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ABSTRACTS

(*Abstracts of papers presented at the European Regional meeting, Hanover, West Germany, August 19–26, 1970.*)

127-4. The distributions of the sample mean and sample variance for a random sample from a mixture of two normal distributions. JAVAD BEHBOODIAN, Pahlavi University.

Let $X_1, X_2, \cdots, X_n$ be a random sample from a mixture of two normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ with the mixing proportions p and $q = 1-p$. In this note we study the distributions of two statistics $\bar{X} = \sum_1^n X_i/n$ and $S^2 = \sum_1^n (X_i - \bar{X})^2/n$. It is shown that the distribution of $\bar{X}$ is a mixture of $n+1$ normal distributions, where the mixing proportions form a binomial probability function $B(n, p)$, and the distribution of nS^2/σ_1^2, for $\sigma_1^2 \leqq \sigma_2^2$, is a mixture of a sequence of chi-square distributions. In the particular case when $\sigma_1^2 = \sigma_2^2 = \sigma^2$ the distribution of nS^2/σ^2 is a mixture of a central chi-square distribution with $n-1$ degrees of freedom and a non-central chi-square distribution with $n-1$ degrees of freedom and non-centrality parameter $|\mu_1 - \mu_2|$, where the mixing proportions are $p^n + q^n$ and $1 - p^n - q^n$. (Received June 3, 1970.)

127-5. A sequential test for shift. PAUL SWITZER, Stanford University.

When two populations must be sampled at different times, we cannot test for shift using the standard SPRT based on paired observations. However, it is still meaningful to obtain the later sample sequentially. A test for shift with one sample-size fixed and the other sample-size random is proposed, some of its properties are developed under normal theory assumptions, and a comparison is made with the completely fixed sample-size procedure. (Received June 9, 1970.)

127-6. Test about the maximum parameter and the superiority of the best population (preliminary report). T. CACOULLOS, University of Athens.

Consider k ($k \geqq 2$) populations of the same parametric family of distributions. The distribution has specified form and involves an unknown scale parameter θ. Let $\theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]}$ be the ordered parameters of the k populations. Fixed-sample tests about $\theta_{[k]}$ are considered. In each case (e.g., normal means or variances) the test is based on the statistics (usually sufficient statistics for θ) used in the corresponding problem of selecting the best population (associated with $\theta_{[k]}$) in the well-known Bechhofer–Sobel formulation of the ranking problems. Under the same assumptions tests about the superiority $\delta = \theta_{[k]} - \theta_{[k-1]}$ (when θ is a location parameter) or $\lambda = \theta_{[k]}/\theta_{[k-1]}$ (when θ is a scale parameter) are also examined. The relevance of these tests to ranking procedures is obvious. (Received June 23, 1970.)

127-7. A class of estimates of location parameter after a preliminary test on regression. A. K. MD. EHANSES SALEH, Carleton University.

In a recent paper Hodges and Lehmann (1963, AMS) proposed a general method of obtaining robust point estimates of location parameter based on rank tests. Adichi (1967, AMS) extends this method to the re-regression model $F(y - \alpha - \beta x_j)$, $j = 1, 2, \cdots, n$ where x_j's satisfy Noether conditions (Hájek (1962)) and defines suitable estimates of α and β which possess excellent properties such as invariance, symmetry of distribution of the estimates asymptotic normality and Pitman's efficiency. In this paper we propose a class of estimates of α when the regression β is doubtful. Under this situation we test on the hypothesis $\beta = 0$ against $\beta > 0$ using Hájek's (1962) statistic and the estimates of α are based on the result of the test. The estimates are shown to be invariant under each condition of the hypothesis and possess nonsymmetric continuous distribution. Hence, the asymptotic "bias" and mean-square error expression of the estimates are obtained.

The estimates are then compared with the usual least squares estimates of α i.e. $\bar{y}-\hat{\beta}_{L-s}\bar{x}$ and the result is studied. As a special case, the two-sample problem of estimating location parameters after a preliminary test on shift parameter is studied based on Wilcoxon signed rank test. (Received June 25, 1970.)

127-8. The occurrence of a renewal process in the theory of numbers. JANOS GALAMBOS, University of Ibadan.

Let $F_1, F_2, \cdots$ be the Fibonacci numbers defined by $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1}+F_{n-2}$ for $n \geqq 3$. It is known that all the integers $N \geqq 1$ can be uniquely represented as $N = F_{i_1}+F_{i_2}+\cdots+F_{i_d}$ with $i_j - i_{j-1} \geqq 2$, $j \geqq 3$. The number d is obviously a function $d(N)$, and similarly $i_j = i_j(N)$. Defining the probability space $(\Omega_n, \mathscr{A}_n, P_n)$ as $\Omega_n = \{F_n, F_n+1, \cdots, F_{n+1}-1\}$ with P_n generating the uniform distribution on the set $\mathscr{A}_n$ of all subsets of Ω_n, $i_j(N)$ and $d(N)$ are random variables. The evaluation of the joint distribution of $i_j(N)$ and $d(N)$ is a combinatorial problem and it turns out that the random variables $m_j = m_j(N) = i_j(N) - i_{j-1}(N)$, $j \geqq 1$ with $i_0(N) = 0$, are asymptotically independent and indentically distributed as $n \to +\infty$, with $\lim P_n(m_j(N) = k) = [\frac{1}{2}(1+5^{\frac{1}{2}})]^{-k}$. Since in our case $i_d = n$ and $i_d = m_1 + m_2 + \cdots + m_d$, the theory of (discrete) renewal processes (Feller **1** XIII) provides a tool to obtain asymptotic results concerning $d(N)$ through the investigation of the stochastic process $m_1, m_2, \cdots$. The method used is to assume first the independence of the m's and then to show that the results do hold asymptotically. This last step is easy since the convergence to the limit is very fast. The asymptotic normality of $d(N)$ is obtained. The method and results extend to the generalised Fibonacci numbers introduced by Daykin (*J. London Math. Soc.* **35** (1960) 143–160). I discovered the applicability of the renewal theory also in a somewhat different content in the field of probabilistic number theory. (Received June 26, 1970.)

127-9. 7/27 replication of a 3^5 factorial experiment. G. BÁNKÖVI AND K. SARKADI, Mathematics Institute, Hungarian Academy of Science.

A $7(3^{5-3})$ factorial plan yielding estimates for the main effects and first order interactions has been constructed. For finding such an arrangement the first task was to determine the identity relationship of the 3^{5-3} fractions. Some reasonable restrictions led to the following two solutions: (a) the identity relationship contains one main effect and no first order interaction, (b) it contains one first order interaction component but no main effect. In both cases the identity relationship—apart from equivalent transformations—is uniquely determined. It seems desirable that the plan should contain three (overlapping) 3^{5-2} fractions. The two cases yield eleven possibilities. Selection among them was made upon the principle that the same effect should desirably not be unconfounded in two different 3^{5-2} fractions. This principle selects three plans out of the eleven. The plan with generators $ABCD_0$ and ABC^2E_0, AB^2DE_0 and $ABCD_0$, ABC^2E_1 and AB^2DE_0 for the three 3^{5-2} fractions, respectively, has the property that the maximum of the variances of the effect estimators is $(21/162)\sigma^2$ where σ^2 is the error variance. (Received June 26, 1970.)

127-10. Comparing means of populations when the other parameters are also unknown. FRANZ STREIT, Universität Bern.

Various techniques for making inference concerning linear combinations of the means of several populations, when also the other parameters are unknown, are examined. A survey of the results in the literature is given and new results are derived. Special emphasis is put on methods yielding explicit results for a wide class of distributions as for instance the method of parametric estimation or the method of structural inference [for information about this procedure see D. A. S. Fraser, *The Structure of Inference*. Wiley, New York (1968)]. Solutions for populations, which are not normally distributed, are given. The problems investigated can be considered as generalizations of the famous Behrens–Fisher problem. (Received June 29, 1970.)

127-11. Tukey and Scheffé's confidence intervals of fixed-width in the interclass correlation model. R. P. BHARGAVA AND M. S. SRIVASTAVA, University of Toronto and University of Connecticut.

The problem of finding Tukey and Scheffé's simultaneous confidence intervals for the differences in the means of a multivariate normal population when the covariance matrix is of the intraclass correlation form, $\Sigma = \sigma^2[(1-\rho)I+\rho ee']$, $e' = (1, 1, \cdots, 1)$, σ^2 unknown but ρ *known*, has been considered in Scheffé [*The Analysis of Variance*. Wiley, New York (1959)] and Miller [*Simultaneous Statistical Influence*. McGraw Hill, New York (1966)]. In this paper these results are extended to the case when σ^2 and ρ both are unknown. Also sequential procedures are given to obtain simultaneous confidence intervals of fixed width $2d$ and given confidence coefficient α. Bounds on the average sample size are obtained, and the cost of not knowing the covariance matrix is shown to be only a finite number of observations which depends on α but does not depend on the mean, σ^2, ρ and d. (Received June 29, 1970.)

127-12. A generalization of a theorem on asymptotic normality of nonparametric tests for independence. F. H. RUYMGAART, Mathematisch Centrum.

On a probability space a sequence of independent pairs of real-valued random variables, each having bivariate distribution H and marginal distributions F, G is given. To test the hypothesis: $H = F \cdot G$ statistics of the type $T_n = \int\int J_n(F_n)K_n(G_n)\,dH_n$ are considered, where $J_n(K_n)$ are sequences of weightfunctions on [0, 1] converging to functions $J(K)$ on (0, 1). Asymptotic normality of the suitably standardized statistic T_n is proved for the class of bivariate distributions H with bounded densities, under conditions on the weightfunctions equivalent to those imposed by Chernoff, Savage (*Ann. Math. Statist.* (1958)) in the location problem. The second derivatives $J''(K'')$ are not needed. In the paper of Bhuchongkul (*Ann. Math. Statist.* (1964)) these conditions are much stronger. For the proof T_n is transformed into an integral over the unit square. In the bivariate Taylor-series up to first order derivatives empirical processes, converging to Brownian bridges, arise. An appropriate "Skorokhod Construction" and precise information about the behavior of the Brownian bridge near 0 and 1, given by Pike, Shorack (*Ann. Math. Statist.* (1968)) complete the proof. (Received July 6, 1970.)

127-13. Weak convergence results for birth and death processes. SIEGFRIED SCHACH, The Johns Hopkins University.

Consider a sequence $\{X_N(t); 0 \leqq t \leqq 1\}$ of stochastic processes with trajectories in $D[0, 1]$. Assume that $(X_N(t_1), \cdots, X_N(t_m)) \to (X(t_1), \cdots, X(t_m))$ for some process $\{X(t); 0 \leqq t \leqq 1\}$ and for all finite collections $t_1, \cdots, t_m \in [0, 1]$. If the sequence of measures induced by $\{X_N(\cdot)\}$ is tight, then $X_N(\cdot) \to X(\cdot)$ in the sense of weak convergence. Using a rather elementary argument, we show that for Markov $X_N(\cdot)$ the conditions $P\{|X_N(t_2) - X_N(t_1)| \geqq \lambda |X_N(t_1)|\} \leqq C\lambda^{-r}(t_2-t_1)(1+|X_N(t_1)|^s)$, $s > 0$, $r > 0$ and $E|X_N(t)|^{3s} \leqq C$, where C is a constant, are sufficient for tightness. This result enables us to obtain weak convergence results for a large class of uni- and multivariate birth and death processes, for which the above inequalities are easily established. The class includes the continuous-time Ehrenfest model, converging to the Ornstein–Uhlenbeck process, as the number of balls gets large. It also includes the infinite server queue, converging to the same limit process, as the traffic intensity approaches infinity at a suitable rate. (Received July 8, 1970.)

127-14. Some results on the sampling distribution of the concentration ratio. GIOVANNI GIRONE, Università Bari.

The limit distribution for samples of size $n \to \infty$ of the concentration ratio R is well known. It tends to normality in some cases and to a form of the χ^2 type in the other cases. We found the exact distribution of R for samples from an exponential distribution: in this case the distribution of

R for samples of size n is equal to the distribution of the average of a sample of size $n-1$ from a rectangular distribution. For samples from a rectangular distribution we found by a direct method the exact distribution of R for sample sizes $n = 2, 3, 4, 5$; in these cases the distribution of R consists of $n-1$ arcs of functions of the type $\sum_{i=1}^{n-1} a/(b_i+R)^2$. For samples from a paretian distribution we did not succeed in finding any analytical result. By the Monte Carlo method we derived the simulated distributions and their (simulated) characteristic values for $n = 5, 10, 20, 25, 50, 100, 200, 300$ and $\alpha = 1,2, 1,4, 1,5, 1,6, 1,8, 2,0$. (Received July 9, 1970.)

(Abstracts of papers presented at the Annual meeting, Laramie, Wyoming, August 25–28, 1970. *Additional abstracts appeared in earlier issues.)*

126-14. Zipf's Law and prior distributions for the composition of a population. BRUCE HILL, University of Michigan.

The limiting distribution of frequencies of frequencies as the population size becomes large is obtained under Bose–Einstein and Maxwell–Boltzmann forms of the classical occupancy problem, when the number of cells is random and has a prior distribution. A rich variety of limiting distributions is obtained, including weak forms of Zipf's Law and the Fisher logarithmic series distribution. Lizards are compared to the weak form of Zipf's Law, and some speculations are made in regard to this law. (Received May 25, 1970.)

126-15. Optimal and admissible designs for polynomial monospline regression (preliminary report). NORMAN T. BRUVOLD, Purdue University.

Consider the regression of the form $\sum_{i=0}^{n} a_i x^i + \sum_{i=1}^{h} \sum_{j=l_i}^{k_i} b_{ij}(x-\xi_i)_+^{n-j}$ where $n-1 \geqq k_i \geqq l_i \geqq 0$, $a < \xi_1, < \cdots < \xi_h < b$, and $x \in [a, b]$. A necessary and sufficient condition for a design μ to be admissible has been found. A design admissible with respect to the above regression is a member of the admissible designs for the case $l_i = 0$, $i = 1, \cdots, h$ considered by Studden and Van Arman (*Ann. Math. Statist.* **40** (1969) 1557–1569). Conditions for admissibility on the spectrum of μ is discussed and given for several cases. Optimal designs for estimating individual regression coefficients are described. (Received May 27, 1970.)

126-16. Distribution of occupation time of a general class of bulk queues. SREEKANTAN S. NAIR AND MARCEL F. NEUTS, Purdue University.

Bulk queues have been studied by several authors. Neuts (*Ann. Math. Statist.* **38** (1967) 759–770) discussed a general class of bulk queues and studied the queue length and busy period. Because of the computational complexity the distribution of occupation time has not been studied so far. Here we closely follow the notation and terminology of Neuts. With the help of a simple lemma on summation of series we study the distribution of occupation time and obtain the steady state moments. (Received June 4, 1970.)

126-17. On the stochastic process basic to a system with scheduled secondary inputs. IZZET SAHIN AND U. NARAYAN BHAT, University of Ottowa and Southern Methodist University.

The stochastic process basic to a stochastic system with two types of inputs has the following characteristics. (i) Duting $(0, \infty)$ the primary and secondary inputs of magnitude $\chi_p^{(n)}$ and $\chi_s^{(n)}$ are realized at epochs $\{p_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ respectively. (ii) The output is linear and continuous. The resulting stochastic process is analyzed for its equilibrium behavior when (a) $\chi_p^{(n)}$ and $\theta_p^{(n)} = p_{n+1} - p_n$ $(n = 1, 2 \cdots)$ are independent and identically distributed (i.i.d) positive random variables with $P(\chi_p^{(n)} \leqq x) = A(x)$, $E[\chi_p^{(n)}] = \alpha_1 < \infty$ and $P(\theta_p^{(n)} \leqq x) = 1 - e^{-\lambda x}$, (b) $s_n = nT$ and $\chi_s^{(n)}$ $(n = 1, 2 \cdots)$ are i.i.d. positive random variables with $P(\chi_s^{(n)} \leqq x) = B(x)$ and $E[\chi_s^{(n)}] = \beta_1 < \infty$

and (c) $\{\chi_p^{(n)}\}$, $\{\chi_s^{(n)}\}$ and $\{\theta_p^{(n)}\}$ are independent sequences. The Laplace–Stieltjes transform (L.S.T.) of the limit of the resulting stochastic process is obtained using known transient results for the mixed type Markov process occurring in between secondary input epochs: $\phi(s) = s_1^{-1}\{1-\lambda\alpha_1+\beta_1/T+\phi(s,0)[1-\beta(s)]/\beta(s)\}$ where $s_1 = s-\lambda+\lambda\alpha(s)$, $\alpha(s)$ and $\beta(s)$ are L.S.T. of $A(x)$ and $B(x)$ and $\phi(s,0) = [\beta(s)-\exp(-s_1 t)]^{-1}\,{}_0\int^T \exp(-s_1 u)F(0,u)\,du$. The function $F(0,u)$ is found to satisfy an integral equation which can be solved by Wiener–Hopf factionization technique. (Received June 9, 1970.)

126-18. Utilizing initial estimates in estimating the coefficients in a general linear model (preliminary report). LAWRENCE S. MAYER AND JAGBIR SINGH, Ohio State University.

A two-stage procedure for estimating the mean of a univariate distribution was introduced by Katti [*Biometrics* **18** (1962) 139–147] and generalized to the multivariate case by Waiker [unpublished manuscript, Florida State University (1969)]. The present paper discusses both the general concept of constructing estimators that utilize an initial estimate and presents specific estimators for the coefficients in several general linear models. The estimators are based on the following scheme. Let β be the unknown parameter and β_0 the initial estimate. An initial sample of observations is gathered and an acceptance region A is constructed. If b_1, the minimum variance unbiased estimate of β based on the initial sample, belongs to A then b_1 is used to estimate β. Otherwise, a second sample is gathered and the minimum variance unbiased estimate based on the combined sample is used to estimate β. The optimal region A is derived based on several criteria, one of which is to minimize the (generalized) expected mean square (EMS) of the estimator. The efficiency of these estimators is computed under the assumption of normality and it is shown that if the guess estimate is accurate the proposed estimators have smaller EMS than the classical estimators. Several miscellaneous results, including an extension of the proposed procedure to sequential estimation, are given. (Received May 25, 1970.)

126-19. Multivariate stable distributions. S. JAMES PRESS, University of Chicago.

A class of multivariate stable distributions is introduced which has the property that linear combinations of the components of independent random vectors belong to the same family of stable laws. Various properties of these laws are discussed and an application of the theory to the problem of optimal allocation of resources is indicated. The class is illustrated with various examples. (Received June 10, 1970.)

126-20. Weakening noise in signal parameter estimation. RAOUL D. LEPAGE, Columbia University.

In his 1947 work "The theory of optimum noise immunity," V. A. Kotel'nikov considered the problem of estimating a real parameter θ, on observing a process $W(t)+m(t,\theta)$ for real values of t. The process W is white noise, and m is a known function of two real variables. Kotel'nikov discovered that if the noise W is (in some sense) weak relative to m, the appropriately defined maximum likelihood estimate θ^* should have approximately the Gaussian law with mean θ and variance $(\int (m'(t,\theta))^2\,dt)^{-1}$, where prime indicates partial derivative in θ. For white noise, this integral represents the reproducing kernel norm squared, $||m'(\ ,\theta)||^2$. Weak noise remains more elusive. Kotel'nikov's proof accepts as weak noise anything guaranteeing that certain Taylor's approximations are accurate. This paper proves that *weakening noise* can be usefully defined as a directed set of Gaussian laws under which the respective downcrossing functions of the derivative of the log-likelihood tend (after scale change by $||m'(\ ,\theta)||$) to the standard normal density function. With this definition, θ^* is well defined and $||m'(\ ,\theta)||\,\theta^*$ tends in law to the standard normal. This weakening noise includes: (1) increasing the amplitude of m without bound, (2) increasing the time domain to include more observations. Some smoothness and singularity conditions are needed. The noise W is assumed to be Gaussian, but not necessarily white or in real time t. (Received June 11, 1970.)

126-21. Parameter estimation for an r-dimensional plane wave observed with additive independent Gaussian errors. MELVIN J. HINICH and PAUL SHAMAN, Carnegie-Mellon University.

Let $\varepsilon(t, \mathbf{x})$ denote a stationary Gaussian process where $E\varepsilon(t, \mathbf{x}) = 0$, $E\varepsilon^2(t, \mathbf{x}) = \sigma^2\ \forall t \in \mathscr{T}$, $\mathbf{x} \in \mathscr{X}$, and $E\varepsilon(t_1, \mathbf{x}_1)\varepsilon(t_2, \mathbf{x}_2) = 0\ \forall t_1 \neq t_2$ or $\mathbf{x}_1 \neq \mathbf{x}_2$. Let $\mathscr{T}$ be the set of integers and $\mathscr{X}$ be a subset of the r-dimensional Euclidean space R^r. Given a coordinate system in R^r and a time origin, observe $y(t, \mathbf{x}) = s(t, \mathbf{x}) + \varepsilon(t, \mathbf{x})$, where $s(t, \mathbf{x}) = \frac{1}{2}a(O) + \sum_{j=1}^{\frac{1}{2}(T-1)} [a(\omega_j) \cos(\omega_j t - \kappa(\omega_j)'\mathbf{x}) + b(\omega_j) \sin(\omega_j t - \kappa(\omega_j)'\mathbf{x})]$, where $\omega_j = 2\pi j/T, j = 1, \cdots, (T-1)/2$ (T odd), and $\kappa(\omega_j)$ is a vector of parameters in R^r. If $\kappa(\omega) = (\omega/v)\mathbf{e}$ where $\mathbf{e}'\mathbf{e} = 1$, $s(t, \mathbf{x})$ is the r-dimensional generalization of a (discrete time) plane wave which is propagating with phase velocity v in a direction parallel to $\mathbf{e}$. For a finite time let the process $y(t, \mathbf{x})$ be simultaneously observed at each $\mathbf{x} \in \mathscr{X} = S_1 \times S_2 \times \cdots \times S_r$, $S_j = \{1, 2, \cdots, n\}$. The maximum likelihood (and periodogram) estimators $\hat{a}_n(\omega_j)$, $\hat{b}_n(\omega_j)$, $\hat{\kappa}_n(\omega_j)$ of $a(\omega_j)$, $b(\omega_j)$, and $\kappa(\omega_j)$, respectively, have a joint limiting normal distribution in which appropriately normalized estimators of the r components of $\kappa(\omega_j)$ are mutually independent, for each $j = 1, \cdots, \frac{1}{2}(T-1)$. The variances of $\hat{a}_n(\omega_j)$ and $\hat{b}_n(\omega_j)$ are $O(n^{-r})$ and the variances of the components of $\hat{\kappa}_n(\omega_j)$ are $O(n^{-r-2})$. The distributions of the estimators for different ω_j's are mutually independent. The estimators and their properties are generalized to the case where $s(t, \mathbf{x})$ is a sum of plane waves, provided there is proper separation between the phase velocities. (Received June 11, 1970.)

126-22. Limit distributions for first passage times. DAVID R. BEUERMAN, Queen's University.

Let $X_1, X_2, X_3, \cdots$ be a sequence of independent and identically distributed random variables which belong to the domain of attraction of a stable law of index α and distribution function G. Let $S_n = \sum_{i=1}^{n} X_i$ and $T_\beta(x) = \min[n: S_n > xn^\beta]$. For the $1 < \alpha \leqq 2$, $E(X_i) > 0$ case of drift, the limit distribution for $T_\beta(x)$ is given in Beuerman (*Canad. Math. Bull.* **12** (1969) 694) for $0 \leqq \beta < 1$. In the case of oscillation, corresponding to either ($1 < \alpha \leqq 2$, $E(X_i) = 0$) or ($0 < \alpha < 1$, G two-sided), we obtain limit distributions for $T_\beta(x)$, using the results of Heyde (*J. Appl. Probability* **6** (1969) 419) for $\beta = 0$; for $\beta > 0$, weak convergence methods are employed. (Received June 11, 1970.)

126-23. Uncorrelated regression residuals and singular value inequalities. STANLEY I. GROSSMAN AND GEORGE P. H. STYAN, McGill University.

Suppose $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$ denotes the usual linear model, where $\mathbf{X}$ is $n \times q$, rank $q \leqq n$. If $\mathbf{A}$ is $n \times n-q$, $\mathbf{A}'\mathbf{X} = \mathbf{0}$, and $\mathbf{A}'\mathbf{A} = \mathbf{I}$, then $\mathbf{A}'\mathbf{y}$ is a vector of uncorrelated regression residuals. Let $\mathbf{v}$ denote the vector of $n-q$ components of the error vector $\mathbf{u}$ which we want $\mathbf{A}'\mathbf{y}$ to approximate, and let Σ_A denote the covariance matrix of the vector of discrepancies $\mathbf{A}'\mathbf{y} - \mathbf{v}$. Then $\operatorname{tr}(\Sigma_A)$ is minimized at $\mathbf{A} = \mathbf{B}$, where $\mathbf{B}'\mathbf{y}$ is the vector of BLUS residuals recently introduced by Theil (*J. Amer. Statist. Assoc.* **60** (1965) 1067–1079). We prove that the BLUS residuals minimize the spectral radius of Σ_A, i.e., $ch_1(\Sigma_A) \geqq ch_1(\Sigma_B)$, where $ch_1(\cdot)$ denotes the largest characteristic root. This inequality also holds for the jth largest root; we use the singular value inequalities $ch_j^{\frac{1}{2}}(\mathbf{G}^*\mathbf{G}) \geqq (\frac{1}{2})ch_j(\mathbf{G} + \mathbf{G}^*)$, for every square complex $k \times k$ matrix $\mathbf{G}$, and $j = 1, \cdots, k$ (cf. Fan & Hoffman, *Proc. Amer. Math. Soc.* **6** (1955) 111–116). The stronger property claimed by Theil (*J. Amer. Statist. Assoc.* **63** (1968) 242–251) that $\Sigma_A - \Sigma_B$ is positive semi-definite is shown not to hold in general. (Received June 15, 1970.)

126-24. Fredholm determinant of a positive definite kernel of a special type and its application. SHASHIKALA SUKHATME, Iowa State University.

Let $\rho(x,y)$ be a positive definite kernel of the form $\rho(x,y) = K(x,y) - \sum_{i=1}^{k} \psi_i(x)\psi_i(y)$, $0 \leqq x, y \leqq 1$, where $K(x,y)$ is a symmetric positive definite kernel defined over the unit square and $\psi_i(x) \in L_2(0,1)$, $1 \leqq i \leqq k$. A method of finding the Fredholm determinant associated with $\rho(x,y)$ is given and it is applied to obtain the characteristic function of the limiting distribution of the modified Cramér–Smirnov statistic in the parametric case. (Received June 15, 1970.)

126-25. Approximate confidence limits for complex systems with exponential component lives. J. M. MYHRE AND SAM C. SAUNDERS, Claremont Men's College and Boeing Scientific Research Laboratories.

The asymptotic distribution of the log-likelihood ratio is shown to provide a method of determining approximate confidence limits for the reliability of any coherent system when each component has an exponential life with unknown failure rate and component performance data are provided in the form: number of failures and total operating time. This method is computationally feasible for arbitrary coherent systems of the order of several hundred, an advantage not enjoyed by any alternative method known at present. This extends the results of the authors, *Annals of Mathematical Statistics* (1968), on confidence limits for coherent structures with binomial data on the component's reliability. Methods similar to those previously utilized are combined with some special properties of the exponential distribution to obtain the results. (Received June 16, 1970.)

126-26. Comparing conditional means and variances in a regression model with measurement errors and known variances. T. W. F. STROUD, Queen's University.

When variables of interest in a normal regression model are masked by measurement errors of known variances, it is possible to obtain inferences about the parameters relating to the regression of the variables of interest. For two-sample problems, we study hypotheses of equality of (i) the regression coefficients and of (ii) the residual variances, assuming a joint normal distribution on the predictor and predicted variables. Since likelihood-ratio tests are very complicated, asymptotic tests are proposed which involve only unrestricted maximum-likelihood estimators, and which are based on a procedure described by Wald. Formulas for the test statistics are presented; their asymptotic null distributions are chi-square. For each problem, an algorithm for evaluating the variance of the asymptotic normal distribution of the normalized test statistic (under the alternative) is given. A large-sample confidence interval for the residual variance of the predicted variable of interest, in the case of one sample, is proposed. (Received June 16, 1970.)

126-27. Certain uncorrelated and independent rank statistics. ROBERT V. HOGG AND RONALD H. RANDLES, University of Iowa.

Some interesting examples of uncorrelated simple linear rank statistics are given; among these is one which shows that a two-sample "location" statistic (like those of Wilcoxon and Van der Waerden) and a two-sample "scale" statistic (like those of Mood, Ansari–Bradley, and Klotz) are uncorrelated, provided the underlying continuous-type distributions are equal. Necessary and sufficient conditions for general rank statistics to be uncorrelated (or have joint distributions which enjoy certain symmetrical properties) are found. One interesting example resulting from these considerations shows that the two-sample statistics of Wilcoxon and Kolmogorov–Smirnov are uncorrelated. Two simple linear rank statistics that are uncorrelated are found, under suitable restrictions, to be asymptotically independent. This is followed by some illustrations in which two rank statistics are exactly independent. One of the examples of independence is, under a suitable null hypothesis, the Kruskal–Wallis statistic (computed on columns) and that of Friedman (computed on rows) and another example is Spearman's correlation coefficient and each of many two-sample rank statistics. The importance of these results about independence (or uncorrelated statistics) is noted. (Received June 17, 1970.)

126-28. Structural solution to the linear calibration problem. ANDREW J. KALOTAY, Bell Telephone Laboratories, Inc.

Suppose that the relationship between a controllable variable x and a response variable y can be described by the linear regression model $y_i = \alpha + \beta x_i + \sigma e_i$, $i = 1, \cdots, n$. In the Linear Calibration (or Inverse Linear Regression) Problem, in addition to the n observations corresponding to the knowns $x_1, \cdots, x_n$, there are m further observations corresponding to an unknown x. The problem is to estimate x. The structural method of analysis proposed by Fraser (1968) is shown to be applicable to this problem. Assuming that the errors are normal and independent, the marginal structural distribution of x is derived. The result is compared with a Bayesian solution proposed by Hoadley, *J. Amer. Statist. Assoc.* (1970) **65** 356. (Received June 18, 1970.)

126-29. A property of Poisson processes and its application to macroscopic equilibrium of particle systems. MARK BROWN, Cornell University.

Consider a Markov process with stationary transition probability, state space $(\mathscr{X}, C)$, and σ-finite stationary measure μ. At time 0 distribute particles according to a Poisson $(\mathscr{X}, C, \mu)$ process, i.e. the joint distribution of the number of particles in non-overlapping sets $C_1 \cdots C_n \in C$ is independent Poisson with parameters $\mu(C_1), \cdots, \mu(C_n)$. Each particle moves around in the state space, independently of the other particles, according to the transition law of the Markov process. Thus a collection of particle paths, i.e. random points in $(X\mathscr{X}_t, XC_t)$, is generated. It turns out under very weak conditions that the collection of particle paths generates a Poisson $(X\mathscr{X}_t, XC_t, \gamma)$ process, where γ is the strictly stationary measure on $(X\mathscr{X}_t, XC_t)$ generated by μ and the transition law of the Markov process. This result was motivated by and includes as a special case a result of Derman, page 545 (*Trans. Amer. Math. Soc.* **79** 541–545) as well as some results of Port. It is derived as an application of a theorem on random transformations of Poisson processes, which in turn is a generalization of a result of Karlin, page 497 (*A First Course in Stochastic Processes*, Academic Press, New York). (Received June 22, 1970.)

126-30. Saturated fractions of 2^n designs of odd resolution. PETER W. M. JOHN AND ROBERT C. JUOLA, University of Texas.

Let $D_{k,n}$ be the set of all treatment combinations from a 2^n factorial design which contain k or fewer letters; the set includes (1). It is shown that $D_{k,n}$ is a saturated fraction of resolution $2k+1$ for each k and all $n \geqq k$. For any n, a saturated resolution V design may also be obtained by taking (1), all pairs $ab, ac, \cdots$ and all $(n-1)$-tuples, i.e. by folding over the points $a, b, \cdots$ in $D_{n,k}$; when $n = 5$ we have the well-known half replicate defined by $I = -ABCDE$. An alternative method of generating saturated resolution V designs is also discussed. (Received June 23, 1970.)

126-31. Asymptotic tail probabilities for a class of functionals of a Gaussian process (preliminary report). NORMAN A. MARLOW, Bell Telephone Laboratories, Inc.

Let $\{y(\tau),\ 0 \leqq \tau \leqq 1\}$ be a zero-mean Gaussian process with continuous paths whose covariance function ρ is continuous and satisfies Fernique's condition (*C.R. Acad. Sci. Paris Sér. A-B.* **259** (1964) 6058–6060). Define $Ax = \int_0^1 \rho(\cdot, s)x(s)\,ds$, $x \in L_2[0,1]$. Let $F: C[0,1] \to \mathscr{R}$

such that (1) $|F[x]-F[y]| \leqq K\cdot(\|x-y\|_\infty)^\beta$ where K and β are constants, (2) the limit $\Gamma[x] = \lim_{\lambda\to\infty}(1/\lambda^\beta)F[x\lambda]$ exists for all $x \in C[0,1]$. Define $\sigma = \sup_{x\in C[0,1]}\{\Gamma[Ax]-(\frac{1}{2})(Ax,x)\}$. Then (i) $\lim_{\lambda\to\infty}(1/\lambda^{2/(2-\beta)})\log E\{\exp(\lambda F[y])\} = \sigma$, (ii) $\limsup_{\alpha\to\infty}(1/\alpha^{2/\beta})\log P\{F[y] \geqq \alpha\} = -(\beta/2)[(2-\beta)/(2\sigma)]^{(2-\beta)/\beta}$, (iii) $0 \leqq \sigma \leqq (\frac{1}{2})(2-\beta)(\beta\|\rho\|_\infty)^{\beta/(2-\beta)}K^{2/(2-\beta)}$. Part (i) is the function space analogue of a one-dimensional Laplace asymptotic formula, and part (ii) follows from a Tauberian argument. Particular asymptotic distributions are obtained for several functionals whose distributions are not known. (Received June 25, 1970.)

126-32. Characterization of *DF* confidence intervals for the coefficients in a linear model (preliminary report). GARY E. MEEK, John Carroll University.

This paper characterizes all distribution-free confidence intervals for the regression coefficients in the general linear model, $Y = \beta_0+\beta_1 x_1+\beta_2 x_2+\cdots+\beta_k x_k+\varepsilon$, where ε is a random variable with a continuous symmetric distribution function and the x_i's are non-random variables. These intervals are shown to be functions of the Pitman or permutation statistics, to be unbiased and to be shortest in the probability sense whenever the corresponding tests are unbiased and MP respectively. A sampling design which uses a multiple comparisons procedure to obtain independent estimates is proposed for the general linear model. Recommendations for using specific forms of the intervals are made based upon the ARE's with respect to intervals based on the t-statistic. (Received June 25, 1970.)

126-33. Ergodic properties of some number-theoretic data sources. S. B. GUTHERY, Bell Telephone Laboratories, Inc.

Number-theoretic expansions of real numbers such as the f-expansions of A. Rényi, which include the continued fraction and M—ary expansions, are viewed in the context of communication theory as data sources. In this paper, the calculation of their entropy number—theoretically via Rohlin's formula is compared with the classical calculation of entropy by C. Shannon. Next, a variety of ergodic properties of the expansions are interpreted within the framework of communication theory. Finally, the definition of a data source is extended to a countably-valued function on a probability space with a measure-preserving transformation and the implications of such concepts as isomorphism of these transformations are considered. (Received June 25, 1970.)

126-34. Estimation of missing data in the multivariate linear model. LYMAN McDONALD, University of Wyoming.

Techniques for the estimation of missing data in multiresponse (multivariate) experiments are suggested and the subsequent analysis of the "completed" data is considered. These techniques are generalizations of the procedures for the analysis of uniresponse (univariate) experiments in which some of the observations (yields) are missing. The techniques require only computational procedures which are already available in the literature for uniresponse experiments. (Received June 25, 1970.)

126-35. Minimum designs in sequential analysis of variance (preliminary report). ROGER D. H. JONES, University of Georgia.

The need for sequential analysis of variance with unequal numbers of observations in experimental units arises in clinical medical trials. An account of the subject with Monte Carlo results was published by Hinkelmann and Jones in "Methods of information in medicine" **8**, 1969. This dealt with one-way classifications, fixed effects models. The authors contributed a paper to the Chapel Hill meeting in May last extending the results to the two-way design without interaction. It is now applied to the two-way design with interaction. (Received June 25, 1970.)

126-36. On a class of random measures that are homogeneous with respect to a fixed measure. R. A. AGNEW AND M. G. FAHEY, Air Force Institute of Technology and Headquarters, USAF.

Given a measurable space, we say that a random measure is homogeneous with respect to a fixed measure on the space if its finite dimensional distributions for disjoint sets depend only on the measures of the sets. (The term "random measure" is used in the sense of Mecke [*Z. Wahrscheinlichkeitstheorie* **9** (1967) 36–58].) Assuming that the underlying space is uncountable and that the fixed measure is nonatomic and sigma-finite, an interesting class of homogeneous random measures containing the class of mixed Poisson processes is characterized and investigated in terms of Laplace functionals. (Received June 25, 1970.)

126-37. Tests of hypotheses for expected life in the presence of an outlier observation. B. K. KALE AND J. R. VEALE, University of Manitoba and Iowa State University.

Consider a life testing experiment in which $(X_1, X_2, \cdots, X_n)$ are such that $(n-1)$ of them are distributed as $f(x, \sigma) = \sigma^{-1}\exp(-x\sigma^{-1})$ and one is distributed as $f(x, \sigma\alpha^{-1})$, $0 < \alpha \leqq 1$. A priori each X_i has probability, $1/n$, of being a heterogeneous observation distributed as $f(x, \sigma\alpha^{-1})$. We consider tests of Hypotheses $H_0: \sigma \leqq \sigma_0$ vs. $H_A: \sigma > \sigma_0$. The standard procedure in the homogeneous case, $\alpha = 1$, is to reject H_0 if $T_n = \sum_{i=1}^n X_i > C$. We show that $\lim_{\alpha\to 0} P\{T_n > c \mid \sigma\} = 1$ for any constant C, and the test based on T_n is useless when an outlier observation is present. We consider test procedure based on $T_m = \sum_{i=1}^{m-1} X_{(i)} + (n-m+1)X_{(m)}$, $m \leqq n-1$. Exact distribution of T_m has been obtained under any (α, σ) and it is shown that $\sup_{0<\alpha\leqq 1} P\{T_m > c \mid \sigma \leqq \sigma_0\} \leqq \delta < 1$. Power function of the test based on T_{n-1} is studied in detail and some approximations are also suggested. These are based on the result that $S_n = \sum_{i=1}^n b_i U_i$, $0 < b_i \leqq 1$, $(U_1, U_2, \cdots, U_n)$ i.i.d. as $f(x, 1)$ can be approximated very well by $\delta(\mathbf{b}) V_{\nu(\mathbf{b})}$ where $\delta(\mathbf{b}) = \sum b_j^2/\sum b_j$ and $\nu(\mathbf{b}) = (\sum b_j)^2/\sum b_j^2$, and V_p is a Gamma p rv. (Received June 25, 1970.)

126-38. Robust sequential confidence intervals for the Behrens–Fisher problem. MALAY GHOSH, University of North Carolina.

The problem of providing a bounded length (sequential) confidence interval for the median of a symmetric (but otherwise unknown) distribution based on a general class of one-sample rank order statistics was investigated by Sen and Ghosh (Institute of Statistics Mimeo Series No. 648, Univ. of North Carolina (1969)). The purpose of the present note is to indicate how the techniques developed there can be extended to the two-sample problem. It has been shown that, in particular, for the Behrens–Fisher situation, when the proposed procedure is based on the "normal-scores" statistic, under very general conditions on the unknown distribution function, it is asymptotically at least as efficient as an analogous procedure suggested by Robbins, Simons and Starr (*Ann. Math. Statist.* **38** 1384–1391). (Received June 26, 1970.)

126-39. Empirical Bayes procedures for multiple decision problems. J. VAN RYZIN, University of Wisconsin.

This paper treats the general theory of multiple decision problems from the empirical Bayes viewpoint of H. Robbins (*Ann. Math. Statist.* **35** (1964) 1–20). Three general theorems are given: (i) conditions under which one has asymptotic optimality of the empirical Bayes rules and two general theorems giving conditions for rates of convergence of varying orders to the optimal Bayes risk. These theorems are then applied to three multiple decision problems: (i) a classification problem, (ii) a monotone multiple decision problem and (iii) a selection problem. Regarding (i) we generalize and give rate results for the earlier results of Robbins (*Ann. Math. Statist.* **35** (1964) 1–20) and Hudimoto (*Ann. Inst. Statist. Math.* **20** (1968) 169–186). The problems as treated in (ii) and (iii) have not been treated from the empirical Bayes viewpoint previously. (Received June 26, 1970.)

126-40. Estimating a quantile of the normal law when loss is squared error. J. V. ZIDEK, University of British Columbia.

Let $S, T_1, \cdots, T_k$ $(k \geqq 1)$ be independent random variables where $S\sigma^{-2}$ has the chi-squared distribution with n degrees of freedom and T_i is normally distributed with mean μ_i and variance σ^2, $-\infty < \mu_i < \infty, \sigma > 0, i = 1, 2, \cdots, k$. Denote by μ, T, and Y, $(\mu_1, \cdots, \mu_k)'$, $(T_1, \cdots, T_k)$ and $TS^{-\frac{1}{2}}$, respectively. Suppose $\eta = (\eta_1, \cdots, \eta_k)'$ is given and $\nu = \mu + \eta\sigma$ is to be estimated under squared error loss with an estimator of the form $S^{-\frac{1}{2}}V(Y)$ when μ and σ are unknown. By examining $E_{\mu,\sigma}((S^{\frac{1}{2}}V(Y) - \mu - \eta\sigma)^2) \mid Y)$ it is shown that there is an estimator $S^{\frac{1}{2}}U(Y)$ for which $E_{\mu,\sigma}\|S^{\frac{1}{2}}U(Y) - \nu\|^2 \leqq E_{\mu,\sigma}\|S^{\frac{1}{2}}V(Y) - \nu\|^2$ unless $P_{\mu,\sigma}(U(Y) \neq V(Y)) > 0$ when strict inequality holds. More precisely, $U(Y) = V(Y) + \min(0, F(Y))Y$ where $F(y) = (\|y\|^2 + 1 + \frac{1}{4}\eta' y(n+k)^{-1} - yV(y)\|y\|^{-2})$ if $\eta' y > 0$ and 0 otherwise. It follows that when $k = 1$, $\eta > 0$, a class of minimax estimators for ν consists of all $S^{\frac{1}{2}}V(Y)$ for which $\min(1, 1 + y^{-2}(\frac{1}{4}\eta^2 y^2(\eta+1)^{-1} - \eta y C + 1))y \leqq V(y) - \eta C \leqq y$, if $y \geqq 0$ and $V(y) = y + \eta C$ if $y < 0$, where $C = \Gamma(\frac{1}{2}(n+1))(2^{\frac{1}{2}}\Gamma(\frac{1}{2}(n+2)))^{-1}$. Also the best affine invariant estimator, $S^{\frac{1}{2}}(Y + \eta C)$ is inadmissible. (Received June 29, 1970.)

126-41. On some problems of estimation and prediction for a class of growth processes (preliminary report). JAMES T. MCCLAVE, University of Florida.

Consider the Markovian growth process $\{X(t): 0 \leqq t \leqq \infty\}$ with growth rate $\lambda(t)$ and $P\{X(0) = x_0\} = 1$. The problem is to hypothesize various $\lambda(t)$, estimate unknown parameters in $\lambda(t)$, and then use this estimated value for predicting future realizations of $\lambda(t)$. If we assume n equispaced observations have been taken, we can make a transformation $X(t) \to Y(t)$ where, for large x_0, $Y(t)$ is approximately normal with variance x_0^{-1} and mean depending on $\lambda(t)$. This transformation can then be used for solving the estimation and prediction problem. If a specific form of $\lambda(t)$ is assumed, e.g. the logistic growth rate, we can consider maximum likelihood estimates of the parameters. In many cases these methods lead to intractable equations, and Monte Carlo experiments using iterative procedures are necessary. (Received June 29, 1970.)

126-42. On a characterization of probability distributions by the joint distribution of some of their linear forms. IGNACY I. KOTLARSKI, Oklahoma State University.

Let X_0, $(X_1, \cdots, X_n)$, $(X_{n+1}, \cdots, X_{n+m})$ be three independent random vectors having dimensions 1, n, m $(n, m = 1, 2, \cdots)$ respectively. Denote the following column vectors (1) $X = [X_0; X_1, \cdots, X_{n+m}]^T$, $Y = [Y_1, \cdots, Y_{n+m}]^T$ (T means transposition), and the matrix (2) $A = [a_{jk}]$, $j = 1, 2, \cdots, n+m$; $k = 0, 1, \cdots, n+m$ having $n+m$ rows and $n+m+1$ columns. Let (3) $Y = A \cdot X$. Denote A_k $(k = 0, 1, \cdots, n+m)$ the square submatrix of A obtained by cancelling the kth column. Suppose (i) X_0, $(X_1, \cdots, X_n)$, $(X_{n+1}, \cdots, X_{n+m})$ are three independent and real random vectors, (ii) their characteristic functions do not vanish, (iii) the matrix A_0, at least one of the matrices $A_1, \cdots, A_n$, and at least one of the matrices $A_{n+1}, \cdots, A_{n+m}$ are nonsingular. Then the joint distribution of $Y = [Y_1, \cdots, Y_{n+m}]^T$, where Y_j are given by (1) and (3) determines the distributions of all the three random vectors X_0, $(X_1, \cdots, X_n)$, $(X_{n+1}, \cdots, X_{n+m})$ up to a shift. (Received June 29, 1970.)

126-43. A chi-square statistic with random cell boundaries. D. S. MOORE, Purdue University.

In testing goodness of fit to a parametric family with unknown parameters it is often desirable to allow the cell boundaries for a chi-square test to be functions of the estimated parameter values.

Suppose M cells are used and m parameters are estimated using BAN estimators based on the raw data. Then A. R. Roy and G. S. Watson showed that in the univariate case the asymptotic null distribution is that of $\sum_1^{M-m-1} Z_i^2 + \sum_{M-m}^{M-1} \lambda_i Z_i^2$ where the Z_i are independent standard normal and the constants λ_i lie between 0 and 1. We extend this result to rectangular cells in any number of dimensions, show that in location-scale problems the λ_i are independent of the parameters if the cell boundaries are chosen in a natural way, and show that in any case all λ_i approach 0 as M is appropriately increased. Finally, we present an effective method of computing the cdf of the asymptotic distribution and give a short table of critical points in the case of testing goodness of fit to the univariate normal family. (Received June 29, 1970.)

126-44. Some admissible empirical Bayes procedures. GLEN MEEDEN, Iowa State University.

Certain empirical Bayes procedures are shown to be inadmissible in the obvious sense. Admissible procedures are found for two examples by using some of the results of J. Rolph [*Ann. Math. Statist.* **39** 1289–1302]. (Received June 29, 1970.)

126-45. Determining subjective probability by sequential choices. ARLO D. HENDRICKSON AND ROBERT J. BUEHLER, University of Minnesota.

A subject has a choice of two prospects: A known prize if event A occurs, or the same prize if event B occurs. Axiomatically, the first choice orders the subjective probabilities by $P(A) \geqq P(B)$. Consider a sequence of choices wherein A is always a fixed outcome whose subjective probability is to be determined; $B_1, B_2, \cdots$, are sequentially chosen outcomes having known probabilities; $g_1, g_2, \cdots$, are the prizes in a sequence of prospects. Hopefully, $P(A)$ can be determined to any accuracy by offering sufficiently many choices, but a complication arises in that the choice at step one may depend not only on $P(A)$ and $P(B_1)$ but also on anticipation of later choices. For a simple choice of $B_1, B_2, \cdots$, the present paper characterizes prize sequences $g_1, g_2, \cdots$, which "encourage honesty," and thereby allow the natural conclusion about $P(A)$ at each step. (Received June 29, 1970.)

126-46. Asymptotic distribution of sample quantiles for exchangeable random variables. K. C. CHANDA, University of Florida.

Let $\{\theta^{(n)}\}$ be a sequence of parameter vectors and let $\{F_n(x_1, \cdots, x_n; \theta^{(n)})\}$ be a sequence of cumulative distribution functions (cdf) defined by $F_n(x_1, \cdots, x_n; \theta^{(n)}) = \int_\Omega F(x_1, \omega; \theta^{(n)}) \cdots F(x_n, \omega; \theta^{(n)})\, dP(\omega)$ where $P(\omega)$ is a probability measure over Ω. Then it is shown that under different conditions on $\{\theta^{(n)}\}$ and F, the rth order statistic $X_{r:n}(\lim_{n\to\infty} r/n = p;\ 0 < p < 1)$ of the random variables $X_1, \cdots, X_n$ with cdf F_n tends asymptotically to different distributions. The particular case of equicorrelated normal distribution with zero means and unit variances is investigated. It is shown that when the common correlation $\rho = \rho_n$ is such that $\lim_{n\to\infty} n\rho_n = 0$, $X_{r:n}$ properly normalized has asymptotically a standard normal distribution. (Received June 29, 1970.)

126-47. Some asymptotic results on tests based on cumulative sums (preliminary report). RICHARD A. JOHNSON, University of Wisconsin.

The theory of weak convergence is used to obtain the asymptotic distribution of the test statistic proposed by Page *Biometrika* **42**. Asymptotic distributions are also obtained for other statistics. Particular emphasis is given to both sequential and fixed sample size tests based on observations from the exponential distributions. The asymptotic effect of serial correlation is studied. (Received June 29, 1970.)

126-48. Admissible clustering procedures. LLOYD FISHER AND JOHN W. VAN NESS, University of Washington.

Selecting a clustering procedure from among the myriad of procedures now proposed is very difficult. One does not usually know enough about a prioris, losses, etc. to determine a "best" procedure. We therefore propose an approach analogous to the one taken in more classical areas of statistics, of restricting one's attention to *admissible* decision rules. This at least eliminates obviously bad rules. Our suggestion is therefore to determine a set of conditions $A, B, C, \cdots$ which a clustering procedure should satisfy. These conditions should be practically meaningful and sufficiently tractable that they can be verified. A procedure which satisfies condition A is called A admissible. A table can then be constructed listing procedures against admissibility conditions. A person wishing to select a clustering method can then decide what conditions are important to his problem, say A, C, and D, and then look in the table for those methods which are A, C, and D admissible. In this paper we define several conditions (some already in use) and construct a table for several clustering procedures. We also obtain some results concerning the limitations of hierarchical procedures. (Received June 29, 1970.)

126-49. Asymptotic expansion for the distribution of the characteristic roots of $\mathbf{S}_1\mathbf{S}_2$ when population roots are not all distinct. HUNG C. LI AND K. C. S. PILLAI, Southern Colorado State College and Purdue University.

The asymptotic expansion of the first two terms for the distribution of the characteristic roots of $\mathbf{S}_1 \mathbf{S}_2^{-1}$ has been obtained by the authors (*Ann. Math. Statist.* **41** (1970)) when all population roots are distinct, both in the real and complex cases, where $\mathbf{S}_1$ and $\mathbf{S}_2$ are independent sample covariance matrices of degrees of freedom n_1 and n_2 respectively from normal populations. However, if the population roots are not all distinct, the formulae derived break down. In the case of q smallest roots equal, the first term of the asymptotic expansion in the real case has also been obtained by the authors (see reference above). In the present paper, we extend this result to the second term. Furthermore, we also derive the corresponding formulae in the complex case, because not all results in the complex case are counterparts of the real. (Received June 29, 1970.)

126-50. Sequential hypothesis tests for semi-Markov processes. ALLAN L. GUTJAHR, Bell Telephone Laboratories, Inc.

Bayesian sequential hypothesis tests are examined for semi-Markov processes where the transition matrix depends on an unknown parameter. Rules that allow stopping at non-transition points are permitted. Under certain general assumptions it is shown that the decision rule is independent of the stopping rule and the optimum decision rule is equivalent to a generalized sequential probability ratio test where the boundaries depend on the time spent in the current state. Truncation in time, rather than truncation in the number of transitions, is used to establish the results stated above. Some other results are generalized from the usual sequential decision problem using this technique. Sequential probability ratio tests are examined for Markov chains and semi-Markov processes. For recurrent, finite state processes it is shown that a sequential probability ratio test can be used to discriminate between two alternative hypotheses provided that the two hypothesized matrices of transition functions differ on at least one state. (Received June 30th, 1970.)

126-51. Heterogeneous questionnaire theory. GEORGE T. DUNCAN, University of Minnesota.

A charging scheme based on the resolution of questions strikes a new direction from the approach of Claude Picard, *Theorie des Questionnaires*, Gauthier-Villars, Paris (1965). The relationship between questionnaire theory and noiseless coding theory is explored. Graph theoretic methods are used to obtain results valid for codes in which words are constructed from arbitrary mixtures of alphabets, as well as arborescence questionnaires, i.e., those having representation as rooted, directed trees. A charge of $\log d$ for each resolution d question is justified by an equity principle. Using this charging scheme an extended noiseless coding theorem shows that the average charge for a heterogeneous questionnaire is bounded below by the Shannon entropy. This result is shown to hold for both finite and countable state spaces. The decision theoretic problem of choosing a questionnaire to resolve a finite state space is examined. Certain admissibility and essentially complete class results are obtained which indicate the structure of optimal heterogeneous questionnaires. A dynamic programming approach is used to provide an algorithm for finding an optimal questionnaire. (Received June 30, 1970.)

126-52. Invariance properties of distribution free tests and their relation to generalized rank tests. C. B. BELL AND VICTOR KUROTSCHKA, University of Michigan.

If a nonparametric hypothesis is expressed in terms of invariance of the Null hypothesis distributions under a transformation group S, of the sample space, then the family of all distribution free (DF) tests coincides with the family of all conditional tests given the S-orbits, provided these are sufficient and complete. A characterization of this family can be given by introducing generalized B-Pitman statistics. If there exists another transformation group G of the sample space which leaves the test problem invariant, then G-invariant tests can be defined as generalized G-rank tests. G and S are called canonical if every data point x can be represented as an intersection of its S- and its G-orbit. One can construct for every B-Pitman function, h, a transformation group, G_h, isomorphic to G, so that G_h leaves the test problem invariant and G_h and S are canonical. This allows one to translate properties of rank tests based on invariance under G into properties of arbitrary DF tests (or strongly DF tests) and, in particular, those of permutation tests. Investigations of relations between rank and arbitrary DF tests can be carried out in terms of properties of the groups G_h and G. (Received June 30, 1970.)

126-53. A Cauchy-type functional equation and a characterization of the multivariate normal distribution. DONALD H. THOMAS, General Motors Technical Center.

Let $x_1, x_2, \cdots, x_n$ be $np \times 1$ vectors in p-dimensional Euclidean space with $n \geqq p$ and define the $p \times n$ matrix X by $X = \{x_1, x_2, \cdots, x_n\}$. In this paper a Cauchy functional equation of the type: $F(XX') = \sum_{i=1}^{n} F(x_i x_i')$, is studied as the basis for establishing a well-known characterization of the multivariate normal distribution. The approach generalizes the use of the ordinary Cauchy equation for the related characterization of the univariate normal which is discussed, for example, in certain classical treatments of Maxwell's law of gases ($n > p = 1$ above). (Received July 1, 1970.)

126-54. The spectral analysis of sampled time series (preliminary report). DAVID R. BRILLINGER, University of California, Berkeley.

Suppose $Y(t)$, $-\infty < t < \infty$, is a real-valued stationary time series with power spectrum $f_{YY}(\lambda)$, $-\infty < \lambda < \infty$. Suppose no continuous stretch of the series is available for analysis, rather the values $Y(\tau_1), \cdots, Y(\tau_n)$ are given, $\tau_1, \cdots, \tau_n$ being the times of events of an independent stationary point process $N(t)$, $-\infty < t < \infty$. We indicate a spectral representation for the sampled series $Y(\tau_j)$ and develop an estimate of $f_{YY}(\lambda)$ based on estima tesof the spectra of $Y(\tau_j)$ and $N(t)$. We also develop estimates for the cross-spectra of sampled bivariate stationary series. (Received July 1, 1970.)

126-55. A pilot course in elementary statistics using a time-sharing computer. LAKSHMI U. TATIKONDA. Louisiana State University in New Orleans.

A computer-oriented introductory statistics course was offered in the Department of Mathematics at Louisiana State University in New Orleans during Spring 1970 using the time-sharing computer equipment. This course was open to all majors. Prerequisites were only high school mathematics. Of the 30 students enrolled, about one-third of them were non-math majors. One of the primary aims of the course was to make the students actively participate in the course through the conversational time-sharing computer, instead of merely teaching about techniques as in the traditional elementary statistics course. Students wrote some computer programs on their own and used several canned programs for various applied statistical problems. Students' reaction to the course was evaluated at the end of the course through an elaborate questionnaire. They were very enthusiastic and felt that they had learned something worthwhile and practical. They also recommended this approach to several other courses. (Received July 1, 1970.)

126-56. A limit theorem for a sequence of exchangeable random variables. W. L. HARKNESS AND A. V. GODAMBE, Pennsylvania State University.

Let $X_{ij}^{(n)}$ be a double sequence of random variables assuming the values 0 and 1 and such that (i) $X_{i1}^{(n)}, \cdots, X_{in}^{(n)}, \cdots$ are i.i.d. for $i = 1, 2, \cdots$; (ii) for each j, $X_{ij}^{(n)} \cdots$, are exchangeable and (iii) the common correlation coefficient ρ_n between $X_{\alpha j}^{(n)}$ and $X_{\beta j}^{(n)}$ is such that $n\rho_n \to 0$ as $n \to \infty$. Further set $X_{i.}^{(n)} = \sum_{j=1}^n X_{ij}^{(n)}$, $X_{.j}^{(n)} = \sum_{i=1}^n X_{ij}^{(n)}$, $Y_L{}^{(n)} = (X_{1.}^{(n)}, \cdots, X_{L.}^{(n)})$ and $Z_L{}^{(n)} = (X_{.1}^{(n)}, \cdots, X_{.L}^{(n)})$. Then the main result of this paper consists in proving that under the assumptions (i), (ii) and (iii), the distribution of $Z_L{}^{(n)}$ $(L = 1, \cdots, n)$, conditional on $Y_n{}^{(n)} = (k, \cdots, k)$ converges (for fixed L) to a multiple Poisson distribution. (Received July 2, 1970.)

126-57. Conversion of convergence in the mean to almost sure convergence by smoothing. W. J. HALL AND JULIAN KIELSON, The University of Rochester.

Consider a sequence of random variables converging in the mean (arbitrary order) to zero. It is shown that, if the rate of convergence is not too slow, then successive averages, and appropriately weighted averages, of the variables converge to zero almost surely. In the context of estimation, it is concluded, for example, that if the mean square error for a sequence of estimators converges to zero, and not too slowly (like $\log^{-r} n$ for some $r > 1$), then a "smoothed" estimator sequence will be *strongly consistent*. (Received July 2, 1970.)

126-58. Non-optimality of preliminary-test estimators for the multinormal mean. STANLEY L. SCLOVE, CARL MORRIS AND R. RADHAKRISHNAN, Carnegie-Mellon University, Rand Corporation and Carnegie-Mellon University.

Let X be a p-variate $(p \geqq 3)$ normal random vector with unknown mean vector θ and covariance matrix $\sigma^2 I$, σ^2 being unknown. Let S be distributed independently of X as σ^2 times a chi-square random variable with n degrees of freedom. The problem is to estimate θ when the loss function is $||\hat{\theta}-\theta||^2/\sigma^2$. The usual estimator, X, has constant risk p for all θ, σ^2, but is dominated by the Stein-James estimator, $(1-cS/X'X)X$, $0 < c < 2(p-2)/(n+2)$. A "preliminary-test" estimator corresponding to the hypothesis $\theta = \mathbf{0}$ takes the value X if $X'X/S > c(>0)$ and equals $\mathbf{0}$ otherwise. It is shown that there are parameter values for which the risk of such an estimator exceeds p, values for which its risk is less than p, and that, for $0 < c < 2(p-2)/(n+2)$, it is dominated by $(1-cS/X'X)^+X$. The results extend immediately to preliminary-test estimators corresponding to any linear hypothesis concerning θ, as well as to the case in which X has completely unknown covariance matrix Σ, S is independent of X and distributed according to a Wishart distribution with parameter Σ, and the loss function is $(\hat{\theta}-\theta)'\Sigma^{-1}(\hat{\theta}-\theta)$. (Received July 3, 1970.)

126-59. Remarks on the unpleasant shape of the likelihood for the model II unbalanced one way layout. JEROME KLOTZ AND JOSEPH PUTTER, University of Wisconsin and Volcani Institute of Agricultural Research.

The problem of obtaining maximum likelihood estimators of variance components for the unbalanced one way layout is investigated for some simple cases. It is found that the likelihood equation may have several roots and the likelihood may be maximized on the boundary of the permissible region rather than at any of the roots. Denote $X_{jk} = \mu + b_j + w_{jk}$; $k = 1, \cdots, K_j$; $j = 1, \cdots, J$ where $b_j : N(0, \sigma_b^2)$ and $w_{jk} : N(0, \sigma_w^2)$ and write $\sigma_t^2 = \sigma_b^2 + \sigma_w^2$, $\rho = \sigma_b^2/\sigma_t^2$. Maximizing the likelihood for fixed ρ, $(0 \leqq \rho \leqq 1)$, we obtain $g(\rho) = 2 \ln L(\hat{\mu}(\rho), \hat{\sigma}_t^2(\rho), \rho) = -N \ln \sigma_t^2(\rho) - \sum_{j=1}^J \ln(1-\rho+K_j\rho) - (N-J)\ln(1-\rho) - N[1+\ln 2\pi]$ where $N = \Sigma K_j$, $\hat{\sigma}_t^2(\rho) = N^{-1}[(1-\rho)^{-1}S_w + \sum_{j=1}^J (1-\rho+K_j\rho)^{-1}K_j(X_j - \hat{\mu}(\rho))^2]$, $\hat{\mu}(\rho) = [\sum_{j=1}^J (1-\rho+K_j\rho)^{-1}K_j X_j]/(\sum_{j=1}^J (1-\rho+K_j\rho)^{-1}K_j)$ and S_w is the within sum of squares. For $J = 2$ this reduces to $g(\rho) = -N\ln(1-\rho+Q(1-R)\rho) + N\ln(1-\rho+Q\rho) + 2\ln(1-\rho) - \sum_{j=1}^2 \ln(1-\rho+K_j) + N[\ln N - 1 - \ln 2\pi - \ln S_t]$ where $Q = 2K_1K_2/(K_1+K_2)$, $R = S_b/S_t$, $S_t = S_b + S_w$, and S_b is the between sum of squares. When $K_1 = 2$, $K_2 = 10$ and $R = .28$ it is found that the derivative equation $g'(\rho) = 0$, equivalent to a cubic equation, has two roots in the interval $0 \leqq \rho \leqq 1$. At $\rho = .036$ there is a relative minimum and at $\rho = .120$ there is a relative maximum. Nevertheless, the absolute maximum occurs on the boundary at $\rho = 0$ and not where the derivative vanishes. (Received July 3, 1970.)

126-60. On just subset selection rules. KLAUS NAGEL, Purdue University.

Let $\prod_1, \prod_2, \cdots, \prod_k$ be stochastically ordered populations from which the independent random variables $X = (X_1, \cdots, X_k)$ are observed. $Y = (Y_1, \cdots, Y_k)$ is called better than X w.r.t. $\prod_i$ if $Y_i \geqq X_i$ and $Y_j \leqq X_j$ for $j \neq i$ hold. A subset selection rule is said to be just if Y better than X w.r.t. $\prod_i$ implies that $\prod_i$ is included in the selected subset with higher probability under Y than under X. Just rules yield their smallest probability of a correct selection (i.e. including the stochastically largest, resp., in case of ties, an arbitrarily tagged population) if the populations are identically distributed. Just rules, that are invariant under permutations, are monotone, i.e. $P\{\text{including} \prod_i\} \geqq P\{\text{including} \prod_j\}$ if $\prod_i$ is stochastically larger than $\prod_j$. Let $(\mathscr{X}, \mathscr{A}, \mathscr{P})$ be a probability space and let $\succ$ denote a partial order relation in $\mathscr{X}$. $A \in \mathscr{A}$ is called an increasing set if $X \in \mathscr{A}$, $Y \succ X$ implies $Y \in A$. $P \in \mathscr{P}$ is called stochastically larger in the generalized sense than $Q \in \mathscr{P}$ if $P(A) \geqq Q(A)$ holds for all increasing sets A. Above results hold for the generalized definition of stochastical ordering. Examples for just rules are given. (Received July 3, 1970.)

126-61. Payoffs of probability forecasters and a theorem of McCarthy. ARLO HENDRICKSON, University of Minnesota.

J. McCarthy (*Proc. Nat. Acad. Sci.* **42** (1956) 654–655) gave a theorem on payoff rules which "keep the forecaster honest." The theorem is corrected by a slight modification and is generalized to a Hilbert space. The domain of the payoff is taken to be a convex set $\mathscr{P}$ of densities with respect to a measure μ on a measure space $(\mathscr{X}, \mathscr{A})$. The Hilbert space $\mathscr{H}$ is taken to be a closed subspace of $\mathscr{L}_2(\mu)$. If $\mathscr{P} \subset \mathscr{H}$ and if a payoff rule f has its range in $\mathscr{H}$, then the expectation function defined by $H(p) = E(f(p) \mid p)$ is the usual inner product $\langle p, f(p) \rangle$. By generalizing Rockafellar's (*Convex Analysis* (1970) Princeton Univ.) definition of subgradient in the finite dimensional case, McCarthy's condition for a payoff rule to encourage honesty becomes equivalent to the condition that the expectation function H have a subgradient $f(p)$ at each point $p \in \mathscr{P}$. The class of continuous, convex and homogeneous functions on the convex cone $\{\lambda p : p \in \mathscr{P}, \lambda \geqq 0\}$ are among the class of expectation functions of payoff rules which encourage honesty. An expectation function H of such a payoff rule f is also characterized as being a support function of a closed set $C \subset \mathscr{H}$, where p is normal to C at $f(p)$ for all $p \in \mathscr{P}$. (Received July 3, 1970.)

126-62. On some subset selection procedures for restricted families of probability distributions (preliminary report). S. PANCHAPAKESAN, Purdue University.

Let $\pi_1, \pi_2, \cdots, \pi_k$ be k independent populations. Associated with π_i is an rv X_i having a continuous distribution F_i, $i = 1, 2, \cdots, k$. It is assumed that there exists one among the k populations, denoted by $F_{[k]}$, which is stochastically larger than any other. We are primarily interested in selecting a subset of the k populations such that the one associated with $F_{[k]}$ is included with a probability at least equal to P^*. Barlow and Gupta (*Ann. Math. Statist.* **40** 905–917) considered this problem in the case where the functional forms of the F_i are not known, but it is assumed that the F_i are partially ordered in some sense w.r.t. a completely known continuous distribution G. The present paper unifies some of the procedures considered by Barlow and Gupta by defining a general partial order relation on the space of distributions. It also generalizes a lemma of Gupta (*Multivariate Analysis*, ed. P. R. Krishnaiah (1966) 457–475) in this context. Some specific results are obtained for special choices of G in the case of tail ordering, a special case of the general ordering relation. (Received July 3, 1970.)

126-63. Sequential estimation of a restricted mean parameter of an exponential family. GEORGE P. MCCABE, JR., Purdue University.

Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables whose common distribution is some unknown member of an exponential family, i.e. $f_\theta(X) = \exp(\theta T(X) - C(\theta))$ is the density function with respect to some σ-finite measure on the real line. The unknown parameter θ is assumed to be a member of the countable set Ω. From a finite number of observations on the sequence of random variables, it is desired to guess the true value of the parameter θ with a uniformly (for all θ) small probability of error. Under certain restrictions on the set Ω, a sequential solution based on the work of Robbins in "Sequential estimation of an integer mean" (to be published in the Herman Wold *Festschrift*) is derived. Corresponding to a given error probability bound ε, a stopping rule and terminal decision function are given. Asymptotic (as $\varepsilon \to 0$) expressions for the expected sample size are calculated and asymptotic optimality is established under certain conditions. (Received July 3, 1970.)

126-64. On the derivation of the asymptotic distribution of the generalized Hotelling's $T_0{}^2$. TAKESI HAYAKAWA, University of North Carolina and Institute of Statistical Mathematics, Tokyo.

The asymptotic distribution of the generalized Hotelling's $T_0{}^2$ in noncentral case was derived by Siotani (Mimeo Series No. 595 (1968) UNC) up to order $N_2{}^{-2}$ by the use of the idea of perturbation of physics. In this report we give another method which is simpler than one of Siotani. To derive it, we prepare three fundamental formulas of weighted sum of the generalized Laguerre polynomials and nine formulas of sum of the univariate Laguerre polynomials. By combining these formulas, we derive an asymptotic probability density function of $T_0{}^2$ which is valued for $|T_0{}^2| < N_2$. (Received July 6, 1970.)

126-65. Some moments of an estimate of Shannon's measure of information. KERMIT HUTCHESON AND L. R. SHENTON, University of Georgia.

An underlying multinomial distribution is assumed and from this the first two moments of an estimate of Shannon's measure of information are derived. A table of the first two moments for varying sample size is given for a trinomial. (Received July 6, 1970.)

126-66. A note on minimax estimation of location and scale parameters (preliminary report). GOVIND S. MUDHOLKAR AND SIDDHARTHA R. DALAL, University of Rochester.

Let $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ be a random sample from a multivariate distribution with parameter $\boldsymbol{\theta} \in R^k$, a k dimensional Euclidean space. Suppose that a statistic T is sufficient for $\boldsymbol{\theta}$ with density function $f_{\boldsymbol{\theta}}^T(\mathbf{t}) = f(\mathbf{t}-\boldsymbol{\theta})$ and in the problem of estimating $\boldsymbol{\theta}$ by estimator $\hat{\boldsymbol{\theta}}$ the loss function is $L(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}) = L(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$, $\hat{\boldsymbol{\theta}} \in R^k$; then often the problem of minimax estimation of $\boldsymbol{\theta}$ can be reduced to that of minimization of $E[L(\mathbf{t}-\mathbf{a}) \mid \boldsymbol{\theta} = \mathbf{0}]$ with respect to $\mathbf{a} \in R^k$. The problem can be solved when f and L are of particular form. More specifically let f and L belong to convex cones generated respectively by characteristic functions of G-invariant, convex, compact sets in R^k and their complements, where G is a group of linear Lebesgue measure preserving transformation of R^k into R^k. Then frequently the minimization problem can be handled with ease and under certain mild regularity conditions minimax estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ can easily be found given a group G. Particular cases of G and applications are considered. The results can be interpreted as robustness properties of certain decision rules. (Received July 6, 1970.)

126-67. Admissibility of certain location invariant multiple decision procedures. MARTIN FOX, Michigan State University.

Random variables X, Y_1, $Y_2, \cdots$ are available for observation with X real valued and Y_1, $Y_2, \cdots$ taking values in arbitrary spaces. The distribution of $Y = (Y_1, Y_2, \cdots)$ is given by μ_j $(j = 1, \cdots, r)$ and the conditional density with respect to Lebesgue measure given $Y_i = y_i$ $(i = 1, \cdots, n-1)$ is $p_{jn}(x-\theta, y)$ where $y = (y_1, y_2, \cdots)$. The parameters j and θ are unknown. A decision $k \in \{1, \cdots, m\}$ is to be made with loss $W(j, k, n, y)$ when n observations are taken. Let $\xi = (\xi_1, \cdots, \xi_r)$ be a prior distribution on $\{1, \cdots, r\}$ with each $\xi_j > 0$. Following Brown's [*Ann. Math. Statist.* **37** (1966) 1087–1136] methods admissibility is proved for the decision procedure which is Bayes in the class of invariant procedures. The result contains that of Lehmann and Stein [*Ann. Math. Statist.* **24** (1953) 473–479]. (Received July 6, 1970.)

126-68. Order statistics for random sample size. S. A. PATIL, K. RAGHUNANDANAN AND J. L. KOVNER, USDA, Forest Service, Temple University, USDA, Forest Service.

The paper presents the distributions of ith and $(n-i+1)$th order statistic when the sample size n has a truncated binomial or Poisson distribution. Epstein (*Ann. Math. Statist.* **20** (1949) 99–103) has studied the case when n has Poisson distribution and the random variable X is defined over a finite range. The joint distribution of ith and jth order statistic is also given. For the uniform distribution, the expressions for the variances and covariances are given. The results are used to find confidence intervals for the population percentiles and the tolerance limits when the sample size is random. An application to sampling insect size, representing a biological population, is given. (Received July 6, 1970.)

126-69. Estimation of monotone parameters and the Kuhn–Tucker conditions. (preliminary report). PETER E. NÜESCH, The Johns Hopkins University.

The van Eeden [*Indag. Math.* **18** (1956) 444–455, ibid. **19** (1957) 128–136 and 201–211]—Brunk [*Ann. Math. Statist.* **29** (1958) 437–454] stepwise approach to obtain the maximum likelihood estimates of parameters restricted by inequalities is related to solving a nonlinear programming problem. The necessary conditions for pooling means are equivalent to the Kuhn–Tucker [*Proc. Second Berkeley Symp. Math. Statist. Prob.* (1951) 481–492] conditions of a saddle point. In case of a quadratic objective function (normal distribution) the pooling conditions are sufficient as are the Kuhn–Tucker conditions for a quadratic program. (Received July 6, 1970.)

126-70. Iterated logarithm laws for sample df and quantile processes. J. KIEFER, Cornell University. (Invited)

Throughout, $n^{-1}T_n$ and $Z_n(p)$ denote the sample df and sample p-tile ($=np$th order statistic) from n i.i.d. rv's uniformly distributed on [0, 1], and $p_n \downarrow 0$. Bahadur [*Ann. Math. Statist.* **37** (1966), 577] studied the relation between the Z_n and T_n processes, from which he derived a $Z_n(p)$ LIL (fixed p). Our refinements [*Ann. Math. Statist.* **38** (1967) 1323; *Proc. First Nonpar. Symp.*, 299 and 349 (1969)], especially considerations of the behavior of $\sup_{x<p_n}[f_n(x)T_n(x)+g_n(x)Z_n(h(x))+r_n(x)]$, are now continued. (Earlier related work is also due to Chung, Baxter, Eicker, Cibisov, LeCam, and others.) Typical consequences: If $np_n = o(\log_2 n)$ and $k_n = \log_2 n/\log[(np_n)^{-1}\log_2 n] \to \infty$, then $\limsup_n T_n(p_n)/k_n = 1$ wp 1; if np_n is constant, this is Baxter's result. (The case of bounded k_n was treated earlier.) If $np_n/\log_2 n \to c =$ positive constant, and $\beta_c > 1$ satisfies $\beta_c(\log\beta_c - 1) = (1-c)/c$, then $\limsup_n T_n(p_n)/c\beta_c \log_2 n = 1$. These and their lim inf analogues yield corresponding results on $Z_n(p_n)$. (Strong form for $Z_n(1/n)$ upper oscillations was recently announced by Robbins and Siegmund, June 1970 abstract, using a more novel technique than our standard binomial or normal estimates.) If $np_n/\log_2 n \to \infty$, then the standard LIL form $p_n \pm [2n^{-1}p_n \log_2 n]^{1/2}$ applies for both $n^{-1}T_n(p_n)$ and $Z_n(p_n)$, partly proved earlier by Eicker using finer but lengthier probability estimates for T_n crossing polygons. (Received July 6, 1970.)

126-71. Some properties of the simultaneous ANOVA tests. GOVIND S. MUDHOLKAR, University of Rochester.

In the general linear model the problem of testing several hypotheses simultaneously can be handled either by the customary analysis of variance procedure or by the simultaneous analysis of variance procedure due to M. N. Ghosh. We study some monotonicity properties of the power function of the simultaneous analysis of variance test for certain restricted alternatives. It is shown that the SANOVA test is more powerful than the ANOVA test against these alternatives. We also study some decision theoretic properties such as the Bayes character of the simultaneous analysis of variance test. (Received July 6, 1970.)

126-72. Admissibility of the *ML* rule in multivariate classification problems based on incomplete samples. J. N. SRIVASTAVA AND N. K. ZAATAR, Colorado State University.

The problem of finding an optimal decision rule for classifying a p-response observation into one of two p-variate normal populations with unknown mean vectors μ_1, μ_2, and known common dispersion matrix Σ, is considered. The inference is based on two random general incomplete samples, one from each population. The units in each sample may be incomplete in the general sense that any proper subset of the p responses may be missing in any unit. The maximum likelihood decision rule is obtained and proven to be admissible with respect to a zero-one loss function. (Received July 6, 1970.)

126-73. On a method of sum composition of orthogonal Latin squares II. A. HEDAYAT AND E. SEIDEN, Cornell University and Michigan State University.

In an abstract published in *Ann. Math. Statist.* **41** (1970), 752 a new method of construction of a pair of orthogonal Latin squares, called sum composition, was introduced. It was shown that for any $n = n_1 + n_2$, $n_1 = p^\alpha$, p a prime greater or equal to 7 and $p \neq 13$, and α a positive integer one can give a uniform rule for constructing a pair of orthogonal Latin squares of order n provided that n_2 assumed its maximum value. Presently the case of $n = n_1 + 3$ is investigated for n_1 satisfying the above conditions. In this case $n_2 = 3$ hence it assumes its minimum value allowed for the construction using the method of sum composition. Again a uniform rule for construction was found for primes of some specific structures. The rule then depends on the structure of the prime but not on its particular value. It is conjectured that the result can be extended to include all primes. This problem is now under investigation. More precisely the following theorem was obtained. THEOREM. *A pair of orthogonal Latin squares of order* $p^{\alpha}+3$, *p a prime greater or equal to* 7 *and* α *a positive integer, can be constructed by a sum composition of a pair of orthogonal Latin squares of order* p^{α} *(based on Galois field) and a pair of orthogonal Latin squares of order* 3 *if p has any of the following forms;* $3m+1$, $8m+1$, $8m+3$, $24m+11$, $60m+23$ *and* $60m+47$, $m \geqq 0$. COROLLARY. *For infinite values of t, one can construct a pair of orthogonal Latin squares of order* $4t+2$, *t a positive integer, which are obtained by a sum composition of a pair of orthogonal Latin squares of order* $4t-1$ *and a pair of orthogonal Latin squares of order* 3. (Received July 6, 1970.)

126-74. An immigration and fragmentation stochastic process. I. N. SHIMI AND D. W. FAIRWEATHER, Florida State University.

A population of different types of particles is considered where immigration of the various particles into the population is allowed. The immigration causes "fragmentation" or branching of the existing target particles in the population. An immigrating particle can also fragment into the different types. By an appropriate choice of the parameters of this model, one can obtain the Harris k-type branching process and a generalization of this process to include immigration. Furthermore, one can obtain a generalization of the Smith and Wilkinson model [*Ann. Math. Statist.* **40** (1969) 814–827] to a k-type process in discrete random environment where immigration is allowed. The joint moment generating function for the number of particles in the mth generation and the expected number of particles in the mth generation are given. Let $\mathbf{N}_m$ be the random vector representing the number of particles of the different types in the mth generation. The almost sure convergence of $\mathbf{N}_m$ and other limits of functions of $\mathbf{N}_m$ for a k-type process in discrete random environment with and without immigration are obtained when $\rho \leqq 1$, where ρ is the largest eigenvalue of a first moment matrix. In particular it is shown that for the process considered by Joffe and Spitzer [*J. Math. Anal. Appl.* **19** (1967) 409–430] $\mathbf{N}_m \to \mathbf{0}$ a.s., not just in probability. (Received July 6, 1970.)

126-75. Characterization and characteristic functions of Dirichlet distributions. H. YASAIMAIBODI, The George Washington University.

The purpose of this paper is to extend the investigations in Yasaimaibodi (Thesis, The George Washington University (1966)), by giving two theorems which characterize, respectively, the Dirichlet and inverted Dirichlet distributions. The theorem for characterizing a Dirichlet distribution (DD) may be stated as follows. The vector random variable (VRV), $\mathbf{X} = (X_1, X_2, \cdots, X_k)$ has the k-variate Dirichlet distribution, $D(v_1, v_2, \cdots, v_k; v_{k+1})$, if, and only if, the VRV $\mathbf{Y} = (Y_\cdot, Y_2, \cdots, Y_{k-1})$ and random variable Z are distributed independently as the $(k-1)$-variate DD, $D(v_1, \cdots, v_{k-1}; v_k)$ and beta distribution $Be(\sum_{j=1}^{k} v_j; v_{k+1})$, respectively, where $X_r = Y_r Z$, and $Z = \sum_{j=1}^{k} X_j$, $r = 1, \cdots, k-1$; cf. Wilks *Mathematical Statistics* (1963), for notations. An operator for generating DD's of lower dimensions is introduced; and, finally, the characteristic functions of above two distributions are obtained. (Received July 6, 1970.)

126-76. Infinite divisibility of a multivariate gamma distribution. S. R. PARANJAPE, Miami University and Aerospace Research Laboratories.

Let $X_1, \cdots, X_k$ be k random variables jointly normally distributed with zero means, unit variances and a correlation matrix $\sum$. The k-dimensional characteristic function of $(X_1^2, \cdots, X_k^2)$ is known to have the form $\phi(t) = [\det |I - 2T_\alpha \Sigma|]^{-1/2}$ where $T_d = \operatorname{diag}(it_1, \cdots, it_k)$, $i = -1^{1/2}$. The question whether $\phi^{\alpha}(t)$ is a characteristic function for any real $\alpha > 0$ has been answered in two special cases. This paper proves that $\phi^{\alpha}(t)$, for any real $\alpha > 0$, is a characteristic function. (Received July 6, 1970.)

126-77. Multivariate stable distributions. JAMES S. PRESS, University of Chicago.

A class of multivariate stable distributions is introduced which has the property that linear combinations of the components of independent random vectors belong to the same family of stable laws. Various properties of these laws are discussed and an application of the theory to the problem of optimal allocation of resources is indicated. The class is illustrated with various examples. (Received July 6, 1970.)

126-78. Combining independent tests of significance. RAMON C. LITTELL AND LEROY FOLKS, University of Florida and Oklahoma State University.

Four methods of combining independent tests of significance are compared via exact Bahadur relative efficiency. The methods considered are Fisher's method, the mean of the normal transforms of the significance levels, the maximum significance level, and the minimum significance level. None of the methods are uniformly more powerful than the others, but, according to Bahadur efficiency, Fisher's method is the most efficient of the four. In some cases, Fisher's method is most efficient of all methods of combining, but this is not generally true. (Received July 7, 1970.)

(Abstracts of papers contributed by title.)

70T-64. Large sample and other multiple comparisons among means. HANS K. URY AND ALVIN D. WIGGINS, California State Department of Health and University of California, Davis.

If the number of multiple comparisons for K large samples is limited to the $\binom{K}{2}$ differences between pairs of means plus "a few" for possible post hoc multiple comparisons, one can frequently obtain shorter confidence intervals using a t-statistic than through the use of Marascuilo's procedure (*Psychological Bulletin* **65** (1966) 280–290) for large samples. The savings in the length of the confidence interval increase with the number of samples and with the confidence level. The length of the t-intervals is compared with the length of Marascuilo's intervals under various conditions in Tables 1–6. Modifications are given for use in the case of moderate and small samples. A numerical example (Marascuilo's Example One) is presented, and two nonparametric analogues of the procedure are given. (Received May 22, 1970.)

70T-66. A set of three orthogonal Latin squares of order 15 associated with a Kirkman–Steiner triple system of order 15. A. HEDAYAT, Cornell University.

In the following by an $O(n, t)$ set we mean a set of t mutually orthogonal Latin squares of order n. Hedayat and Raktoe [Kirkman–Steiner triple systems and sets of mutually orthogonal Latin squares: presented at the 125th meeting of IMS held at Chapel Hill, North Carolina] have shown that every Kirkman–Steiner triple system of order $v = 3 \pmod 6$ implies a special $O(v, 2)$ set. In this paper we have shown that the case $v = 15$ leads to a new result, namely the existence of an $O(15, 3)$ set, by showing that one of the Latin squares in the derived $O(15, 2)$ set corresponds to the Kirkman–Steiner triple system given by H. J. Ryser [see page 102 of *Combinatorial Mathematics*, Wiley, 1963] admits a special orthogonal mate which can be embedded in an $O(15, 3)$ set. Whether or not this $O(15, 3)$ set can be embedded in a larger set is under investigation. (Received June 3, 1970.)

70T-67. Minimax estimators for the mean of a multivariate normal distribution. HUI-LIANG TSAI AND PI-EHR LIN, Florida State University.

Baranchik (*Ann. Math. Statist.* **41** 642–645) obtained a family of minimax estimators for the mean μ of a p-variate ($p > 2$) normal distribution with covariance matrix $\Sigma = \sigma^2 I$. Here we treat the more general case when Σ is positive definite but otherwise unknown. Let $\mathbf{X} \sim N(\mu, \Sigma)$ and $S \sim W(n, \Sigma)$, a Wishart distribution with parameters (n, Σ), independent of $\mathbf{X}$. We show that $\delta(\mathrm{X}, S) = [1 - r(\mathrm{X}'S^{-1}\mathrm{X})/\mathrm{X}'S^{-1}\mathrm{X}]\mathrm{X}$ is a minimax estimator of μ with loss function $[\delta(\mathrm{X}, S) - \mu]'\Sigma^{-1}[\delta(\mathrm{X}, S) - \mu]$, if $r(\cdot)$ is a nonnegative, non-decreasing measurable real function less than or equal to $2(p-2)(n-p+3)^{-1}$. (Received June 3, 1970.)

70T-68. Invariance theorems for first passage time random variables (preliminary report). ADHIR KUMAR BASU, Queen's University.

Let $x_1, x_2, \cdots$ be i.i.d. rv with $Ex = \mu > 0$, and $E(x-\mu)^2 = \sigma^2 \dot{<} \infty$. Let $S_k = x_1 + \cdots + x_k$ and $\nu_x = \max\{k : S_k \leqq x\}$, $x \geqq 0$ and $\nu_x = 0$ if $x_1 > x$. Billingsley proved if $x_1 \geqq 0$ then $Z_n(x, w) = (\nu_{nx}(w) - nx/\mu)/\sigma\mu^{-3/2}n^{1/2}$ converges weakly to the Weiner measure W. Let $\tau_x(w) = \inf\{k \geqq 1 \mid S_k > x\}$. In Section two we proved that $T_n(x, w) = (\tau_{nx}(w) - nx/\mu)/\sigma\mu^{-3/2}n^{1/2}$ converges weakly to the Weiner measure when the x's may not necessarily be nonnegative. Also we indicated that this result can be extended to nonidentical case. In Section three we proved that certain first passage time random variables of partial sums of i.i.d. rv with mean zero (with positive mean) and finite variance tend to corresponding first passage time rv of Brownian motion (with positive drift). (Received June 4, 1970.)

70T-69. Mean squared error properties of a family of Stein–James estimators. SERGE L. WIND, Columbia University.

Cogburn (*Bernoulli Bayes Laplace Anniversary Volume* (1965) 24–29) discussed the properties of the risks of a family of estimators of a multivariate location parameter, under the assumption that the covariance matrix of the observed random variable was known. In this paper, bounds on the risks, under squared error loss, of such estimators are given for both fixed and random unknown location parameters when the covariance matrix is unknown. Results are also presented for the instance when a prior (Bayes) distribution is imposed on α when the covariance matrix is αI. The decision function proposed by James and Stein (*Proc. Fourth Berkeley Symp. Math Statist. Prob.* **1** (1961) 361–379) for estimating the mean of a multivariate normal distribution is a member of the family considered, but here no specific parametric distribution is assumed on either the underlying distribution or on any prior distribution. (Received June 9, 1970.)

70T-70. An empirical Bayes approach to the multiple linear regression problem. SERGE L. WIND, Columbia University.

We consider estimation, subject to a quadratic loss function, of the coefficient parameters β of the multiple linear regression model $Y = X\beta + \varepsilon$, with $E\varepsilon = 0$ and $\operatorname{Cov}\varepsilon = \alpha I$ (but with no parametric assumptions on the distribution of the error variable ε). β is assumed to be a vector random variable, distributed according to an unknown prior distribution. Relative to equivalence classes or subsets of the space of all prior distributions which group distributions with the same specified moments, restricted minimax solutions (see Cogburn, *Ann. Math. Statist.* **38** (1967) 447–463) are exhibited. Given that the regression problem occurs repeatedly and independently with the same prior throughout, the classic empirical Bayes formulation, we determine restricted asymptotically optimal estimators—i.e. decision functions whose Bayes risk approaches the risk of the restricted minimax decision at each component stage. Criteria for handling nuisance parameters which vary with the component problem are proposed and then applied to the regression problem where the variances of the error variables and the set of independent observations vary. Restricted minimax and restricted asymptotically optimal estimators are given for one approach which assumes a joint prior on all the parameters. (Received June 9, 1970.)

70T-71. On a Markovian queue with a general bulk service rule. J. MEHDI AND A. BORTHAKUR, University of Gauhati.

Neuts (*Ann. Math. Statist.* (1967)) has considered a single-channel bulk queue with Poisson input and a general service rule. Here the server will start service only when a minimum number A of customers is present; if the number lies between A and B, entire queue length is taken for service and if the number exceeds B, the first B customers are taken into service. This rule is quite general and other bulk service rules can be treated as particular cases of this. Neuts has obtained many elegant results which include the probability of the number in the system. In this paper we consider a Markovian queue under this general bulk service rule. By straightforward use of Laplace transforms we find the probability of the number in the queue as well as the number of channels in operation. This result is not implied from Neut's results. The two channel case has also been investigated and extension to the multi-channel case has been indicated. (Received June 10, 1970.)

70T-72. Estimating the age of a Galton–Watson branching process. STEPHEN M. STIGLER, University of Wisconsin.

Point estimates of, confidence intervals for, and hypothesis tests about the age (in generations) of a Galton–Watson process with known offspring distribution are given for the supercritical and critical cases. Extensions to other branching processes are discussed briefly. (Received June 15, 1970.)

70T-73. On some properties of Hammersley's estimator of an integer mean and related sequential procedures. RASUL A. KHAN, Columbia University.

Let $X_1, X_2, \cdots, X_n$ be i.i.d. $N(i, 1)$, $i = 0, \pm 1, \pm 2, \cdots$ Hammersley (*J. Roy. Statist. Soc.*, Ser. B **12** (1950) 192–240) proposed $d = n.i.(\bar{X}_n)$ (nearest integer to the sample mean) as an estimator of i. Lindley suggested that the estimator is minimax relative to 0–1 loss, and Stein conjectured its minimaxity relative to square error loss (see discussion in the above paper). It is proved that d is admissible and minimax relative to 0–1 loss while relative to square error loss the estimator is neither admissible nor minimax. Robbins (Sequential estimation of an integer mean, to appear in the Herman Wold Festschrift) gave a sequential rule N which is asymptotically optimal and better than fixed sample size procedure relative to 0–1 loss. It is shown that N is exponentially bounded i.e. $P_i(N > n) \leqq C\rho^n$, $C > 0$, $0 < \rho < 1$ which implies that (i) $E_i N < \infty$ (ii) $E_i \exp(tN) < \infty$ for some $t > 0$ and hence (iii) $E_i N^k < \infty$ for all $k \geqq 1$ and all i. Moreover, we consider non-countable family of normal distributions $\{N(i, \sigma^2): i = 0, \pm 1, \pm 2, \cdots, 0 < \sigma < \infty\}$ (i and σ unknown) and give a simple two-stage sequential procedure for reaching a decision about unknown i with the property $\sup_{i,\sigma^2} P_{i,\sigma^2}$ (error) $\leqq \varepsilon$ for any given ε $(0 < \varepsilon < 1)$. (Received June 16, 1970.)

70T-74. On sequential distinguishability. RASUL A. KHAN, Columbia University.

Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables governed by a member of a countable family $\mathscr{P} = \{P_\theta : \theta \in \Omega\}$ of probability measures having densities, $f(\Omega) = \{f_\theta(x) : \theta \in \Omega\}$ with respect to some σ-finite measure μ (such a measure μ trivially exists). The family $\mathscr{P}$ is said to be sequentially distinguishable if there exists a stopping rule t and a terminal decision $\delta(X_1, \cdots, X_t)$ such that $P_\theta(t < \infty) = 1 \,\forall\, \theta \in \Omega$ and $\sup_{\theta \in \Omega} P_\theta(\delta(X_1, \cdots, X_t) \neq \theta) \leqq \varepsilon$ for any given ε $(0 < \varepsilon < 1)$. A stopping time N defined essentially in terms of likelihood ratio (though mixed up with an appropriate double indexed sequence of constants) has been studied and conditions have been found so that the family $\mathscr{P}$ is sequentially distinguishable through N under different modes of sequential distinguishability (cf. Hoeffding and Wolfowitz, *Ann. Math. Statist.* **29** (1958) 700–718). The stopping time N was suggested by Prof. Robbins and the motivation is the Wald's SPRT. This work therefore generalizes, in a sense, the SPRT to countable many simple hypotheses with uniformly small probability of error. The essential tools in the investigation are certain measures of divergence, namely total variation metric, Hellinger's integral and Kullback–Leibler information measure etc. and the usual conditions of strong consistency of likelihood ratio. (Received June 16, 1970.)

70T-75. On sequential distinguishability for some exponential distributions. RASUL A. KHAN, Columbia University.

The methods developed in Khan (see abstract, On sequential distinguishability) are applied to some important distributions. Let $X_1, X_2, \cdots$ be i.i.d. Poisson (P_λ) random variables assumed to be governed by a member of a countable family $\mathscr{P} = \{P_\lambda : \lambda \in \Lambda\}$ where $\Lambda = \{\lambda_i : i = 1, 2, \cdots, 0 < \lambda_1 < \lambda_2 < \cdots, \lambda_i - \lambda_{i-1} \geqq 1\}$ and by convention $\lambda_0 = 0 \notin \Lambda$. Based on the observations we want to decide in favour of a member of $\mathscr{P}$ with uniformly small probability of error. A stopping rule N (depending on $\alpha > 1$) leading to a terminal decision is obtained. The procedure is such that (i) $P_\lambda(N < \infty) = 1 \ \forall\ \lambda \in \Lambda$ (ii) $\sup_{\lambda \in \Lambda} P_\lambda(\text{error}) \leqq 2/(\alpha - 1)$ (iii) $P_{\lambda_i}(N > n) \leqq C\rho^n$, $C > 0, 0 < \rho < 1$ etc. (iv) $E_{\lambda_i} N \sim K_i \log \alpha$ as $\alpha \to \infty$ where $K_i = [\min\{(1 - \lambda_i(\lambda_{i+1} - \lambda_i)^{-1} \log \lambda_{i+1}/\lambda_i), (\lambda_i(\lambda_i - \lambda_{i-1})^{-1} \log \lambda_i/\lambda_{i-1} - 1)\}]^{-1}$. The rule is asymptotically optimal. The problem is also done without the restriction $\lambda_i - \lambda_{i-1} \geqq 1$. Similar techniques are applied to $\mathscr{P} = \{N(\theta, 1) : \theta \in \Omega\}$ where $\Omega = \{\theta_i : i = 0, \pm 1, \pm 2, \cdots, -\infty < \cdots < \theta_{-1} < \theta_0 < \theta_1 < \cdots < \infty, \theta_i - \theta_{i-1} \geqq 1\}$ and a stopping rule similar to that of Robbins (Sequential estimation of an integer mean, to appear in the Herman Wold Festschrift) is obtained with properties (i) through (iii) and (iv) $E_{\theta_i} N \sim 2C_i \log \alpha$ as $\alpha \to \infty$ where $C_i = [\min\{(\theta_{i+1} - \theta_i), (\theta_i - \theta_{i-1})\}]^{-1}$. The methods can be applied to more general exponential models. (Received June 16, 1970.)

70T-76. Testing the mean of a normal population using weighted means (preliminary report). S. R. SRIVASTAVA, Lucknow University.

Given two independent samples each of size n from two normal populations $N(\mu_1, \sigma^2)$, $N(\mu_2, \sigma^2)$, σ^2 known, Mosteller (*J. Amer. Statist. Assoc.* **43** (1948) 231–242) has given an estimate of μ_1 obtained by pooling of observations on the basis of a preliminary test of significance and also a maximum likelihood pooling estimate when $\mu_1 - \mu_2$ is assumed a priori distributed as $N(0, a^2\sigma^2)$. In the present report we have studied the power function of the test statistics based on these two estimates of μ_1 for the hypothesis $H_0 : \mu_1 = \mu_0$ vs $H_1 : \mu_1 \neq \mu_0$. The results obtained show that the test based on maximum likelihood pooling estimate has more power than that of a newer pool test for H_0. (Received June 16, 1970.)

70T-77. Sufficient partitions of the sample space and generalized B-Pitman statistics. C. B. BELL AND VIKTOR KUROTSCHKA, University of Michigan.

Let (S, Σ) be a topological transformation group of the sample space $(\mathscr{X}, \mathfrak{A})$ where $\mathfrak{A}$ is the Borel σ-algebra generated by some "nice" topology on $\mathscr{X}$. S will then be called a symmetry group for the family $\mathscr{P} := \{p_\theta; \theta \in \Omega\}$ of probability measures on $\mathfrak{A}$ if for every $s \in S$ and every $\theta \in \Omega$ yields $p_\theta \circ s = p_\theta$. Let $\sigma(\Sigma)$ denote the Borel σ-algebra generated by Σ. Then compactness of (S, Σ) is equivalent to the existence of finite Haar measure on $\sigma(\Sigma)$, which allows one to prove that $\mathfrak{A}^s := \{A \in \mathfrak{A}; s(A) = A \text{ for all } s \in S\}$, is a sufficient σ-algebra for $\mathscr{P}$. In other words, every maximal invariant with respect to a compact symmetry group S for $\mathscr{P}$ and in particular the mapping $T : \mathscr{X} \ni x \to S(x) := \{s(x); s \in S\} \in \mathscr{X}/S$ which partitions $\mathscr{X}$ in S-orbits is sufficient for $\mathscr{P}$. Besides the well-known case of finite symmetry groups (which are trivially compact) this result also covers compact infinite symmetry groups such as the orthogonal group, products of orthogonal groups, and wreath products of orthogonal groups with the symmetric group, acting on Euclidean sample spaces. These examples appear in problems related to spherically symmetric experiments. In addition if $\mathscr{P}$ is S-complete then a characterization of the family of all $\mathscr{P}$-similar sets as well as $\mathscr{P}$-distribution-free statistics can be obtained by introducing generalized B-Pitman statistics defined in terms of the above mentioned Haar measure. Applications to nonparametric problems where families of underlying probability distributions are exhibited by symmetry properties are obvious. (Received June 19, 1970.)

70T-78. Finding a single defective in binomial group-testing. S. KUMAR AND MILTON SOBEL, University of Wisconsin-Milwaukee and University of Minnesota.

The problem of finding a single defective item from an infinite binomial population when the group-testing is possible, i.e., when we test any number of units x simultaneously and find out if all the x are good or if at least one of x defective is present. An optimal procedure is obtained in the sense that it minimizes the expected number of tests required to find one defective. Upper and lower bounds are derived using information theory and the relation of our procedure to the Huffman algorithm and the corresponding cost is studied. (Received June 22, 1970.)

70T-79. Estimating the scale parameter of the exponential distribution with unknown location. J. V. ZIDEK, University of British Columbia.

Let $X_1, \cdots, X_n$ be i.i.d. random variables each with a density which is $a^{-1}\exp(b-x)$ or 0 according as $x \geqq b$ or $x < b$ and $-\infty < b < \infty$, $a > 0$ are unknown constants. Let $\bar{X} = n^{-1}. \Sigma X_i$ and $M = \min X_i$. The maximum likelihood estimator, $\hat{t}$, of a is $\hat{t} = \bar{X} - M$. If loss is measured by any member of a certain class, L, of strictly convex functions, then $\hat{t}$ is inadmissible. The result is proved by showing that if the loss is in L, that $\hat{t}$ is not the pointwise limit of any sequence of non-randomized estimators which are Bayes within the class of scale invariant estimators, D. It is a necessary condition for the admissibility of $\hat{t}$, a scale invariant estimator, that it be such a limit. This necessary condition is stated and proved in a general setting. Squared error loss is an element of L. If it is chosen, a simple argument reveals that if $\hat{u} = (\bar{X}-M)T(M(\bar{X}-M)^{-1})$; where $T(y) = 1$, $y < 0$ and $\min(1, n(1+y)(n+1)^{-1}) \leqq T(y) \leqq 1$, $y \geqq 0$, then $\hat{u}$ is minimax. Using a generalization of the author's conditions for admissibility [*Ann. Math. Statist.* **41** (1970) 446–457] a class, B, of generalized Bayes estimators within D are obtained with each member of B admissible in D. The improper measures determining members of B have densities on the orbit space, R, created in the parameter space by the action of the group of scale changes. These prior densities, g, satisfy certain regularity conditions and $\int_1^\infty (t^2g(t))^{-1}\,dt = \int_{-\infty}^{-1} (t^2g(t))^{-1}\,dt = \infty$. (Received June 29, 1970.)

70T-80. Information through uncertainty functions. GEORGE T. DUNCAN, University of Minnesota.

Certain information theoretic results based on the Shannon entropy function are extended to results about uncertainty functions, as defined by DeGroot (*Ann. Math. Statist.* **33** (1962) 404–419). Results of Rényi (*Studia Sci. Math. Hungar.* **2** (1967) 249–256) concerning data reduction and sufficiency are generalized. Countable state space results are achieved through a version of Jensen's inequality which is valid for a function from sequence space. Payment schedules for a forecaster which allow no profit in dishonesty and promote diligence are studied. The relationship between uncertainty which are Bayes risk functions and payment schedules which emphasize the value of information are studied. Information in an observable random variable X about a random parameter θ is defined as the average reduction in uncertainty about θ given X. Minimum average questionnaire charge is examined as an uncertainty function. Questionnaire information is compared to Shannon information. By simultaneously determining a sufficient large number of parameter realizations, the questionnaire information per parameter realization may be made arbitrarily close to the Shannon information per parameter realization. (Received June 30, 1970.)

70T-81. Orthant probabilities for some special Green's and equal correlation structures and their connection with two results in probability theory. PETER E. NÜESCH, The Johns Hopkins University.

If $X_1, X_2, \cdots, X_p$ is a set of identically distributed and equally correlated random variables, then (I) $P[\bigcap_{i=1}^{p-1} \{X_i > X_p\}] = p^{-1}$ and (II) $P[\bigcap_{i=1}^{p-1} \{i^{-1}S_i > p^{-1}S_p\}] = p^{-1}$, where $S_i = X_1 + X_2 + \cdots + X_i$. These two results can be reformulated as orthant probabilities of a $(p-1)$-dimensional random vector with special equal correlation structure (case I) and Green's (following the definition in Karlin, *Total Positivity*, **1**, page 110) correlation structure (case II). The purpose of the paper is to devise a method which proves both results simultaneously. (Received July 6, 1970.)

70T-82. Products of distributions attracted to extreme value laws (preliminary report). SIDNEY I. RESNICK, Israel Institute of Technology.

If $H_1(x), \cdots, H_m(x)$ are right continuous df's, when is $\prod_{i=1}^m H_i(x)$ attracted to an extreme value law $\phi(x)$ ($\prod_{i=1}^m H_i(x) \in \phi(x)$)? Set $x_0{}^i = \inf\{x \mid H_i(x) = 1\}$ and to avoid any of the $H_i(x)$ being extraneous suppose $x_0{}^i = x_0 \leqq \infty$. Define the A-function of $H_i(x)$ to be $A_i(z) = \int_z^{x_0} 1 - H_i(t)\,dt/z(1-H_i(z))$. Then $\prod_{i=1}^m H_i(x) \in \phi(x)$ iff $H_i(x) \in \phi(x)$ and $A_i(z) \sim A_j(z)$, $z \to x_0$, $1 \leqq i, j \leqq m$. In particular, $\prod_{i=1}^m H_i(x) \in \Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$, $x \geqq 0$ iff for each i, $1 - H_i(x)$ is regularly varying with exponent $-\alpha$. Similar results hold for $\phi(x) = \Psi_\alpha(x) = \exp\{-(-x)^\alpha\}$, $x \leqq 0$. The method of analysis shows that the maxima of a sequence of rv's defined on a Markov chain with semi-Markov matrix $\{p_{ij} H_i(x)\}$, $1 \leqq i, j \leqq m$, have a limit law $\phi(x)$ iff $H_i(x) \in \phi(x)$ and $A_i(z) \sim A_j(z)$, $z \to x_0$ (Cf. Resnick and Neuts, *Advances in Appl. Probability* (1970)). Equivalence of A-functions partitions the domain of attraction of $\phi(x)$ into equivalence classes. Tail equivalence considerations imply that these equivalence classes, besides being closed under products, are closed under convex combinations. They are not closed under passages to the limit (complete convergence). (Received July 6, 1970.)

70T-83. A polynomial approximation to Mill's ratio. JOHN S. WHITE, University of Minnesota.

Let X be a reduced normal variable with density function $f(x) = \exp(-x^2/2)/2\pi^{\frac{1}{2}}$. The asymptotic expansion $\text{Prob}(X > x) = Q(x) = f(x)/x(1 - 1/x^2 + 3/x^4 - 15/x^6 + \cdots)$ is well known. If $z(x) = Q(x^{\frac{1}{2}})x^{\frac{1}{2}}/f(x)$, then $z(x)$ satisfies the differential equation $2x^2z'' - (x^2+x)z' + z = 0$. An approximation to $z(x)$ may be obtained by solving the differential equation $2x^2z_n'' - (x^2+x)z_n' + z_n = \tau T_n(1 - 2x_0/x)$, where $T_0(s) = 1$, $T_1(s) = s$, $T_{j+1}(s) = 2sT_j(s) - T_{j-1}(s)$ are the Tchebychef polynomials, x_0 is a positive constant, and τ is determined by the boundary condition $z_n(x) \to 1$ as $x \to \infty$. The solution $z_n(x)$ is of the form $z_n(x) = 1 + a_1 t + \cdots + a_n t^n$ where $t = 1/x$. Meinardus (*Computing* (1966) pages 39–49) has shown that, for $x > x_0$, $|z(x) - z_n(x)| < \tau/3x_0{}^2(x_0 - 1)$. A computer program which generates these approximations has been written and various examples are given. For example, for $n = 5$, $x_0 = 9$, the approximation $z_5(x) = 1 - (275061t - 812808t^2 + 3470040t^3 - 13063680t^4 + 25194240t^5)/275116$, has error less than 10^{-10}. (Received July 8, 1970.)

70T-84. Simplicial distributions (SD) I. ANDRE G. LAURENT, Wayne State University.

Let $\mathbf{X}$ and $\mathbf{U}'$ be n-dimensional vectors, $\mathbf{U}'$ known, $\mathbf{X}$ random with df $f(\mathbf{x}) = g_n(\mathbf{u}'\mathbf{x})$, $u_i > 0$, $x_i \geqq 0$. We then say that $f(\mathbf{x})$ is a simplicial df (special case: x_i's independent with negative exponential distributions). Without loss of generality one will consider $f(\mathbf{x}) = g_n(s_n)$, $s_n = x_1 + \cdots + x_n$, $S_n = X_1 + \cdots + X_n$, sufficient, complete. The conditional df of $\mathbf{X}$, given $S_n = s_n$, is uniform on the E_n simplex, of maximum entropy, and the normalized multivariate Dirac function $\Delta_n(S_n, s_n)$. SD's are obtained by averaging Δ_n by means of df of S_n and generalize the trivial multivariate negative exponential df when conditional uniformity given S_n, but not component independence, is postulated. Let $\mathbf{X}' = (\mathbf{Y}', \mathbf{Z}')$, where $\mathbf{Y}$ is k-dimensional. One gives the cond. df $f_k(\mathbf{y} \mid s_n) = w(\mathbf{y}, s_n)$ of $\mathbf{Y}$ given $S_n = s_n$, which is independent from f, provides the MVU estimator $w(\mathbf{y}, S_n)$ of the df of $\mathbf{Y}$, and the parameter free df of $\mathbf{Y}S_n^{-1}$; and others such as the joint df $f(\mathbf{y}, s_n) = (s_n - s_k)^{n-k-1} g(s_n)/(n-k-1)!$. The latter induces a symbolic calculus with many applications: e.g. when $k = 1$ the df of Y is the $(n-1)$-repeated integral of g_n. Conversely $f_n(\mathbf{x}) = (-1)^{n-1} g_1^{(n-1)}(s_n)$, if a df is an SD which is the multivariate generalization of the univariate $f_1(y) = g_1(y)$. (Received July 13, 1970.)

70T-85. Simplicial distributions (SD) II. ANDRE G. LAURENT, Wayne State University.

Families of SD and their projections are studied as well as statistics whose df is invariant within the class. The paper gives the characteristic function of the uniform distribution on the E_n simplex, i.e., of the normalized simplicial Dirac function, and, more generally, the average of $h(\mathbf{U}'\mathbf{X})$ on the simplex when h is an entire function. A modified multivariate Mellin transform $\mathrm{ML}(f_n; \mathbf{r})$ is presented as the "natural" tool to study SD's. For example, $\mathrm{ML}(f_n; \mathbf{r}) = \mathrm{ML}(g_n; r_1 + \cdots + r_n)$. This transform plays a part similar to the one played by the Hankel transform in the study of spherical distributions. (Received July 13, 1970.)

70T-86. A note on the joint distribution of correlated quadratic forms (preliminary report). C. G. KHATRI, P. R. KRISHNAIAH AND P. K. SEN, Aerospace Research Laboratories and University of North Carolina.

Let $(\mathbf{X}'_{1j}, \cdots, \mathbf{X}'_{pj})$, $(j = 1, 2, \cdots, n)$, be distributed independently and identically as a multivariate normal with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ. Also, let $\mathbf{X}'_{ij}$ be of order $1xm_i$. In addition, let $Q_i = \sum_{j=1}^{n} \mathbf{X}'_{ij} A_i \mathbf{X}_{ij}$ for $i = 1, 2, \cdots, p$, where $A_1, \cdots, A_p$ are symmetric, positive definite matrices. In this note, we derived the joint density of $Q_1, \cdots, Q_p$. Some applications of this distribution in simultaneous test procedures are also discussed. (Received July 13, 1970.)

70T-87. Application of Hahn–Banach theorem to a moment problem. JAME⁻ SULLIVAN AND J. S. RUSTAGI, Ohio State University.

The Hahn–Banach theorem is applied for solving a moment problem of minimizing the expectation of a function satisfying certain conditions, over a class of probability distributions having prescribed moments. This is a generalization of a problem discussed by Rubin and Isaacson [Stanford Univ. Technical Report (1954)]. Comparison of these results has been made in some cases with those of Harris [*Ann. Math. Statist.* (1959) 521–528] who used the technique of eometry of moment spaces. (Received July 8, 1970.)