A.R.E. of the $\mathfrak{L}_N(\Phi)$ and $\mathfrak{L}_N(R)$ tests with respect to the Sen-David D_N test and the analysis of variance $\mathfrak{F}$-test. From [14], the Sen-David D_N test and the analysis variance $\mathfrak{F}$-test have asymptotically the non-central chi square distributions with $2(t-1)$ degrees of freedom and the non-centrality parameter Δ_D and $\Delta_{\mathfrak{F}}$ respectively, where

(5.1) $$\Delta_D = 4[t(1-\theta^2)]^{-1} \sum_{i=1}^{t} [\vartheta_{1,i}^2 f_1^2(0) - 2\theta\vartheta_{1,i}\vartheta_{2,i} f_1(0) f_2(0) + \vartheta_{2,i}^2 f_2^2(0)]$$

and

(5.2) $$\Delta_{\mathfrak{F}} = 1[t(1-\rho^2)]^{-1} \sum_{i=1}^{t} [(\vartheta_{1,i}/\sigma_1)^2 - 2\rho\vartheta_{1,i}\vartheta_{2,i}/(\sigma_1\sigma_2) + (\vartheta_{2,i}/\sigma_2)^2]$$

where $\sigma_j^2 = \text{Var}(F_j), j = 1, 2$; and $\rho = \text{Corr}(F_1, F_2)$. Hence, using a theorem of Hanan (1956), the Pitman efficiency of the $\mathfrak{L}_N(\Phi)$ relative to the $\mathfrak{L}_{N(R)}$ test, the Sen-David D_N test and the $\mathfrak{F}$-test are

(5.3) $$e_{\mathfrak{L}_{N(\Phi)},\mathfrak{L}_{N(R)}} = \Delta_{\mathfrak{L}(\Phi)}/\Delta_{\mathfrak{L}(R)}, \qquad e_{\mathfrak{L}_{N(\Phi)},D} = \Delta_{\mathfrak{L}(\Phi)}/\Delta_D, \qquad e_{\mathfrak{L}_{N(\Phi)},\mathfrak{F}} = \Delta_{\mathfrak{L}(\Phi)}/\Delta_{\mathfrak{F}},$$

when $\Delta_{\mathfrak{L}(\Phi)}$, $\Delta_{\mathfrak{L}(R)}$, Δ_D and $\Delta_{\mathfrak{F}}$ are given by (4.35), (4.38), (5.1) and (5.2) respectively.

The above efficiencies depend upon not only the underlying distribution function $\Pi(x, y)$ but also on the parameters $\vartheta_{1,i}, \vartheta_{2,i}, i = 1, \cdots, t$ and t, the number of treatments. Thus, unlike the univariate situation where one usually arrives at a simple numerical measure of the asymptotic (Pitman) relative efficiency, the multivariate case offers substantial complications. However, in some special cases, useful information about the relative performance of the test procedures may be obtained.

Case 1. *Bivariate normal case.* Let us assume that the underlying distribution function $\Pi(x, y)$ is non-singular bivariate normal with mean vector zero and covariance matrix $\Sigma = (\rho\sigma_1\sigma_2)$. In such a case, proceeding as in Sen and Puri (1967) and, Chatterjee and Sen (1964), it can easily be shown that

$$e_{\mathfrak{L}_{N(\Phi)},\mathfrak{F}} = 1; \qquad 0.87 \leqq e_{\mathfrak{L}_{N(R)},\mathfrak{F}} \leqq 0.96; \qquad 1 \leqq e_{\mathfrak{L}_{N(\Phi)},\mathfrak{L}_{N(R)}} \leqq 1.15;$$
$$0 \leqq e_{D,\mathfrak{L}_{N(R)}} \leqq 0.73.$$

These results indicate that when the underlying distribution is bivariate normal, the $\mathfrak{L}_N(R)$ test is asymptotically more efficient than the Sen-David test; the $\mathfrak{L}_N(\Phi)$ test is asymptotically as efficient as the $\mathfrak{F}$-test, and more efficient than any other test.

Case 2. *Independent coordinates.* Let $\Pi(x, y)$ have independent coordinates. Then $\rho_\Phi = \rho_R = \rho = \theta = 0$. In such a case the results are the same as in corresponding univariate theory. For example, in such a case

$$\inf_{\Pi\varepsilon\mathfrak{F}_0} \inf_\vartheta e_{\mathfrak{L}_{N(\Phi)},\mathfrak{F}} = 1, \qquad \inf_{\Pi\varepsilon\mathfrak{F}_0} \inf_\vartheta e_{D,\mathfrak{F}} = \tfrac{1}{3},$$
$$\inf_{\Pi\varepsilon\mathfrak{F}_0} \inf_\vartheta e_{\mathfrak{L}_{N(\Phi)},\mathfrak{L}_{N(R)}} = \tfrac{1}{6}\pi, \qquad \inf_{\Pi\varepsilon\mathfrak{F}_0} \inf_\vartheta e_{\mathfrak{L}_{N(R)},\mathfrak{F}} = 0.864$$

where $\mathfrak{F}_0$ is the class of all absolutely continuous diagonally symmetric unimodal distributions.

The case when $\Pi(x, y)$ has identical marginals can similarly be dealt with.

6. Appendix: The dispersion matrix of $[N_\alpha^{\frac{1}{2}}(B_{1N,1}^{(\alpha)} + B_{2N,1}^{(\alpha)}, B_{1N,2}^{(\alpha)} + B_{2N,2}^{(\alpha)})$, $\alpha = 1, \cdots, c]$. Denote

$$A_j^{(\alpha)}(x, y) = F_j^{(\alpha)}(x)[1 - F_j^{(\alpha)}(y)]; \tag{6.1}$$

$$B_{jk}(x, y) = dJ_j(H_j(x))\, dJ_k(H_k(y)), \qquad j, k = 1, 2$$

$$C_{jk}^{(\alpha)}(x, y) = d[F_j^{(\alpha)} + F_j^{(\alpha)}(-x)]\, d[F_k^{(\alpha)}(y) + F_k^{(\alpha)}(-y)], \quad j, k = 1, 2 \tag{6.2}$$

$$D_{j,k}^{(\alpha)}(x, y) = dH_j(x)\, d[F_k^{(\alpha)}(y) + F_k^{(\alpha)}(-y)], \qquad j, k = 1, 2 \tag{6.3}$$

$$U_{jk}(x, y) = J_j'[H_j(x)]J_k'[H_k(y)], \qquad j, k = 1, 2 \tag{6.4}$$

$$E_\alpha(x, y) = [\Pi_\alpha(x, y) - F_1^{(\alpha)}(x)F_2^{(\alpha)}(y)]. \tag{6.5}$$

First we compute the variance of $B_{1N,j}^{(\alpha)} + B_{2N,j}^{(\alpha)}$, $j = 1, 2$. Integrating the right hand side of (3.6) by parts, we obtain

$$B_{1N,j}^{(\alpha)} = D_{1N,j}^{(\alpha)} + D_{2N,j}^{(\alpha)} \tag{6.6}$$

where

$$D_{1N,j}^{(\alpha)} = -\int_{x=0}^{\infty} [F_{j,N}^{(\alpha)}(x) - F_j^{(\alpha)}(x)]J_j'(H_j(x))\, dH_j(x) \tag{6.7}$$

and

$$D_{2N,j}^{(\alpha)} = -\int_{x=0}^{\infty} [F_{j,N}^{(\alpha)}(-x-) - F_j^{(\alpha)}(-x)]J_j'(H_j(x))\, dH_j(x). \tag{6.8}$$

Since $E(B_{1N,j}^{(\alpha)} + B_{2N,j}^{(\alpha)}) = 0$ we obtain

$$\begin{aligned} &\operatorname{Var}(B_{1N,j}^{(\alpha)} + B_{2N,j}^{(\alpha)}) \\ &= E[D_{1N,j}^{(\alpha)} + D_{2N,j}^{(\alpha)} + B_{2N,j}^{(\alpha)}]^2 \\ &= E(D_{1N,j}^{(\alpha)})^2 + E(D_{2N,j}^{(\alpha)})^2 + E[B_{2N,j}^{(\alpha)}]^2 + 2E[D_{1N,j}^{(\alpha)}D_{2N,j}^{(\alpha)}] + 2E[D_{1N,j}^{(\alpha)}B_{2N,j}^{(\alpha)}] \\ &\quad + 2E[D_{2N,j}^{(\alpha)}B_{2N,j}^{(\alpha)}]. \end{aligned} \tag{6.9}$$

Now

$$\begin{aligned} E[D_{1N,j}^{(\alpha)}]^2 &= E\{\textstyle\int_{x=0}^{\infty}\int_{y=0}^{\infty} [F_{j,N}^{(\alpha)}(x) - F_j^{(\alpha)}(x)][F_{j,N}^{(\alpha)}(y) - F_j^{(\alpha)}(y)] \\ &\qquad \cdot J_j'[H_j(x)]J_j'(H_j(y))\, dH_j(x)\, dH_j(y)\} \\ &= 2N_\alpha^{-1} \textstyle\int_{0<x<y<\infty}\int A_j^{(\alpha)}(x, y)B_{jj}(x, y). \end{aligned} \tag{6.10}$$

Note that the application of Fubini's Theorem permits the interchange of integral and expectation. Similarly,

$$\begin{aligned} &E(D_{2N,j}^{(\alpha)})^2 \\ &= 2N_\alpha^{-1} \textstyle\int_{0<x<y<\infty}\int A_j^{(\alpha)}(-y, -x)B_{jj}(x, y), \end{aligned} \tag{6.11}$$

$$E(B_{2N,j}^{(\alpha)})^2 = 2N^{-1}\sum_{r=1}^{c}\rho_N^{(r)}[\int_{0<x<y<\infty}\int\{A_j^{(r)}(x,y)+A_j^{(r)}(-y,-x)\} \cdot U_{jj}(x,y)D_{jj}^{(\alpha)}(x,y)+\int_{x=0}^{\infty}\int_{y=0}^{\infty}A_j^{(r)}(-y,x)U_{jj}(x,y) \cdot D_{jj}^{(\alpha)}(x,y)], \tag{6.12}$$

$$E(D_{1N,j}^{(\alpha)}D_{2N,j}^{(\alpha)}) = N_\alpha^{-1}\int_{x=0}^{\infty}\int_{y=0}^{\infty}A_j^{(\alpha)}(-y,x)B_{jj}(x,y), \tag{6.13}$$

$$\begin{aligned} E(D_{1N,j}^{(\alpha)}B_{2N,j}^{(\alpha)} + D_{2N,j}^{(\alpha)}B_{2N,j}^{(\alpha)}) &= -N^{-1}[\int_{0<x<y<\infty}\int\{A_j^{(\alpha)}(x,y)-A_j^{(\alpha)}(-y,-x)\}U_{jj}(x,y) \\ &\cdot D_{jj}^{(\alpha)}(x,y)+\int_{0<y<x<\infty}\int\{A_j^{(\alpha)}(y,x)-A_j^{(\alpha)}(-x,-y)\} \\ &\cdot U_{jj}(x,y)D_{jj}^{(\alpha)}(x,y)-\int_{x=0}^{\infty}\int_{y=0}^{\infty}\{A_j^{(\alpha)}(-y,-x) \\ &- A_j^{(\alpha)}(-x,y)\}U_{jj}(x,y)D_{jj}^{(\alpha)}(x,y)]. \end{aligned} \tag{6.14}$$

Hence using (6.10) to (6.14), we obtain

$$\begin{aligned} \sigma_{N,jj,\alpha\alpha} &= \mathrm{Var}\,[N_\alpha^{\frac12}(B_{1N,j}^{(\alpha)}+B_{2N,j}^{(\alpha)})] \\ &= 2[\int\int_{0<x<y<\infty}\{A_j^{(\alpha)}(x,y)+A_j^{(\alpha)}(-y,-x)\}B_{jj}(x,y) \\ &+ \int_{x=0}^{\infty}\int_{y=0}^{\infty}A_j^{(\alpha)}(-y,x)B_{jj}(x,y)+\rho_N^{(\alpha)}\sum_{r=1}^{c}\rho_N^{(r)} \\ &\cdot[\int\int_{0<x<y<\infty}\{A_j^{(r)}(x,y)+A_j(-y,-x)\}U_{jj}(x,y)C_{jj}^{(\alpha)}(x,y) \\ &- \int_{x=0}^{\infty}\int_{y=0}^{\infty}A_j^{(r)}(-y,x)U_{jj}(x,y)C_{jj}^{(\alpha)}(x,y)] \\ &- \rho_N^{(\alpha)}[\int\int_{0<x<y<\infty}\{A_j^{(\alpha)}(x,y)+A_j^{(\alpha)}(-y,-x)\}U_{jj}(x,y)D_{jj}^{(\alpha)}(x,y) \\ &- \int\int_{0<y<x<\infty}\{A_j^{(\alpha)}(y,x)+A_j^{(\alpha)}(-x,-y)\}U_{jj}(x,y)D_{jj}^{(\alpha)}(x,y) \\ &+ \int_{x=0}^{\infty}\int_{y=0}^{\infty}\{A_j^{(\alpha)}(-y,x)+A_j^{(\alpha}(-x,y)\}U_{jj}(x,y)D_{jj}^{(\alpha)}(x,y)]]. \end{aligned} \tag{6.15}$$

Proceeding analogously, we obtain

$$\begin{aligned} \sigma_{N,12,\alpha\alpha} = \sigma_{N,21,\alpha\alpha} &= \mathrm{Cov}\,[N_\alpha^{\frac12}(B_{1N,1}^{(\alpha)}+B_{2N,1}^{(\alpha)}), N_\alpha^{\frac12}(B_{1N,2}^{(\alpha)}+B_{2N,2}^{(\alpha)})] \\ &= \int_{x=0}^{\infty}\int_{y=0}^{\infty}\{E_\alpha(x,y)+E_\alpha(x,-y)+E_\alpha(-x,y)+E_\alpha(-x,-y)\} \\ &\cdot B_{12}(x,y)-\rho_N^{(\alpha)}\int_{x=0}^{\infty}\int_{y=0}^{\infty}\{E_\alpha(x,y)-E_\alpha(x,-y) \\ &+ E_\alpha(-x,y)-E_\alpha(-x,-y)\}D_{12}^{(\alpha)}(x,y)-\rho_N^{(\alpha)}\int_{x=0}^{\infty}\int_{y=0}^{\infty} \\ &\cdot\{E_\alpha(x,y)+E_\alpha(x,-y)-E_\alpha(-x,y)-E_\alpha(-x,-y)\} \\ &\cdot D_{21}^{(\alpha)}(y,x)+\rho_N^{(\alpha)}\sum_{r=1}^{c}\rho_N^{(r)}\int_{x=0}^{\infty}\int_{y=0}^{\infty} \\ &\cdot\{E_r(x,y)-E_r(x,-y)-E_r(-x,y)+E_r(-x,-y)\} \\ &\cdot U_{12}(x,y)C_{12}^{(\alpha)}(x,y), \end{aligned} \tag{6.16}$$

$$\begin{aligned}
\sigma_{N,12,\alpha\beta} &= \operatorname{Cov}[N_\alpha^{\frac{1}{2}}(B_{1N,1}^{(\alpha)} + B_{2N,1}^{(\alpha)}), N_\beta^{\frac{1}{2}}(B_{1N,2}^{(\beta)} + B_{2N,2}^{(\beta)})] \qquad \alpha \neq \beta \\
&= -({\rho_N}^{(\alpha)}{\rho_N}^{(\beta)})^{\frac{1}{2}}[\textstyle\int_{x=0}^\infty \int_{y=0}^\infty \{E_\alpha(x,y) + E_\alpha(-x,y) - E_\alpha(x,-y) \\
&\quad - E_\alpha(-x,-y)\}U_{12}(x,y)D_{12}^{(\alpha)}(x,y) \\
&\quad + \textstyle\int_{x=0}^\infty \int_{y=0}^\infty \{E_\beta(x,y) + E_\beta(x,-y) - E_\beta(-x,y) \qquad (6.17)\\
&\quad - E_\beta(-x,-y)\}U_{12}(x,y)D_{21}^{(\alpha)}(y,x) \\
&\quad - \textstyle\sum_{r=1}^c {\rho_N}^{(r)} \int_{x=0}^\infty \int_{y=0}^\infty \{E_r(x,y) - E_r(x,-y) \\
&\quad - E_r(-x,y) + E_r(-x,-y)\}U_{12}(x,y)c_{12}^{(\alpha)}(x,y)],
\end{aligned}$$

$$\begin{aligned}
\sigma_{N,jj,\alpha\beta} &= \operatorname{Cov}[N_\alpha^{\frac{1}{2}}(B_{1N,j}^{(\alpha)} + B_{2N,j}^{(\alpha)}), N_\beta^{\frac{1}{2}}(B_{1N,j}^{(\beta)} + B_{2N,j}^{(\beta)})] \\
&= -({\rho_N}^{(\alpha)}{\rho_N}^{(\beta)})^{\frac{1}{2}}[\textstyle\iint_{0<x<y<\infty} \{{A_j}^{(\alpha)}(x,y) - {A_j}^{(\alpha)}(-y,-x)\} \\
&\quad \cdot U_{jj}(x,y)D_{jj}^{(\beta)}(x,y) \\
&\quad + \textstyle\iint_{0<y<x<\infty} \{{A_j}^{(\alpha)}(y,x) - {A_j}^{(\alpha)}(-x,-y)\}U_{jj}(x,y)D_{jj}^{(\beta)}(x,y) \\
&\quad - \textstyle\int_{x=0}^\infty \int_{y=0}^\infty \{{A_j}^{(\alpha)}(-y,x) - {A_j}^{(\alpha)}(-x,y)\}U_{jj}(x,y)D_{jj}^{(\beta)}(x,y) \qquad (6.18)\\
&\quad + \textstyle\iint_{0<x<y<\infty} \{{A_j}^{(\beta)}(x,y) - {A_j}^{(\beta)}(-y,-x)\}U_{jj}(x,y)D_{jj}^{(\alpha)}(y,x) \\
&\quad + \textstyle\iint_{0<y<x<\infty} \{{A_j}^{(\beta)}(y,x) - {A_j}^{(\beta)}(-x,-y)\}U_{jj}(x,y)D_{jj}^{(\alpha)}(y,x) \\
&\quad - \textstyle\int_{x=0}^\infty \int_{y=0}^\infty \{{A_j}^{(\beta)}(-x,y) - {A_j}^{(\beta)}(-y,x)\}U_{jj}(x,y)D_{jj}^{(\alpha)}(y,x) \qquad (6.18)\\
&\quad + \textstyle\sum_{r=1}^c {\rho_N}^{(r)}\{\iint_{0<x<y<\infty} {H_j}^{(r)}(x)(1 - {H_j}^{(r)}(y))U_{jj}(x,y) \\
&\quad \cdot d({F_j}^{(\alpha)}(x) + {F_j}^{(\alpha)}(-x))\,d({F_j}^{(\beta)}(y) + {F_j}^{(\beta)}(-y)) \\
&\quad + \textstyle\iint_{0<y<x<\infty} {H_j}^{(r)}(y)(1 - {H_j}^{(r)}(x))U_{jj}(x,y)\,d({F_j}^{(\alpha)}(x) \\
&\quad + {F_j}^{(\alpha)}(-x))\,d({F_j}^{(\beta)}(y) + {F_j}^{(\beta)}(-y))\}].
\end{aligned}$$

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REFERENCES

[1] BENNET, B. M. (1962). On multivariate sign tests. *J. Roy. Statist. Soc., Ser. B* **24** 155–161.

[2] CHATTERJEE, S. K. and SEN, P. K. (1964). Nonparametric tests for the bivariate two sample location problem. *Calcutta Statist. Assoc. Bull.* **13** 18–58.

[3] CHATTERJEE, S. K. and SEN, P. K. (1969). Nonparametric tests for the multivariate multisample location problem. To appear in *Essays in Probability and Statistics*, Univ. North Carolina Press.

[4] DAVID, H. A. (1963). *The Method of Paired Comparisons*. Charles Griffen, London.

[5] DAVIDSON, R. R. and BRADLEY, R. A. (1969). Multivariate paired comparisons: The extension of a univariate model and associated estimation and test procedures. *Biometrika* **56** 81–95.

[6] ESSEN, C. G. (1965). Fourier analysis of distribution functions. *Acta Math.* **77** 1–125.

[7] HANNAN, E. J. (1956). The asymptotic power of tests based upon multiple correlations. *J. Roy. Statist. Soc., Ser B* **18** 227–233.

[8] NOETHER, G. E. (1960). Remarks about a paired comparison model. *Psychometrika* **25** 357–367.
[9] PURI, M. L. (1964). Asymptotic efficiency of a class of c-sample tests. *Ann. Math. Statist.* **35** 102–121.
[10] PURI, M. L. and SEN, P. K. (1966). On a class of multivariate multisample rank order tests. *Sankhyā Ser. A* **28** 353–376.
[11] PURI, M. L. and SEN, P. K. (1967). On some optimum non-parametric procedures in two-way layouts. *J. Amer. Statist. Assoc.* **63** 1214–1229.
[12] PURI, M. L. and SEN, P. K. (1969). On the asymptotic normality of one sample rank order test statistics. *Teoria Veroyatnostey iee Primenenyia.* **14** 167–172.
[13] RAO, C. R. (1965). *Linear Statistical Inference and Its Applications.* Wiley, New York.
[14] SEN, P. K. and DAVID, H. A. (1968). Paired comparisons for paired characteristics. *Ann. Math. Statist.* **39** 200–208.
[15] SEN, P. K. and PURI, M. L. (1967). On the theory of rank order tests for location in the multivariate one sample problem. *Ann. Math. Statist.* **38** 1216–1228.
[16] SVERDRUP, E. (1952). The limit distribution of a continuous function of random variables. *Skand. Aktuarietidskr.* **35** 1–10.