The Annals of Mathematical Statistics
1972, Vol. 43, No. 2, 676-680

A NOTE ON POISSON-SUBORDINATION

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Pseudo-Poisson processes can be obtained from discrete time Markov processes by subordination. A continuous time analogue of a random walk is defined by

$$Y(t) = S[T(t)]$$

where $S(n)$ is the partial sum of a sequence of independent identically distributed random variables and $T(t)$ a process with stationary independent increments, independent of $S(n)$ and taking values in the non-negative integers. It is then shown that $Y(t)$ is a compound Poisson process; furthermore the supremum of $Y(t)$ is Poisson-subordinated to the maximum of $S(n)$ if and only if $T(t)$ is a Poisson process.

1. Pseudo-Poisson processes. In [5] Feller defines a *pseudo-Poisson process* as a continuous time process with stationary transition probabilities

$$Q_t(x, \Gamma) = P\{X(t+s) \in \Gamma \mid X(s) = x\} \tag{1}$$

satisfying

$$Q_t(x, \Gamma) = e^{-\mu t} \sum_{n=0}^{\infty} \frac{(\mu t)^n}{n!} K^{(n)}(x, \Gamma) . \tag{2}$$

Here $x \in \Sigma$, the sample space, Γ is a Borel set in Σ and $\mu > 0$. $K(x, \Gamma)$ is a stochastic kernel inducing a Markov chain $\{Z_n, n = 0, 1, 2, \cdots\}$ governed by

$$K^{(n)}(x, \Gamma) = P\{Z_{n+m} \in \Gamma \mid Z_m = x\} \tag{3}$$

for all $m = 0, 1, 2, \cdots$.

If $\{Z_n\}$ is a Markov chain formed by successive sums of independent identically distributed random variables, then $X(t)$ is called a *compound Poisson process*.

Looking at (1) from the point of view of subordination theory [2, 3, 5, 9, 11] we can write $X(t) = Z_{T(t)}$ where $\{T(t), t \geqq 0\}$ is a Poisson process, independent of $\{Z_n\}$, and with

$$P[T(t) = n] = e^{-\mu t} \frac{(\mu t)^n}{n!} .$$

Feller expresses the relationship between $\{Z_n\}$ and $\{X(t)\}$ by saying that $\{X(t)\}$ is *subordinated* to $\{Z_n\}$ using $\{T(t)\}$ as a *directing process*. Starting with an arbitrary Markov process $\{Z_n\}$ the process $X(t) = Z_{T(t)}$ is again Markovian if $T(t)$ is a process with stationary independent increments, independent of $\{Z_n\}$. As such the probabilities $a_n(t) \equiv P[T(t) = n]$ satisfy the representation formula

Received June 7, 1971; revised September 8, 1971.

for infinitely divisible processes ([4] page 280):

$$\log\{\sum_{n=0}^{\infty} a_n(t)z^n\} = \alpha t[p(z) - 1] \tag{4}$$

where $\alpha > 0$, $p(z) = \sum_{n=0}^{\infty} p_n z^n$, $p_n \geqq 0$ and $p(1) = 1$.

THEOREM 1. *A continuous time Markov process $\{X(t),\ t \geqq 0\}$ with transition probabilities* (1) *is a pseudo-Poisson process if and only if it is subordinated to a discrete time Markov process.*

PROOF. We only have to prove that if Z_n is a discrete time Markov process, and $T(t)$ satisfies (4), then $X(t) = Z_{T(t)}$ has transition probabilities of the form (2). Let indeed

$$M^{(n)}(x, \Gamma) = P\{Z_{n+m} \in \Gamma \mid Z_m = x\}$$

then by total probability

$$P_t(x, \Gamma) = \sum_{n=0}^{\infty} a_n(t) M^{(n)}(x, \Gamma)\,.$$

Now put $G(x, \Gamma) = \sum_{l=0}^{\infty} p_l M^{(l)}(x, \Gamma)$ where $\{p_l\}$ is the discrete distribution involved in (4). An easy induction argument shows that

$$G^{(n)}(x, \Gamma) = \sum_{l=0}^{\infty} p_l^{(n)} M^{(l)}(x, \Gamma) \tag{5}$$

where $\{p_l^{(n)}\}$ is the n-fold convolution of $\{p_l\}$. On the other hand (4) implies readily that

$$P_t(x, \Gamma) = e^{-\alpha t} \sum_{m=0}^{\infty} \frac{(\alpha t)^m}{m!} G^{(m)}(x, \Gamma)$$

which proves the desired formula.

The above theorem is not surprising if one compares it with a result of J. W. Cohen [3] where it is proved that a *Poisson-subordination* of a discrete Markov chain leads to a conservative continuous time Markov chain and conversely. For applications of this idea, see [6, 10].

2. Continuous time random walks. The above theorem illustrates that a natural way to obtain a continuous time analogue from a discrete time process can be based on Poisson-subordination. As another example we define a continuous time analogue of a discrete time random walk.

Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables with partial sums $S_0 = 0$ a.s. and $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$. Let $T(t)$ be a process governed by (4). We define a *continuous time random walk* $Y(t)$ by the subordination $Y(t) = S_{T(t)}$. Clearly then

$$P[Y(t) \leqq x] = \sum_{n=0}^{\infty} a_n(t) P[S_n \leqq x]\,. \tag{6}$$

LEMMA. *A continuous time random walk is a compound Poisson process.*

PROOF. Let $U_1, U_2, \cdots$ be a sequence of discrete valued independent identically distributed random variables, independent of $\{S_n\}$ and with distribution

p_k from (4); $T_o = 0$ a.s. and $T_n = U_1 + U_2 + \cdots + U_n$ for $n \geqq 1$. Then the process $S_m' = S_{T_m}$ is a random walk subordinated to S_n with distributions

$$G^{(m)}(x) = P[S_m' \leqq x] = \sum_{n=0}^{\infty} p_n^{(m)} P[S_n \leqq x] .$$

Finally put $Y(t) = S'_{T'(t)}$ where $T'(t)$ is a Poisson process. Then by Theorem 1

$$P[Y(t) \leqq x] = e^{-\alpha t} \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!} P\{S_n' \leqq x\} \tag{7}$$

or $\{Y(t)\}$ is compound Poisson.

Instead of deriving the analogues of Spitzer's random walk theory [8] for the $Y(t)$ process we restrict ourselves to the supremum functional.

3. The supremum of $Y(t)$. In this section we prove the rather surprising

THEOREM 2. *For all x and $T \geqq 0$*

$$P[\sup_{0 \leqq t \leqq T} Y(t) \leqq x] = e^{-\alpha T} \sum_{n=0}^{\infty} \frac{(\alpha T)^n}{n!} P[\max_{0 \leqq m \leqq n} S_m' \leqq x] . \tag{8}$$

PROOF. Put $\sigma_Y(x, T) = P[\sup_{0 \leqq t \leqq T} Y(t) \leqq x]$. Now $Y(t)$ is a separable process with stationary independent increments and $Y(o) = 0$ a.s. The double Laplace-Stieltjes transform of $\sigma_Y(x, T)$ has then been obtained by Baxter-Donsker [1]. For $u \geqq 0$

$$\log\{u \int_o^{\infty} dT \int_{o-}^{\infty} e^{-\alpha\lambda - uT} \sigma_Y(d\alpha, T)\} \equiv f(\lambda, u) = \int_u^{\infty} ds \int_o^{\infty} e^{-st}[\psi_Y(\lambda, T) - 1]\, dT \tag{9}$$

where $\psi_Y(\lambda, T) - 1 = \int_o^{\infty} [e^{-\alpha\lambda} - 1] d_\alpha P[Y(T) \leqq \alpha]$.

By (7) we have

$$\psi_Y(\lambda, T) - 1 = e^{-\alpha T} \sum_{n=0}^{\infty} \frac{(\alpha T)^n}{n!} \int_{o+}^{\infty} [e^{-\lambda x} - 1] G^{(n)}(dx) .$$

By Fubini's theorem the right-hand side of (9) can be written as

$$f(\lambda, u) = \sum_{n=0}^{\infty} \int_{o+}^{\infty} [e^{-\lambda x} - 1] G^{(n)}(dx) \int_u^{\infty} ds \int_o^{\infty} e^{-(s+\alpha)T} \frac{(\alpha T)^n}{n!} dT .$$

We drop the term with $n = 0$ for $G^{(o)}(x) = U(x)$, the unit step function at $x = 0$. After some manipulations using

$$\int_o^{\infty} e^{-(s+\alpha)T} (\alpha T)^n\, dT = n!\, \alpha^n (\alpha + s)^{-n-1} \tag{10}$$

we obtain

$$f(\lambda, u) = \sum_{n=1}^{\infty} \frac{t^n}{n} \int_{o+}^{\infty} [e^{-\lambda x} - 1] G^{(n)}(dx)$$

where $t = \alpha(\alpha + u)^{-1}$. This can be rewritten in the form

$$f(\lambda, u) = \sum_{n=1}^{\infty} \frac{t^n}{n} \int_{o+}^{\infty} e^{-\lambda x} G^{(n)}(dx) + \sum_{n=1}^{\infty} \frac{t^n}{n} P[S_n' \leqq 0] + \log(1 - t) . \tag{11}$$

On the other hand the distribution of $\max_{1 \leqq m \leqq n} S_m'$ is well known from a Spitzer identity ([7] page 218], i.e. for $|t| < 1$ and $\lambda > 0$.

$$(12) \qquad \log \sum_{n=1}^{\infty} t^n \int_{o-}^{\infty} e^{-\lambda x} d_x P[\max_{0 \leqq m \leqq n} S_m' \leqq x] = \sum_{n=1}^{\infty} \frac{t^n}{n} P[S_n' \leqq 0] + \sum_{n=1}^{\infty} \frac{t^n}{n} \int_{o+}^{\infty} e^{-\lambda x} G^{(n)}(dx) .$$

Comparing (9), (11) and (12) we obtain

$$u \int_o^\infty dT \int_{o-}^\infty e^{-\lambda x - uT} \sigma_Y(dx, T) = (1 - t) \sum_{n=0}^\infty t^n \int_{o-}^\infty e^{-\lambda x} d_x P[\max_{0 \leqq m \leqq n} S_m' \leqq x] .$$

Use $1 - t = u(u + \alpha)^{-1}$ again together with (10). The relation (8) follows from the uniqueness theorem for the Laplace-Stieltjes transform.

It follows from the proof that a duality as given by (7) and (8) is only possible for Poisson-subordination.

As pointed out by the referee, the last theorem can be proved by using more explicitly some properties of the Poisson process. By the lemma

$$Y(t) = S'_{T_t'}$$

where $T_o' = 0$ a.s. and T_t' is Poisson; hence we have with probability one that for every $m = 1, 2, \cdots$

$$\{T_0' = 0,\ T_t' = m\} = \{0 = T_0' < T'_{t_1} < \cdots < T'_{t_{m-1}} < T_t' = m, 0 < t_1 < \cdots < t_{m-1} < t\} .$$

Consequently, with probability one

$$\{\sup_{0 \leqq \tau \leqq t} Y(\tau) \leqq x\} = \bigcup_{m=0}^\infty \{\sup_{0 \leqq \tau \leqq t} S'_{T_\tau'} \leqq x,\ T_t' = m\} = \bigcup_{m=0}^\infty \{\sup_{0 \leqq n \leqq m} S_m' \leqq x,\ T_t' = m\}$$

from which relation (8) follows immediately by the independence of the processes T_t' and S_n'.

A special case of Theorem 2 is due to Täcklind [10] and is mentioned by Spitzer [8].

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