*The Annals of Mathematical Statistics*
1972, Vol. 43, No. 1, 358-362

# A LAW OF THE ITERATED LOGARITHM FOR THE ASYMMETRIC STABLE LAW WITH CHARACTERISTIC EXPONENT ONE

BY J. L. MIJNHEER

*University of Leiden*

A law of the iterated logarithm is proved using an extension of the Borel-Cantelli lemma and the asymptotic expansion of the left tail of this stable law.

**1. Introduction.** Let $\{X_i\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with characteristic function (ch.f).

$$f(t) = \exp(-|t| - (2/\pi)it \log |t|) . \tag{1.1}$$

From this ch.f. it is easy to see that $S_n = X_1 + \cdots + X_n$ and $nX_1 + (2/\pi)n \log n$ have the same distribution. In [8] Skorokhod gives the following asymptotic expansions for the tails of the distribution function $F$ of this stable law:

$$1 - F(x) \sim x^{-1} \quad \text{for} \quad x \to \infty , \tag{1.2}$$

$$F(x) \sim 2^{\frac{1}{2}}P(U \geqq 2(\pi e)^{-\frac{1}{2}} \exp(-\pi x/4)) \quad \text{for} \quad x \to -\infty , \tag{1.3}$$

where $U$ is the normal random variable with $EU = 0$ and $EU^2 = 1$.

The main result of this note is the following law of the iterated logarithm (LIL).

THEOREM. *Let* $\varphi$ *be a positive monotonically increasing function on* $(0, \infty)$, $\{X_i\}$ *a sequence of* i.i.d. *random variables with* ch.f. (1.1) *and* $S_n = X_1 + \cdots + X_n$. *Then*

$$P((S_n - (2/\pi)n \log n)/n \leqq -\varphi(n) \text{ i.o.}) = 0 \quad \textit{or} \quad 1 \qquad \textit{according as}$$

$$\int^{\infty} A(t) \exp(-A^2(t)/2)t^{-1}\, dt < \infty \quad \textit{or} \quad = \infty , \qquad \textit{where} \tag{1.4}$$

$$A(t) = 2(\pi e)^{-\frac{1}{2}} \exp(\pi\varphi(t)/4) .$$

For random variables with finite second moment a version of the LIL of this type is proved by Feller [4]. M. Lipschutz [6] has proved similar theorems for continuous and positive random variables in the domain of attraction of the asymmetric stable laws with characteristic exponent $0 < \alpha < 1$ and $1 < \alpha < 2$ with some assumptions on the tails. In [1] L. Breiman suggests a LIL of this type should hold for the stable law (1.1).

**2. Proof.** In the case of convergence of (1.4) we follow the proof for coin tossing in Feller's first book [5]. In this way we can also extend Lipschutz's result to the asymmetric stable law with $1 < \alpha < 2$ (this random variable is

Received December 22, 1970.

not positive a.e.). In the case of divergence of (1.4) the proof resembles the proof of Lipschutz [6].

We apply the following extension of the Borel-Cantelli lemma.

LEMMA. *Let* $\{D_n\}$ *be any sequence of events with* $\sum P(D_n) = \infty$; *then* $P(\limsup D_n) \geqq c^{-1}$ *if*

(2.1) $$\liminf \{\textstyle\sum_{r=1}^{n} P(D_r)\}^{-2} \sum_{r=1}^{n} \sum_{s=1}^{n} P(D_r \cap D_s) \leqq c\ .$$

See for the proof for example Spitzer [9].

PROOF OF THE THEOREM. It is no loss of generality to suppose that

(2.2) $$(\log\log t)^{\frac{1}{2}}/2 \leqq A(t) \leqq 2\,(\log\log t)^{\frac{1}{2}}\ .$$

The proof of this assertion follows the proofs of similar statements in lemmas $a$, $d$, $a'$ and $d'$ in [6]. Then follows

(2.3) $$\varphi(t) \sim 2/\pi \log\log\log t \quad \text{for} \quad t \to \infty\ .$$

By estimate (1.3) we have

(2.4) $$\begin{aligned} P((S_n - (2/\pi)n\log n)/n \leqq -\varphi(n)) &\sim 2^{\frac{1}{2}}P(U \geqq A(n)) \\ &\sim \pi^{-\frac{1}{2}}e^{-\frac{1}{2}A^2(n)}/A(n) \qquad \text{for} \quad n \to \infty\ . \end{aligned}$$

Assume the integral (1.4) converges. Define a sequence of integers $\{n_r\}$ by $n_r = [\exp(r/\log r)]$ and consider the events:

$$\begin{aligned} A_n &: \ (S_n - (2/\pi)n\log n)/n \leqq -\varphi(n)\ , \\ B_r &: \ \min\nolimits_{n_r < n \leqq n_{r+1}} ((S_n - (2/\pi)n\log n)/n) \leqq -\varphi(n_r)\ , \\ C_r &: \ (S_{n_{r+1}} - (2/\pi)n_{r+1}\log n_{r+1})/n_{r+1} \leqq -\varphi(n_r)\ . \end{aligned}$$

Then $\limsup A_n \subset \limsup B_r$. Therefore it suffices to prove $\sum P(B_r) < \infty$. Next we define the event $P_{r,i}$ that $i$ is the smallest number with $n_r < i \leqq n_{r+1}$ for which

$$(S_i - (2/\pi)i\log i)/i \leqq -\varphi(n_r)\ ;$$

for $i = n_r + 1, \cdots, n_{r+1} - 1$ we let $Q_{r,i}$ be the event that

$$(S_{n_{r+1}} - (2/\pi)n_{r+1}\log n_{r+1})/n_{r+1} - (S_i - (2/\pi)i\log i)/i \leqq 0\ .$$

Then

$$B_r = \textstyle\sum_{i=n_r+1}^{n_{r+1}} P_{r,i}$$

($\sum$ denotes the union of mutually exclusive events) and

$$C_r \supset \textstyle\sum_{i=n_r+1}^{n_{r+1}-1} P_{r,i}Q_{r,i} + P_{r,n_{r+1}}\ .$$

It is easy to verify that $P(Q_{r,i}) \geqq \frac{1}{2}$ for all $i$ and therefore

(2.5) $$P(B_r) = \textstyle\sum_{i=n_r+1}^{n_{r+1}} P(P_{r,i}) \leqq 2P(C_r)\ .$$

Convergence of the integral (1.4) is well known to be equivalent to (see [5] and [6])

$$\sum \exp(-A^2(n_r)/2)/A(n_r) < \infty \, . \tag{2.6}$$

From (2.4) and (2.6) it follows that $\sum P(C_r) < \infty$ and as a result $\sum P(B_r) < \infty$ by (2.5). Assume now that the integral (1.4) diverges, which implies divergence of the sum in (2.6).

With the same sequence $\{n_r\}$ as before, we define the event $D_r$ by

$$0 \leqq S_{n_r} \leqq (2/\pi) n_r \log n_r - n_r \varphi(n_r) \, .$$

It obviously suffices to prove $P(D_r \text{ i.o.}) = 1$.

The divergence of the sum in (2.6) together with (2.4) implies that $\sum P(D_r) = \infty$. Since $P(D_r \text{ i.o.})$ is either zero or one, the lemma ensures that it suffices to show that the lim inf in (2.1) is finite.

Consider $P(D_r \cap D_s)$ for $s > r$.

$$\begin{aligned} P(D_r \cap D_s) &= P(0 \leqq S_{n_r} \leqq -n_r \varphi(n_r) + (2/\pi) n_r \log n_r \cap 0 \leqq S_{n_s} \\ &\leqq -n_s \varphi(n_s) + (2/\pi) n_s \log n_s) \\ &\leqq P(D_r) P(n_r \varphi(n_r) - (2/\pi) n_r \log n_r \leqq S_{n_s} - S_{n_r} \\ &\leqq -n_s \varphi(n_s) + (2/\pi) n_s \log n_s) \leqq P(D_r) P(X \leqq -n_s \varphi(n_s)/(n_s - n_r) \\ &\qquad + (2/\pi)(n_s/(n_s - n_r)) \log n_s - (2/\pi) \log (n_s - n_r)) \, . \end{aligned}$$

When $n_r/n_s$ goes to zero

$$\begin{aligned} -n_s \varphi(n_s)/(n_s - n_r) + (2/\pi)(n_s/(n_s - n_r)) \log n_s - (2/\pi) \log (n_s - n_r) \\ = -\varphi(n_s) + O((n_r/n_s) \log n_s) \, . \end{aligned}$$

With some calculations it follows that for each $\delta > 0$ and $\varepsilon > 0$ there exists a number $r_0$ such that for all $r \geqq r_0$ and $s \geqq r + (\log r)^{2+\delta}$ we have

$$P(D_r \cap D_s) \leqq (1 + \varepsilon) P(D_r) P(D_s) \, . \tag{2.7}$$

Next we define a function $\psi(n)$ by

$$2(\pi e)^{-\frac{1}{2}} \exp(\pi \psi(n)/4) = (2\lambda \log \log n)^{\frac{1}{2}} \quad \text{with} \quad \lambda > 1 \, . \tag{2.8}$$

Then by (1.3) for all $r$

$$P((S_{n_r} - (2/\pi) n_r \log n_r)/n_r \leqq -\psi(n_r)) \leqq C r^{-\lambda_1} \quad \text{with} \quad 1 < \lambda_1 < \lambda$$

and hence

$$\begin{aligned} P(D_r \cap D_s) \leqq P((2/\pi) n_r \log n_r - n_r \psi(n_r) \leqq S_{n_r} \leqq (2/\pi) n_r \log n_r - n_r \varphi(n_r) \cap \\ 0 \leqq S_{n_s} \leqq (2/\pi) n_s \log n_s - n_s \varphi(n_s)) + C r^{-\lambda_1} \\ \leqq P(D_r) P(S_{n_s - n_r} \leqq (2/\pi) n_s \log n_s - n_s \varphi(n_s) \\ - (2/\pi) n_r \log n_r + n_r \psi(n_r)) + C r^{-\lambda_1} \, . \end{aligned}$$

Following the calculations in [6] we get for every $r$

$$\sum_s^* P(D_r \cap D_s) \leqq k_2 P(D_r) + C \sum_s^* r^{-\lambda_1} \leqq k_2 P(D_r) + C(\log r)^{2+\delta} r^{-\lambda_1}, \tag{2.9}$$

where $*$ restricts the summation to indices $s$ with $r < s < r + (\log r)^{2+\delta}$. Combining (2.7) and (2.9), we get

$$\liminf \{\sum_{r=1}^n P(D_r)\}^{-2} \sum_{r=1}^n \sum_{s=1}^n P(D_r \cap D_s) \leqq 1 + \varepsilon .$$

**3. Remarks.**

REMARK 1. Take $\varphi(t) = +(2/\pi) \log (\pi e \lambda \log \log t) - (2/\pi) \log 2$ i.e., $A(t) = (2\lambda \log \log t)^{\frac{1}{2}}$, then

$$P((S_n - (2/\pi) n \log n)/n \leqq -\varphi(n) \text{ i.o.}) = 0 \quad \text{or} \quad 1, \quad \text{according as } \lambda > 1 \quad \text{or} \quad \lambda \leqq 1 . \tag{3.1}$$

This yields

$$\liminf \{(S_n - (2/\pi) n \log n)/n + (2/\pi) \log (\pi e \log \log n)\} = (2/\pi) \log 2 \quad \text{a.s.} \tag{3.2}$$

REMARK 2. Dividing by $(2/\pi)\, n \log n$ throughout, (3.1) yields

$$\liminf S_n/((2/\pi) n \log n) = 1 \quad \text{a.s.} \tag{3.3}$$

The result (3.3) was proved by Miller in [7] for positive random variables in the domain of attraction of the stable law (1.1). For all distributions, considered in [7], the LIL (3.1) holds. From the expansion of the right tail of the distribution, it is easy to prove

$$S_n/((2/\pi) n \log n) \to_P 1 ,$$

$EX_1 = \infty$ and hence $S_n/n \to \infty$ a.s. From the paper of Chow and Robbins [2] we know that

$$\limsup S_n/((2/\pi) n \log n) = \infty \quad \text{a.s.}$$

REMARK 3. The theorem holds also for random variables in the domain of attraction of the stable law (1.1) if we make some assumptions on the tails. See for example Cramér [3] and Lipschutz [6]. It is difficult to give simple conditions in terms of the tails of the distribution function of the random variable in the domain of attraction.

## REFERENCES

[1] BREIMAN, L. (1968). A delicate law of the iterated logarithm for non-decreasing stable processes. *Ann. Math. Statist.* **39** 1818–1824.

[2] CHOW, Y. S. and ROBBINS, H. (1961). On sums of independent random variables with infinite moments and "fair" games. *Proc. Nat. Acad. Sci. U.S.A.* **47** 330–335.

[3] CRAMÉR, H. (1963). On asymptotic expansions for sums of independent random variables with a limiting stable distribution. *Sankhyā Ser. A.* **25** 13–24.

[4] FELLER, W. (1943). The general form of the so-called law of the iterated logarithm. *Trans. Amer. Math. Soc.* **54** 373-402.

[5] FELLER, W. (1957). *An Introduction to Probability Theory and its Applications*, **1**. Wiley, New York.

[6] LIPSCHUTZ, M. (1956). On strong bounds for sums of independent random variables which tend to a stable distribution. *Trans. Amer. Math. Soc.* **81** 135-154.

[7] MILLER, H. D. (1967). A note on sums of independent random variables with infinite first moment. *Ann. Math. Statist.* **38** 751-758.

[8] SKOROKHOD, A. V. (1961). Asymptotic formulas for stable distribution laws. *Select. Trans. Math. Statist. Prob.* **1** 157-162.

[9] SPITZER, F. (1964). *Principles of Random Walk*. Van Nostrand, Princeton.