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HAUSDORFF DIMENSION IN A PROCESS WITH STABLE COMPONENTS—AN INTERESTING COUNTEREXAMPLE[1]

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Let $X_{\alpha_1}(t)$ and $X_{\alpha_2}(t)$ be independent stable processes in R_1 of stable index α_1 and α_2 respectively, where $1 < \alpha_2 < \alpha_1 \leqq 2$. Let $X(t) \equiv (X_{\alpha_1}(t), X_{\alpha_2}(t))$ be a process in R_2 formed by allowing X_{α_1} to run on the horizontal axis and X_{α_2} on the vertical axis; $X(t)$ is called a process with stable components. The Blumenthal-Getoor indices of $X(t)$ satisfy $\alpha_2 = \beta'' < \beta' = 1 + \alpha_2 - \alpha_2/\alpha_1 < \beta = \alpha_1$. Denote by $\dim E$ the Hausdorff dimension of E. It is shown that if $E = [0, 1]$ and F is any fixed Borel set for which $\dim F \leqq 1/\alpha_1$ then (with probability 1) we have $\dim X(E) = \beta' \dim E$ and $\dim X(F) = \beta \dim X(F)$. This shows that the results of Blumenthal and Getoor (1961) for the bounds on $\dim X(E)$ for arbitrary processes X and fixed Borel sets E are the best possible, and that their conjecture that $\dim X(E) = \dim X[0, 1] \cdot \dim E$ is incorrect.

1. Introduction. In [2], Blumenthal and Getoor studied sample path properties of stochastic processes $X(t)$ in R_d with stationary independent increments. They defined various indices (β'', β' and β) which are uniquely determined by a given process and which can be used to characterize many aspects of the sample function behavior. We will not repeat the definition of the indices here but point out that the authors showed that they satisfy the inequalities:

$$0 \leqq \beta'' \leqq \beta' \leqq \beta$$

and that for some processes the indices are distinct. On the other hand, for stable processes of index α we have $\beta = \beta'' = \alpha$.

We use the notation $\dim E$ to denote the Hausdorff-Besicovitch dimension of Borel sets $E \subset R_d$. With this understanding, Blumenthal and Getoor [2] (Theorem 8.1) showed that for an arbitrary process $X(t)$ in R_d and fixed Borel subset $E \subset [0, 1]$ the following relations hold (with probability 1) between $\dim E$ and $\dim X(E)$, where $X(E)$ is the range of the process on the set E:

$$\dim X(E) \leqq \beta \dim E \qquad \text{if} \quad \beta < 1 \tag{1}$$

$$\dim X(E) \geqq \beta' \dim E \qquad \text{if} \quad \beta' \leqq d\,. \tag{2}$$

Recently Millar [9] (Theorem 5.1) removed the restriction of $\beta < 1$ in (1). For stable processes of index $\alpha \leqq d$ we thus have $\dim X(E) = \alpha \dim E$. In the final section of [2], Blumenthal and Getoor suggest that possibly a relation

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of the following type holds (with probability 1) for arbitrary processes $X(t)$ and fixed Borel subsets $E \subset [0, 1]$:

$$\dim X(E) = \dim X[0, 1] \cdot \dim E . \tag{3}$$

In the stable case of index $\alpha \leqq d$, (3) is seen to be true. The purpose of this paper is to provide an example of a process which shows that the bounds in (1) and (2) are the best possible and for which the conjecture (3) fails. Specifically, we shall show that for a certain process $X(t)$ we have $0 < \beta'' < \beta' < \beta < d$ and that there are fixed subsets E and F of $[0, 1]$ for which

$$\dim X(E) = \beta' \dim E \qquad \text{and} \qquad \dim X(F) = \beta \dim F \tag{4}$$

hold with probability 1. Moreover, we shall choose $E = [0, 1]$.

The process we consider will be a type of process with stable components (see [6] and [11]) and will be defined in Section 2. In Section 2 we also give the indices for our process and state the pertinent properties of the characteristic function and probability density of our process. Section 3 is devoted to a verification of the claims made concerning our example. We conclude, in Section 4, with several further observations about our process. In order to keep the paper of modest length we shall refer frequently to other papers for definitions, previously stated results, or straightforward calculations. To this end, see Blumenthal and Getoor [1] or [2] for the now familiar definition of Hausdorff dimension.

2. Preliminaries. We suppose (see [7]) that the characteristic function of a one-dimensional stable process $X_\alpha(t)$ of index $\alpha \neq 1$ has the form $\exp[t\psi(y)]$, where (for any real y)

$$\psi(y) = -|y|^\alpha[1 + i\beta \operatorname{sgn}(y) \tan \pi\alpha/2]$$

and β is a parameter such that $-1 \leqq \beta \leqq 1$ and $0 < \alpha \leqq 2$. We assume that $X_\alpha(t)$ is a standard Markov process and that the strong Markov property holds. Now let $1 < \alpha_2 < \alpha_1 < 2$ and suppose that two independent one-dimensional stable processes $X_{\alpha_1}(t, \omega)$ and $X_{\alpha_2}(t, \omega)$ are defined on some probability space. If the two one-dimensional spaces in which the $X_{\alpha_i}(t)$ take their values are orthogonal, the process $X(t)$ defined by

$$X(t, \omega) \equiv (X_{\alpha_1}(t, \omega), X_{\alpha_2}(t, \omega)) \tag{2.1}$$

is called a process in R_2 with stable components. Any stable process in R_1 of index $\alpha > 1$ has a density $p_\alpha(t, x)$ which satisfies the scaling property:

$$p_\alpha(t, x) = r^{\alpha^{-1}} p_\alpha(rt, r^{\alpha^{-1}}x) \qquad \text{for all } r > 0 . \tag{2.2}$$

Moreover, $p_\alpha(t, x)$ is positive, continuous, and bounded in x for each fixed t.

The density of $X(t)$ will be denoted by $p(t, y)$ and is of course computed by:

$$p(t, y) \equiv p_{\alpha_1}(t, x_1)p_{\alpha_2}(t, x_2) ,$$

where $y = (x_1, x_2) \in R_2$.

According to [6] and [11], the various indices for the above process satisfy:

$$\beta'' = \alpha_2 < \beta' = 1 + \alpha_2 - \alpha_2/\alpha_1 < \beta = \alpha_1 .$$

In [11], Pruitt and Taylor determined the Hausdorff measure function of $X(t)$ and thereby found that $\dim X([0, 1]) = 1 + \alpha_2 - \alpha_2/\alpha_1$. The reason for choosing the stable indices both greater than 1 is that, as Pruitt and Taylor find, if $0 < \alpha_2 < \alpha_1 < 1$ we have $\beta' = \beta = \alpha_1$ and if $\alpha_1 \leqq 1$ the measure function of $X(t)$ leads to a Hausdorff dimension of the range of $X(t)$ of α_1.

3. Dimension results for $X(t) \equiv (X_{\alpha_1}(t), X_{\alpha_2}(t))$. We now state the main result of this paper and indicate how the proof can be obtained.

THEOREM 1. *Let $X(t)$ be defined as in* (2.1). *Suppose that $E = [0, 1]$ and that F is any fixed Borel subset of* $[0, 1]$ *for which* $\dim F \leqq \alpha_1^{-1}$. *Then, with probability* 1 *the following relations are satisfied:*

$$\dim X(E) = \beta' \dim E \tag{3.1}$$

$$\dim X(F) = \beta \dim F . \tag{3.2}$$

PROOF. The proof of (3.1) follows from the work of Pruitt and Taylor [11] who give the index β' as $1 + \alpha_2 - \alpha_2/\alpha_1$ and find that this is the exponent which occurs in their determination of the exact Hausdorff measure of the sample paths. (3.2) is proved by first noting that Millar's result states that $\dim X(F) \leqq \beta \dim F$ for arbitrary $F \subset [0, 1]$.

Finally, we show that (if $\dim F \leqq \alpha_1^{-1}$)

$$\dim X(F) \geqq \alpha_1 \dim F \qquad \text{with probability one.} \tag{3.3}$$

Once this is done the proof is complete, since $\alpha_1 = \beta$. The proof of (3.3) follows the same line of argument as Blumenthal and Getoor [1] (pages 371–372) used to establish the lower bound of $\dim X(E)$ in the stable case. They point out in [2] (page 507) that the proof must be slightly modified because of an incorrect assertion. Their proof uses a theorem by Davies [3], one by Frostman [4] concerning β-capacity, and one by McKean [8]. Even under the slight modification mentioned above, the details in the present case go through in exactly the same fashion as established in [1]. We indicate only the change that must be made.

Blumenthal and Getoor [1] (page 372) use the scaling property of the stable processes $X_\alpha(t)$ of index α to establish that:

$$E|X_\alpha(t) - X_\alpha(s)|^{-\delta} = c|t - s|^{-\delta/\alpha} \tag{3.4}$$

holds for some finite positive constant c (independent of s and t) whenever $0 < s < t < \infty$ and $d > \delta > 0$. They use this to show that $\dim X_\alpha(F) \geqq \delta$ provided $\delta < \alpha \dim F$ (and $\alpha \dim F \leqq 1$ if the process is one-dimensional). From this they conclude that $\dim X_\alpha(F) \geqq \alpha \dim F$.

To take the place of (3.4) we establish the following lemma.

LEMMA 1. *Let $X(t)$ be given as in the statement of Theorem 1. Then if $0 < \delta < 1$, a positive constant c_1 (independent of s and t) can be found such that*

$$E|X(t) - X(s)|^{-\delta} < c_1|t - s|^{-\delta/\alpha_1} \tag{3.5}$$

whenever $0 < s < t < \infty$.

PROOF OF LEMMA 1. If we let $t - s = \tau$ and $0 < \delta < 1$ we obtain:

$$\begin{aligned} E|X(t) - X(s)|^{-\delta} &= E|X(t-s)|^{-\delta} = \int_{R_1}\int_{R_1} \frac{p(\tau, (x_1, x_2))}{\{x_1^2 + x_2^2\}^{\delta/2}}\, dx_1\, dx_2 \\ &= \int_{R_1}\int_{R_1} \frac{\tau^{-(\alpha_1^{-1}+\alpha_2^{-1})} p_{\alpha_1}(1, \tau^{-\alpha_1^{-1}}x_1) p_{\alpha_2}(1, \tau^{-\alpha_2^{-1}}x_2)}{\{x_1^2 + x_2^2\}^{\delta/2}}\, dx_1\, dx_2 \\ &= \int_{R_1}\int_{R_1} \frac{p_{\alpha_1}(1, u_1) p_{\alpha_2}(1, u_2)}{\{\tau^{2/\alpha_1}u_1^2 + \tau^{2/\alpha_2}u_2^2\}^{\delta/2}}\, du_1\, du_2 \\ &< \tau^{-\delta/\alpha_1} \int_{R_1} \frac{p_{\alpha_1}(1, u_1)}{|u_1|^\delta}\, du_1 < c_1 \tau^{-\delta/\alpha_1}, \end{aligned}$$

by use of the scaling property for the p_{α_i}, a change of variable, and the fact that $p_{\alpha_1}(1, u_1)$ is bounded and continuous. This proves the lemma.

Then $\dim X(F) \geqq \delta$ provided $\delta < \alpha_1 \dim F$, where we assume $\alpha_1 \dim F \leqq 1$ in order to guarantee $\delta < 1$. Hence, $\dim X(F) \geqq \alpha_1 \dim F$ whenever $\dim F \leqq \alpha_1^{-1}$ and the proof of the theorem is complete.

4. Remarks. (i) The index γ of Pruitt [10] is in this case $1 + \alpha_2 - \alpha_2/\alpha_1$, since Pruitt established that $\dim X[0, 1] = \gamma$ for arbitrary processes with stationary and independent increments. Thus, a result of the type $\dim X(E) = \gamma \dim E$ cannot in general hold.

(ii) As we indicated at the outset, for a fixed Borel set E we have $\dim X_\alpha(E) = \alpha \dim E$ (with probability 1) for stable processes X_α in R_d of index $\alpha \leqq d$. In [5] Hawkes used an interesting method to show that

$$P[\dim X_\alpha(E) \geqq \alpha \dim E \quad \text{for all Borel sets} \quad E] = 1 \tag{4.1}$$

for such processes. If $X(t) \equiv (X_{\alpha_1}(t), X_{\alpha_2}(t))$ as in the statement of our theorem and we use the delayed hitting probabilities for small spheres as derived in [6] (Lemmas 3.3 and 3.4) and the estimates of Pruitt and Taylor [11] (Lemma 5.1 and the Corollary to Lemma 6.1) we can apply Hawkes' technique. In Hawkes' Lemma 3 we use $[2^{1-n}d^{\frac{1}{2}}]^{\alpha_2(1-1/p)}$ in place of $[2^{1-n}d^{\frac{1}{2}}]^{\alpha(1-1/p)}$ for l_n and we choose k_p as the smallest integer greater than $p(1+\rho)/(2-\rho)$, where $\rho = 1+\alpha_2-\alpha_2/\alpha_1$, instead of the smallest integer greater than $p(1+\alpha)/(2-\alpha)$. His argument then leads to

$$(4.2) \qquad P[\dim X(E) \geqq \alpha_2 \dim E \quad \text{for all Borel sets} \quad E] = 1 \cdot$$

It would be interesting to know if there are inequalities involving upper bounds in (4.1) and (4.2), and what the best possible bounds are in results of the above type.

(iii) In [6], we showed that for a process of the type considered in the theorem that:

$$\sup\{\alpha : |X(t)|t^{-1/\alpha} \to +\infty \quad \text{almost surely as} \quad t \to 0\} = \alpha_2$$

is in this case distinct from the Hausdorff dimension of the sample paths. Hence we need not have equality of these two numbers. The present paper provides another instance of when $(X_{\alpha_1}(t), X_{\alpha_2}(t))$ can serve as a useful counterexample and suggests that this process might frequently serve such an end.

REFERENCES

[1] Blumenthal, R. M. and Getoor, R. K. (1960). A dimension theorem for sample functions of stable processes. *Illinois J. Math.* **4** 370–375.

[2] Blumenthal, R. M. and Getoor, R. K. (1961). Sample functions of stochastic processes with stationary independent increments. *J. Math. and Mech.* **10** 493–515.

[3] Davies, R. O. (1952). Subsets of finite measure in analytic sets. *Indag. Math.* **14** 488–489.

[4] Frostman, O. (1935). Potential d'equilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. *Medd. Lunds Univ. Mat. Sem.* **3**.

[5] Hawkes, J. (1971). Some dimension theorems for the sample functions of stable processes. *Indiana Univ. Math. J.* **20** 733–738.

[6] Hendricks, W. J. (1970). Lower envelopes near zero and infinity for processes with stable components, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **16** 261–278.

[7] Lévy, P. (1937). *Theorie de l'Addition der Variables Aleatories.* Gauthier Villars, Paris,

[8] McKean, H. P. (1955). Sample functions of stable processes. *Ann. of Math.* **61** 564–579.

[9] Millar, P. W. (1971). Path behavior of processes with stationary independent increments. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **17** 53–73.

[10] Pruitt, W. E. (1969). The Hausdorff dimension of the range of a process with stationary independent increments. *J. Math. and Mech.* **19** 371–378.

[11] Pruitt, W. E. and Taylor, S. J. (1969). Sample path properties of processes with stable components. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **12** 267–289.