The Annals of Mathematical Statistics
1972, Vol. 43, No. 3, 1020-1026

RIESZ DECOMPOSITION FOR WEAK BANACH-VALUED QUASI-MARTINGALES

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In this paper we present the concept of a weak Banach-valued quasi-martingale. The concept of a strong Banach-valued quasi-martingale was introduced by the author in an earlier paper. We prove that a weak Banach-valued quasi-martingale under certain conditions has a weak Riesz decomposition, i.e., it can be decomposed in an essentially unique way as a sum of a weak martingale and a weak quasi-potential.

Introduction. In an earlier paper [9], [10], the author has introduced the concept of a strong Banach-valued quasi-martingale and obtained its strong Riesz decomposition. In this paper is presented the concept of a weak Banach-valued quasi-martingale and corresponding weak Riesz decomposition. Real-valued quasi-martingales were treated in papers of D. L. Fisk [4], S. Orey [8] and K. M. Rao [11]. Also, the Riesz decomposition for real-valued super-martingales was obtained by P. A. Meyer [7].

The setting. Let $(\Omega, \mathscr{F}, P)$ be a given probability space and let $\mathscr{X}$ be a Banach space which is weakly sequentially complete. Let $\mathscr{X}^*$ be the dual of $\mathscr{X}$ with card $(\mathscr{X}^*) = c$, $T \equiv [0, +\infty)$ and $(\mathscr{F}_t; t \in T)$ be an increasing family of σ-sub-algebras of $\mathscr{F}$, i.e., $\forall s, t \in T$, $s \leqq t$ implies $\mathscr{F}_s \subseteqq \mathscr{F}_t$. Finally, let $\forall t \in T$, $X_t : \Omega \to \mathscr{X}$ be a family of weakly integrable (Gel'fand-Pettis) random variables [5], page 77, such that $\forall t \in T$, X_t is $\mathscr{F}_t$-weakly measurable [5], page 72. In [6], page 240, M. Metivier has introduced the following.

DEFINITION. A family $(X_t, \mathscr{F}_t; t \in T)$, with X_t's and $\mathscr{F}_t$'s having the properties described above, is a weak martingale if:

$$(*) \qquad \forall \Lambda, \Lambda \in \mathscr{F}_t : \int_\Lambda X_t \, dP = \int_\Lambda X_{t'} \, dP , \qquad t \leqq t' ,$$

where the integrals in question are weak (Gel'fand-Pettis) integrals.

Now we can introduce the following concept:

DEFINITION 1. A family $(X_t, \mathscr{F}_t; t \in T)$ is called a weak quasi-martingale if there exists an $M > 0$ such that for every $x^* \in S^*$ (S^* denotes the unit sphere in $\mathscr{X}^*$), and every strictly increasing sequence $(t_n)_{n=1}^{+\infty}$, $t_n \in T$, $n = 1, 2, \cdots, t_n \uparrow +\infty$ as $n \to +\infty$, one has:

$$(1) \qquad \sup \sum_{i=1}^{n} |E(x^*(X_{t_i}) - x^*(X_{t_{i+1}}))| \leqq M < +\infty ,$$

where the supremum is taken over all possible sequences $(t_n)_{n=1}^{+\infty}$ from T described above.

It follows from (1) that Banach-valued strong and weak martingales are weak

Received June 17, 1971; revised October 11, 1971.

quasi-martingales. Indeed, if $(X_t, \mathscr{F}_t; t \in T)$ is a weak martingale, (*) implies that $\forall x^* \in \mathscr{X}^*: \int_\Omega x^*(X_{t_i})\, dP = \int_\Omega x^*(X_{t_{i+1}})\, dP$, i.e., $E(x^*(X_{t_i}) - x^*(X_{t_{i+1}})) = 0$, so that (1) holds. If $(X_t, \mathscr{F}_t; t \in T)$ is a strong martingale then

$$(+) \qquad X_{t_i} = E(X_{t_{i+1}} | \mathscr{F}_{t_i}) \quad \text{a.e.} \quad (P),$$

where $E(\ |\ )$ denotes the strong conditional expectation as introduced in [12], page 353. Further, (*) implies that $\int_\Omega x^*(X_{t_{i+1}})\, dP = \int_\Omega x^*(E(X_{t_{i+1}} | \mathscr{F}_{t_i})\, dP = \int_\Omega x^*(X_{t_i})\, dP$, so,

$$\forall x^* \in \mathscr{X}^*: E(x^*(X_{t_i}) - x^*(X_{t_{i+1}})) = \int_\Omega x^*(X_{t_i} - E(X_{t_{i+1}} | \mathscr{F}_{t_i}))\, dP = 0,$$

because of (+). Therefore, (1) holds showing that $(X_t, \mathscr{F}_t; t \in T)$ is a weak quasi-martingale.

Also, a strong quasi-martingale [9], [10], is a weak quasi-martingale. Indeed, if $(X_t, \mathscr{F}_t; t \in T)$ is a strong quasi-martingale then there exists a constant $M > 0$ such that $\sup \sum_{i=1}^n E(\|X_{t_i} - E(X_{t_{i+1}} | \mathscr{F}_{t_i})\|) \leqq M$, where $t_n \in T$, $n = 1, 2, \cdots$, and $t_n \uparrow +\infty$ as $n \to +\infty$. This implies that $\forall x^* \in S^*$, one has that:

$$\begin{aligned} |E(x^*(X_{t_i}) - x^*(X_{t_{i+1}}))| &= |E(E(x^*(X_{t_i}) | \mathscr{F}_{t_i})) - E(E(x^*(X_{t_{i+1}}) | \mathscr{F}_{t_i}))| \\ &= |E(x^*(X_{t_i}) - E(x^*(X_{t_{i+1}}) | \mathscr{F}_{t_i}))| \\ &\leqq E(\|x^*\|\, \|X_{t_i} - E(X_{t_{i+1}} | \mathscr{F}_{t_i})\|) \\ &\leqq \|x^*\| E(\|X_{t_i} - E(X_{t_{i+1}} | \mathscr{F}_{t_i})\|) \\ &= E(\|X_{t_i} - E(X_{t_{i+1}} | \mathscr{F}_{t_i})\|) \end{aligned}$$

which implies that $\forall x^* \in S^*$; $\sup \sum_{i=1}^n |E(x^*(X_{t_i}) - x^*(X_{t_{i-1}}))| \leqq \sup \sum_{i=1}^n E(\|X_{t_i} - E(X_{t_{i+1}} | \mathscr{F}_{t_i})\|) \leqq M$, as claimed.

DEFINITION 2. A family $(X_t, \mathscr{F}_t; t \in T)$ is a weak quasi-potential if $\forall x^* \in S^*$, $\lim_{t \to +\infty} E(|x^*(X_t)|) = 0$.

One can easily show that a strong quasi-potential [9], [10] is also a weak quasi-potential.

DEFINITION 3. A family $(X_t, \mathscr{F}_t; t \in T)$ has a weak Riesz decomposition if $\forall t \in T$, $X_t = X_t^{(1)} + X_t^{(2)}$ a.e. (P), where $(X_t^{(1)}, \mathscr{F}_t; t \in T)$ is a weak martingale and $(X_t^{(2)}, \mathscr{F}_t; t \in T)$ is a weak quasi-potential.

DEFINITION 4. Let $\mathscr{X}^* = (x_\gamma^*; \gamma \in \Gamma)$ where $\Gamma \subseteqq R^1$. We say that a family $(X_t, \mathscr{F}_t; t \in T)$ is weakly separable if for every fixed $t \in T$ the real-valued random process $(x_\gamma^*(X_t); \gamma \in \Gamma)$ is separable with respect to closed intervals in R^1, [2], page 53.

PROPOSITION 1. *Let $(X_t, \mathscr{F}_t; t \in T)$ be a weakly separable family which has a weak Riesz decomposition: $\forall t \in T$, $X_t = X_t^{(1)} + X_t^{(2)}$ a.e. (P). Then, this decomposition is essentially unique.*

PROOF. Assume that $(X_t, \mathscr{F}_t; t \in T)$ has two weak Riesz decompositions, namely, that $\forall t \in T$:

$$(2) \qquad X_t^{(1)} + X_t^{(2)} = Y_t^{(1)} + Y_t^{(2)}$$

where $(X_t^{(1)}, \mathscr{F}_t)$, $(Y_t^{(1)}, \mathscr{F}_t)$ are weak martingales and $(X_t^{(2)}, \mathscr{F}_t)$, $(Y_t^{(2)}, \mathscr{F}_t)$ are weak quasi-potentials. Then, from (2) and (*) it follows that for every $x_\gamma^* \in S^*$, $(x_\gamma^*(X_t^{(1)}) - x_\gamma^*(Y_t^{(1)}), \mathscr{F}_t; t \in T)$ is a real-valued sub-martingale and therefore $E(|x_\gamma^*(X_t^{(1)}) - x_\gamma^*(Y_t^{(1)})|)$ is a non-decreasing function on t. On the other hand, $\lim_{t\to+\infty} E(|x_\gamma^*(X_t^{(1)}) - x_\gamma^*(Y_t^{(1)})|) = \lim_{t\to+\infty} E(|x_\gamma^*(Y_t^{(2)}) - x_\gamma^*(X_t^{(2)})|) = 0$, which implies that $E(|x_\gamma^*(X_t^{(1)}) - x_\gamma^*(Y_t^{(1)})|) \equiv 0$, that is,

$$P(x_\gamma^*(X_t^{(1)}) - x_\gamma^*(Y_t^{(1)}) = 0) = 1\,, \qquad \gamma \in \Gamma\,. \tag{3}$$

Note that a null set in (3) on whose complement $x_\gamma^*(X_t^{(1)}) = x_\gamma^*(Y_t^{(1)})$, depends on a functional x_γ^*.

From (3) and the fact that $(x_\gamma^*(X_t^{(1)}) - x_\gamma^*(Y_t^{(1)}); \gamma \in \Gamma)$ is a separable random process for every fixed $t \in T$, it follows that there exists a sequence $(\gamma_j)_{j=1}^{+\infty}$ from Γ such that

$$P(x_{\gamma_j}^*(X_t^{(1)}) - x_{\gamma_j}^*(Y_t^{(1)}) = 0; j \geqq 1) = 1\,. \tag{++}$$

Finally, using the result in [2], page 55, it follows from the assumed separability and relation $(++)$ that $P(x_\gamma^*(X_t^{(1)}) - x_\gamma^*(Y_t^{(1)}) = 0, \gamma \in \Gamma) = 1$, i.e., that a weak Riesz decomposition is essentially unique, as claimed. Further on, the subscript γ is going to be omitted from the notation of a linear functional.

REMARK. It follows easily that a strong Riesz decomposition [9], [10], is also a weak Riesz decomposition.

Let $(\mathscr{F}_t; t \in T)$ be an increasing family of σ-sub-algebras of $\mathscr{F}$ and assume, moreover, that $\forall t \in T$, $(\Omega, \mathscr{F}_t, P)$ is an atomic probability space. Further, $\forall t \in T$, let us assume that $X_t : \Omega \to \mathscr{X}$ is weakly integrable and $\mathscr{F}_t$-strongly measurable. Then, using the result in [1], page 268, one has the following representation for X_t's:

$$\forall t \in T\colon X_t = \textstyle\sum_{j=1}^{+\infty} y_j^{(t)} I_{E_j^{(t)}} \tag{4}$$

where $y_j^{(t)} \in \mathscr{X}$, $j = 1, 2, \cdots$, $E_j^{(t)} \in \mathscr{F}_t$ and the series

$$\textstyle\sum_{j=1}^{+\infty} y_j^{(t)} P(E \cap E_j^{(t)}) \tag{5}$$

is unconditionally convergent for every $E \in \mathscr{F}$. Moreover, from the same paper [1], page 269, one gets the following representation for a weak conditional expectation of random variables X_t having representations (4):

$$E(X_t \mid \mathscr{F}_{t'}) = \textstyle\sum_{j=1}^{+\infty} y_j^{(t)} P(E_j^{(t)} \mid \mathscr{F}_{t'})\,, \tag{6}$$

provided that the series in (6) is unconditionally convergent a.e. (P). ($\mathscr{F}_{t'}$ is a σ-subalgebra of $\mathscr{F}$, $P(E_j^{(t)} \mid \mathscr{F}_{t'}) = E(I_{E_j^{(t)}} \mid \mathscr{F}_{t'})$ is a real-valued conditional expectation.)

If $(\Omega, \mathscr{F}_{t'}, P)$ is an atomic probability space then it has at most countably many atoms, say $(A_k)_{k=1}^{+\infty}$, and (6) can be written as

$$E(X_t \mid \mathscr{F}_{t'})(\omega) = \textstyle\sum_{j=1}^{+\infty} y_j^{(1)} \sum_{k=1}^{+\infty} (P(E_j^{(t)} \cap A_k)/P(A_k)) I_{A_k}(\omega)\,, \qquad \omega \in \Omega\,,$$

or, for every $\omega \in \Omega$, there is a positive integer $k_0(\omega)$ such that

$$E(X_t \mid \mathscr{F}_{t'})(\omega) = (1/P(A_{k_0(\omega)})) \sum_{j=1}^{+\infty} y_j^{(t)} P(E_j^{(t)} \cap A_{k_0(\omega)}) ,$$

and which is unconditionally convergent a.e. (P) due to (5). Therefore, under assumptions made on page 1022 concerning a family $(\mathscr{F}_t;\ t \in T)$, it follows that there exist weak conditional expectations $E(X_t \mid \mathscr{F}_{t'})$, $\forall t, t' \in T$. (X_t's are weakly integrable and strongly measurable.)

REMARK. Recently L. Schwartz has shown (not yet published result) that in a general case a weak conditional expectation does not exist.

PROPOSITION 2. *If $(X_t, \mathscr{F}_t;\ t \in T)$ is a weak martingale where X_t's and $\mathscr{F}_t$'s have the properties described on pages* 1022, 1023, *then $\forall s, t \in T$, $s \leqq t$,*

$$X_s = E(X_t \mid \mathscr{F}_s) \quad \text{a.e.} \quad (P) , \tag{7}$$

where $E(\ \mid\ )$ denotes a weak conditional expectation.

PROOF. From (*) and (4) it follows that $\forall x^* \in \mathscr{X}^*$,

$$\forall \Lambda \in \mathscr{F}_s:\ \int_\Lambda x^*(X_s)\, dP = \int_\Lambda x^*(X_t)\, dP = \sum_{j=1}^{+\infty} x^*(y_j^{(t)}) P(\Lambda \cap E_j^{(t)}) ,$$

(this series is absolutely convergent because the series without linear functional is unconditionally convergent; this is due to the Orlicz-Pettis theorem [5], page 62), or,

$$\int_\Lambda x^*(X_s)\, dP = \sum_{j=1}^{+\infty} x^*(y_j^{(t)}) \int_\Lambda P(E_j^{(t)} \mid \mathscr{F}_s)\, dP ,$$

wherefrom by using (6) one gets that $\int_\Lambda x^*(X_s)\, dP = \int_\Lambda x^*(E(X_t \mid \mathscr{F}_s))\, dP$, which implies (7).

PROPOSITION 3. *Let $(X_t, \mathscr{F}_t;\ t \in T)$ be an $\mathscr{X}$-valued random process with X_t's and $\mathscr{F}_t$'s described as in Proposition* 2, *i.e., as on pages* 1022, 1023. *If, moreover, there exists a constant $M > 0$ such that $\forall x^* \in S^*$:*

$$\sup \sum_{i=1}^{n} E(|x^*(X_{t_i}) - x^*(E(X_{t_{i+1}} \mid \mathscr{F}_{t_i}))|) \leqq M < +\infty , \tag{8}$$

then the process $(X_t, \mathscr{F}_t;\ t \in T)$ is a weak quasi-martingale.

PROOF. The conclusion in the proposition follows immediately from

$$\begin{aligned} |E(x^*(X_{t_i}) - x^*(X_{t_{i+1}}))| &= |E(x^*(X_{t_i})) - E(E(x^*(X_{t_{i+1}}) \mid \mathscr{F}_{t_i}))| \\ &= |E(x^*(X_{t_i}) - x^*(E(X_{t_{i+1}} \mid \mathscr{F}_{t_i}))| \\ &\leqq E(|x^*(X_{t_i}) - x^*(E(X_{t_{i+1}} \mid \mathscr{F}_{t_i}))|) , \end{aligned}$$

for $i = 1, 2, \cdots$.

Now we have the following

THEOREM. *Every weakly separable $\mathscr{X}$-valued weak quasi-martingale $(X_t, \mathscr{F}_t;\ t \in T)$, which satisfies assumptions of Proposition* 3, *possesses a weak Riesz decomposition. Moreover, this decomposition is essentially unique.*

PROOF. Let $t_n \in T$, $n = 1, 2, \cdots, t_n \uparrow +\infty$ as $n \to +\infty$ and define $u(n) = X_{t_n} - E(X_{t_{n+1}} \mid \mathscr{F}_{t_n})$ for $n = 1, 2, \cdots$. Then, for every $x^* \in S^*$ it follows from (8) that

$\sum_{n=1}^{+\infty} E(|x^*(u(n))|) \leqq M$. Let $t \in T$ be fixed and assume that $t_n \geqq t$, $n = 1, 2, \cdots$. Define $v_t(n) = E(X_{t_n} | \mathscr{F}_t)$. Then, using the properties of the weak conditional expectation it follows that for every $x^* \in S^*$:

$$E(x^*(u(n)) | \mathscr{F}_t) = x^*(v_t(n)) - x^*(v_t(n+1)) \quad \text{a.e.} \quad (P),$$

which implies that $\sum_{n=1}^{+\infty} E(|x^*(v_t(n)) - x^*(v_t(n+1))|) \leqq \sum_{n=1}^{+\infty} E(|x^*(u(n))|) \leqq M < +\infty$. Without loss of generality we may assume that $E(|x^*(v_t(n) - v_t(n+1))|) \leqq (\frac{1}{2})^n$, $n = 1, 2, \cdots$. Put $v_t(0) = 0$, and define:

$$g_t(n) = \sum_{k=1}^{n} |x^*(v_t(k) - v_t(k-1))|, \qquad n = 1, 2, \cdots. \tag{9}$$

Then, $g_t(n)$, $n = 1, 2, \cdots$, are integrable due to the fact that $v_t(k)$, $k = 1, 2, \cdots$, are weakly integrable and, moreover, $0 \leqq g_t(n) \uparrow$ as $n \to +\infty$, and $t \in T$ is fixed. From (9) it follows that $E(g_t(n)) < 1 + E(|x^*(v_t(1))|)$ for all n. Hence by the monotone convergence theorem there exists an integrable function g_t such that $g_t(n) \uparrow g_t$ a.e. (P), as $n \to +\infty$. Using (9) it follows that $|x^*(v_t(n))| \leqq g_t(n)$, which implies that $|x^*(v_t(n))| \leqq g_t$ a.e. (P) for all $n = 1, 2, \cdots$. Finally, define $h_t(n) = v_t(n) - v_t(n-1)$, $n = 1, 2, \cdots$. Then, one gets that $\sum_{k=1}^{n} |x^*(h_t(k))| = g_t(n) \leqq g_t$ a.e. (P), for $n = 1, 2, \cdots$; hence for every $x^* \in S^*$ the series $\sum_{k=1}^{+\infty} |x^*(h_t(k))|$ is convergent a.e. (P). Now, $\mathscr{X}$ being weakly sequentially complete implies that there exists an $A(t) \in \mathscr{X}$ such that for every $x^* \in \mathscr{X}^*$, $x^*(A(t)) = \sum_{k=1}^{+\infty} x^*(h_t(k))$. Further, one has that:

$$\forall x^* \in \mathscr{X}^*: |x^*(A(t)) - x^*(v_t(n))| \to 0 \quad \text{a.e.} \quad (P), \qquad \text{as} \quad n \to +\infty. \tag{10}$$

This relation shows that $A(t)$ is weakly $\mathscr{F}_t$-measurable for every $t \in T$. Since $|x^*(v_t(n))| \leqq g_t$ a.e. (P), it follows that $|x^*(A(t))| \leqq g_t$ a.e. (P), hence $A(t)$ is a weakly integrable function for every $t \in T$. Finally, let $s, t \in T$, $s \leqq t$, and $t_n \uparrow +\infty$ as $n \to +\infty$ such that $t_n \geqq t$, $n = 1, 2, \cdots$. Then, using the Lebesgue dominated convergence theorem and representation (6) for a weak conditional expectation one gets that for every $\Lambda \in \mathscr{F}_s$:

$$\begin{aligned}
\int_\Lambda x^*(A(s))\, dP &= \int_\Lambda (\lim_{n\to+\infty} x^*(v_s(n)))\, dP = \lim_{n\to+\infty} \int_\Lambda x^*(v_s(n))\, dP \\
&= \lim_{n\to+\infty} \int_\Lambda x^*(E(X_{t_n} | \mathscr{F}_s))\, dP \\
&= \lim_{n\to+\infty} \sum_{j=1}^{+\infty} x^*(y_j^{(t_n)}) \int_\Lambda P(E_j^{(t_n)} | \mathscr{F}_s)\, dP \\
&= \lim_{n\to+\infty} \sum_{j=1}^{+\infty} x_*(y_j^{(t_n)}) P(\Lambda \cap E_j^{(t_n)}) \\
&= \lim_{n\to+\infty} \sum_{j=1}^{+\infty} x^*(y_j^{(t_n)}) \int_\Lambda P(E_j^{(t_n)} | \mathscr{F}_t)\, dP \\
&= \lim_{n\to+\infty} \int_\Lambda x^*(E(X_{t_n} | \mathscr{F}_t))\, dP = \lim_{n\to+\infty} \int_\Lambda x^*(v_t(n))\, dP \\
&= \int_\Lambda (\lim_{n\to+\infty} x^*(v_t(n)))\, dP = \int_\Lambda x^*(A(t))\, dP, \quad \text{a.e.} \quad (P),
\end{aligned}$$

which shows that the family $(A(t), \mathscr{F}_t; t \in T)$ is a weak martingale. Also, by the dominated convergence theorem a.e. (P), one has that

$$E(|x^*(A(t)) - x^*(v_t(n))|) \to 0 \qquad \text{as} \quad n \to +\infty. \tag{11}$$

Now, given $\varepsilon > 0$ there is an integer n_0 such that for every $x^* \in S^*$:

$$\sum_{n=n_0}^{+\infty} E(|x^*(X_{t_n}) - x^*(E(X_{t_{n+1}} | \mathscr{F}_{t_n}))|) < \varepsilon/2. \tag{12}$$

Let $k \geqq n_0$. Then (11) implies that $E(|x^*(A(t_k)) - x^*(v_{t_k}(m))|)$ is small for m sufficiently large. Let $m(k)+1$ be an integer for which:

(13) $$E(|x^*(A(t_k)) - x^*(v_{t_k}(m(k)+1))|) < \varepsilon/2 .$$

Taking into account (12) one gets that

(14) $$E(|x^*(X_{t_k}) - x^*(A(t_k))|) \leqq E(|x^*(X_{t_k}) - x^*(v_{t_k}(m(k)+1))|) + \varepsilon/2 .$$

Now for $n \geqq k$ it follows that $E(x^*(u(n)) \mid \mathscr{F}_{t_k}) = x^*(v_{t_k}(n) - v_{t_k}(n+1))$, so that $E(|x^*(v_{t_k}(n) - v_{t_k}(n+1))|) \leqq E(|x^*(u(n))|)$. Therefore it follows that

(15) $$\begin{aligned} E(|x^*(X_{t_k}) &- x^*(v_{t_k}(m(k)+1))|) \\ &\leqq \textstyle\sum_{n=k}^{m} E(|x^*(v_{t_k}(n) - v_{t_k}(n+1))|) \\ &\leqq \textstyle\sum_{n=k}^{m} E(|x^*(u(n))|) \leqq \sum_{n=n_0}^{+\infty} E(|x^*(u(n))|) \leqq \varepsilon/2 , \end{aligned}$$

where the last inequality in (15) is obtained from (12). Finally, from (13) and (15) it follows that for every $x^* \in S^*$:

(16) $$\lim_{n\to+\infty} E(|x^*(X_{t_n}) - x^*(A(t_n))|) = 0 .$$

Further we have to show that the weak martingale $(A(t), \mathscr{F}_t; t \in T)$ is independent of a particular choice of the increasing sequence $(t_n)_{n=1}^{+\infty}$ as well as that the limit in (16) is independent of a choice of this sequence. To prove this, let us assume otherwise. Then there exists a strictly increasing sequence, say $(s_k)_{k=1}^{+\infty}$, such that $s_k \uparrow +\infty$ as $k \to +\infty$, and $\varepsilon_0 > 0$ such that for every $x^* \in S^*$:

(17) $$\lim_{k\to+\infty} E(|x^*(X_{s_k}) - x^*(A(s))|) \geqq \varepsilon_0 > 0 .$$

Let us form the increasing sequence $(p_k)_{k=1}^{+\infty}$, $p_k \uparrow +\infty$ as $k \to +\infty$ by interlacing the sequences $(t_n)_{n=1}^{+\infty}$ and $(s_k)_{k=1}^{+\infty}$ satisfying (16) and (17), respectively. Then by applying the first part of the proof, there exists a weak martingale $(B(t), \mathscr{F}_t; t \in T)$ such that:

(18) $$\forall x^* \in S^* : \lim_{k\to+\infty} E(|x^*(X_{p_k}) - x^*(B(p_k))|) = 0 .$$

It follows that $(A(t) - B(t), \mathscr{F}_t; t \in T)$ is a weak martingale which implies that for every $x^* \in S^*$, $(|x^*(A(t)) - x^*(B(t))|, \mathscr{F}_t; t \in T)$ is a real-valued sub-martingale, which further implies that $E(|x^*(A(t) - B(t))|)$ is a non-decreasing function on t. On the other hand: $E(|x^*(A(t_k)) - x^*(B(t_k))|) \leqq E(|x^*(X_{t_k}) - x^*(A(t_k))|) + E(|x^*(X_{t_k}) - x^*(B(t_k))|)$, or, $E(|x^*(A(t_k)) - x^*(B(t_k))|) \leqq E(|x^*(X_{t_k}) - x^*(A(t_k))|) + E(|x^*(X_{p_k}) - x^*(B(p_k))|) \to 0$ a.e. (P) as $k \to +\infty$. This fact, together with the earlier conclusion that $E(|x^*(A(t) - B(t))|)$ is a non-decreasing function on t, implies that:

(19) $$\forall t \in T\colon A(t) = B(t) \quad \text{a.e.} \quad (P) .$$

The relations (18) and (19) imply that $E(|x^*(X_{p_k}) - x^*(A(p_k))|) \to 0$ as $k \to +\infty$, which contradicts (17). Therefore, $\forall x^* \in S^* : \lim_{t\to+\infty} E(|x^*(X_t) - x^*(A(t))|) = 0$. Finally, put $\forall t \in T\colon Y_t = A(t)$ a.e. (P), $Z_t = X_t - Y_t$ a.e. (P). Then, we have a weak Riesz decomposition $X_t = Y_t + Z_t$ for the weak quasi-martingale $(X_t,$

$\mathscr{F}_t$; $t \in T$). This decomposition is essentially unique due to Proposition 1, which terminates the proof of the theorem.

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