The Annals of Mathematical Statistics
1970, Vol. 41, No. 2, 651–654

AN $M/G/\infty$ ESTIMATION PROBLEM[1]

BY MARK BROWN

Cornell University

0. Introduction. We shall consider the problem of estimating the cdf G in an $M/G/\infty$ queue, when the data are arrival and departure epochs without identification of customers. That is, the experimenter observes the epochs customers arrive and depart without keeping track of individual customers. From such data taken over a finite time interval he wishes to estimate the cdf G of the service time distribution. We shall define a sequence of estimators $\{G_n, n \geqq 1\}$ which depend on the data up to the nth output and then we shall show that

$$\Pr(\lim_{n\to\infty} \sup_x |G(x) - G_n(x)| \to 0) = 1. \tag{1}$$

The intensity λ of the Poisson arrival process, and the number of customers initially in service are not assumed to be known. However, we assume that the expected service time μ_G exists.

A consequence of (1) is that G is completely identifiable from a single sample path of the process $\{Q(t), t > 0\}$, where $Q(t)$ is the number of customers in service at time t. It is interesting to contrast this with an identifiability result of Kendall and T. Lewis [6]. They show that in a $GI/G/\infty$ queue if one is given the output epochs along with the serial number of the input corresponding to each output, then the input distribution is completely identifiable and the output distribution is identifiable up to a local parameter.

The problem of inferring properties of G from a finite sample path of the process $\{Q(t), t \geqq 0\}$ has received some attention in the literature. A brief survey of papers in this area and applications can be found in Cox [4] pages 302–303. The present paper appears to be the first in which G itself is estimated rather than individual quantities associated with G.

The work in this paper was motivated by problems of statistical estimation in low density Poisson traffic streams. Assume that vehicles enter a highway according to a Poisson process and choose distances to travel i.i.d. as D, and velocity patterns i.i.d. as $\{V(t), t \geqq 0\}$, the velocity patterns and distances being independent. The assumption of independence of velocity patterns ignores vehicular interactions. It is used in approximations to low density traffic, where interactions occur infrequently (see [3], [8]). Under the above assumptions the amounts of time spent by vehicles on the highway are i.i.d. random variables. We interpret the system as an $M/G/\infty$ queue in which each vehicle is a customer and the amount of time spent by a vehicle on the highway is its service time. If one wishes to estimate the service time cdf, our procedure would enable him to do so, without having to keep track of individual vehicles.

Received March 17, 1969.
[1] Research supported by the Department of Transportation, Federal Highway Administration Bureau of Public Roads under Contract No. FH–11–6913 with Cornell University.

1. Theory. Consider a Poisson process on the real line of intensity λ. The points of increase of the process will be referred to as input points. The output process is obtained by translating the input points by i.i.d. random variables with cdf G. It is assumed that $\mu_G = \int x\,dG(x) < \infty$. The output process is also a Poisson process of intensity λ, [5] page 404. Choose any output point and label it Y_0. Then define Y_n for $n = 0, \pm 1, \cdots$ as the nth output point to the right of Y_0. Define Z_n for $n = 0, \pm 1, \cdots$ as the distance from Y_n to the nearest input to the left of Y_n.

Our argument proceeds as follows. We show that the sequence $\{Z_i\}$ is stationary and ergodic. This enables us to estimate the cdf H of Z_i. Since H and G are related by a simple expression (Lemma 3) we can modify the estimator of H to obtain an estimator of G.

LEMMA 1. *The sequence $\{Z_n, n = 0, \pm 1, \cdots\}$ is stationary and ergodic.*

PROOF. Stationarity follows from the fact that a shift of the Z_n sequence is equivalent to shifting the origin to a new output point.

It is shown in [2] page 119 that every invariant event of a stationary sequence is a tail event. Therefore to prove ergodicity it is sufficient to prove that every tail event has probability 0 or 1. This will follow if we show that for all m, the tail σ-field of $\{Z_i\}$ is independent of $\{Z_i, i \leqq m\}$. Now for any m, with probability 1, there are a finite number of customers in service at time Y_m. This is assured since the finiteness of μ_G implies that the system attains equilibrium. Thus, the after-affects of inputs prior to Y_m is finite with probability 1. Also, inputs subsequent to Y_m and their service times are independent of the history prior to Y_m. Therefore the tail σ-field of $\{Z_i\}$ depends only on the inputs subsequent to Y_m and their service times. The tail σ-field is thus independent of $\{Z_i, i \leqq m\}$.

Let H be the cdf of Z_i and λ the intensity of the input process.

LEMMA 2. $G(x) = 1-(1-H(x))\,e^{\lambda x}$.

PROOF. The probability that $Z_i \leqq x$ given W_i, the distance from Y to the input which led to Y, and K, the number of input points in $(Y_i - W_i, Y_i)$ is given by:

$$H_{Z\,|\,W,K}^{(z\,|\,w,k)} = 1-(1-x/w)^k, \qquad x < w;$$
$$= 1, \qquad x \geqq w.$$

It follows that

$$H_{Z\,|\,W}^{(x\,|\,w)} = 1-e^{-\lambda x}, \qquad x < w;$$
$$= 1, \qquad x \geqq w;$$

and thus that $H(x) = 1-(1-G(x))\,e^{-\lambda x}$.

LEMMA 3. *If $\{F_n, n = 1, 2, \cdots\}$ is a sequence of random* cdf's *and F is a non-random* cdf *such that for all real x*

$$\Pr(F_n(x) \to F(x), \qquad F_n(x-0) \to F(x-0)) = 1$$

then F_n converges uniformly a.s. *to F.*

PROOF. The result follows by the same argument as found in the proof of the Glivenko–Cantelli theorem in [7] page 20.

Let $\{\hat{\lambda}_n, n = 1, 2, \cdots\}$ be a sequence of random variables converging a.s. to λ. For example $\hat{\lambda}_n$ may be the sample input intensity in $[Y_0, Y_n]$.
Define

$$\begin{aligned} A_i(x) &= 1, \qquad Z_i \leqq x; \\ &= 0, \qquad Z_i > x; \\ H_n(x) &= n^{-1} \textstyle\sum_0^{n-1} A_i(x); \\ V_n(x) &= 1-(1-H_n(x))\, e^{\hat{\lambda}_n x}; \\ G_n(x) &= \sup_{0 \leqq y \leqq x} V_n(y). \end{aligned}$$

LEMMA 4. *H_n converges uniformly* a.s. *to H.*

PROOF. Since the sequence $\{Z_i\}$ is stationary and ergodic, so is $\{A_i(x)\}$ [2] page 119. Therefore by the ergodic theorem, [2] page 118, $H_n(x)$ converges a.s. to $H(x)$ for all x. Similarly $H_n(x-0)$ converges a.s. to $H(x-0)$ for all x. The conclusion now follows from Lemma 3.

LEMMA 5. *The sequence of random functions $\{V_n\}$ converges uniformly on bounded intervals* a.s. *to G.*

PROOF. Easy consequence of the definitions of V_n and G together with Lemmas 3 and 4.

THEOREM 1. *$G_n \to G$ uniformly* a.s.

PROOF. Note that $G(x) = \sup_{0 \leqq y \leqq x} G(y)$. Lemma 5 implies that $P(G_n(x) \to G(x), G_n(x-0) \to G(x-0)) = 1$ for all x. For each n, G_n is a random cdf since G_n is monotone nondecreasing, $G_n(0) = 0$ and $G_n(y) = 1$ for $y \geqq \max(Z_0, \cdots, Z_{n-1})$. The proof is completed by applying Lemma 3.

2. Comments and additions.

(i) Although we have obtained an estimator for the cdf G, it is clearly not the best estimator in any sense because we do not use all the information. The problem of finding a best estimator (according to any criterion) is still open.

(ii) It would be of interest to obtain the asymptotic distribution of $\sup(G_n - G)$ suitably normalized, and weak convergence of the sequence of processes $\{G_n - G, n = 1, 2, \cdots\}$ suitably normalized. The author has been unable to verify the mixing conditions given by Billingsley [1] page 197. Were one to verify the mixing conditions there would still remain the difficulty of computing the covariance kernel of the limiting process.

(iii) It would also be interesting to estimate the service time cdf G in the $G \cdot I \cdot / G / \infty$ queue, or at least to prove identifiability of G from $\{Q(t), t > 0\}$. Some type of restriction on the input cdf would be necessary.

Acknowledgment. I am grateful to the referee. His suggestions led to several improvements in my paper.

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.
[2] BREIMAN, L. (1968). *Probability*. Addison–Wesley, Reading.
[3] BROWN, M. (1969). Some results on a traffic model of Rényi. *J. Appl. Prob.* **6** 293–301.
[4] COX, D. R. (1965). Some problems of statistical analysis connected with congestion. *Proc. Symp. on Congestion Theory*. Univ. of North Carolina Press.
[5] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.
[6] KENDALL, D. G. and LEWIS, T. (1965). On the structural information contained in the output of $GI/G/\infty$. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*. **4** 144–148.
[7] LOÈVE, M. (1963). *Probability Theory*. Van Nostrand, Princeton.
[8] RÉNYI, A. (1964). On two mathematical models of the traffic on a divided highway. *J. Appl. Probability* **1** 311–320.