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ON WEAK CONVERGENCE OF EXTREMAL PROCESSES FOR RANDOM SAMPLE SIZES[1]

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The results of Dwass (1964) [*Ann. Math. Statist.* **35** 1718–1725] and Lamperti (1964) [*Ann. Math. Statist.* **35** 1726–1737] on the weak convergence of extremal processes (in the Skorokhod J_1-topology) to appropriate Markov processes are extended here for random sample sizes.

1. Introduction. Let $\{X_1, X_2 \cdots\}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathscr{A}, P)$, where each X_i has a common distribution function (df) $F(x)$, $x \in R$, the real line $(-\infty, \infty)$. For the sample maxima

$$M_n = \max\{X_i : 1 \leqq i \leqq n\}, \qquad n \geqq 1, \tag{1.1}$$

Gnedenko (1943) has determined all the three possible types of nondegenerate df $G(x)$ which can appear in

$$\lim_{n\to\infty} P\{(M_n - a_n)/b_n \leqq x\} = G(x), \qquad x \in R, \tag{1.2}$$

where a_n and $b_n(>0)$ are suitable constants; see (2.8). All such G are continuous. Extensions of (1.2) for random sample sizes are due to Berman (1962), Barndorff-Nielsen (1964) and Lamperti (1963), among others.

Dwass (1964) and Lamperti (1964) have considered the so called extremal stochastic processes $\{m_n(t) : 0 \leqq t < \infty\}$, $n \geqq 1$, where

$$\begin{aligned} m_n(t) &= (M_{[nt]} - a_n)/b_n, & t > n^{-1}, \\ &= (X_1 - a_n)/b_n, & 0 \leqq t \leqq n^{-1}, \end{aligned} \tag{1.3}$$

($[s]$ being the largest integer contained in s), and have shown that (1.2) insures the weak convergence (in the Skorokhod J_1-topology) of $\{m_n(t)\}$ to an appropriate Markov process. The object of the present investigation is to show that for random sample sizes, under the usual convergence condition (viz., Blum *et al.* (1963) and Mogyorodi (1965)), this weak convergence holds. This extension is comparable to Theorem 17.2 of Billingsley (1968) which extends the classical Donsker Theorem for random sample sizes.

2. The main result. Let $\{N_n; n \geqq 1\}$ be a sequence of nonnegative integer-valued random variables such that

$$n^{-1}N_n \to \lambda, \quad \text{in probability,} \qquad \text{as} \quad n \to \infty, \tag{2.1}$$

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where λ is a positive random variable having an arbitrary distribution and defined on the same probability space $(\Omega, \mathscr{A}, P)$. For every $n \geqq 1$, we define a stochastic process

$$(2.2)\qquad \begin{aligned} m_{N_n}(t) &= m_k(t) && \text{if} \quad N_n = k \geqq 1\,, \\ &= -\infty\,, && \text{if} \quad N_n = 0\,, \qquad 0 \leqq t < \infty\,; \end{aligned}$$

where $m_k(t)$ is defined by (1.3). Consider then a Markov process $\{m(t): 0 \leqq t < \infty\}$ for which for nonnegative t and s,

$$(2.3)\qquad P\{m(t) \leqq x\} = [G(x)]^t\,,$$

$$(2.4)\qquad \begin{aligned} P\{m(t+s) \leqq y \mid m(t) = x\} &= 0\,, && \text{if} \quad y < x\,, \\ &= [G(y)]^s\,, && y \geqq x\,, \end{aligned}$$

where $G(x)$ is defined by (1.2). Further, consider the space $D[0, 1]$ of real functions $y(t)$ defined on $[0, 1]$ with the properties that (i) $y(t-0)$ and $y(t+0)$ exist for $0 < t < 1$, $y(t) = y(t+0)$, $0 \leqq t < 1$, and (ii) $y(t)$ is continuous at $t = 0$ and $t = 1$. Also, let Λ be the class of strictly increasing, continuous mapping of $[0, 1]$ onto itself. Then, with $D[0, 1]$, we associate the Skorokhod J_1-topology

$$(2.5)\qquad \rho_D(x, y) = \inf_{\lambda \in \Lambda} [\sup_t |x(t) - y(\lambda(t))| + \sup_t |\lambda(t) - t|]\,,$$

where both x and y belong to $D[0, 1]$. We also denote by

$$(2.6)\qquad m = \{m(t), t \in [0, 1]\}\,, \qquad m_n = \{m_n(t), t \in [0, 1]\}\,;$$

$$(2.7)\qquad m_{N_n} = \{m_{N_n}(t), t \in [0, 1]\} \qquad \text{and} \qquad m_{[n\lambda]} = \{m_{[n\lambda]}(t), t \in [0, 1]\}\,,$$

where $m_n(t)$, $m_{N_n}(t)$ and $m(t)$ are defined earlier, and $m_{[n\lambda]}(t)$ is defined as in (2.2) with N_n being replaced by $[n\lambda]$; all these processes belong to $D[0, 1]$ and are non-decreasing in t. Finally, we consider the three types of $G(x)$ that can appear in (1.2). These are respectively

$$(2.8)\qquad \begin{aligned} G_1(x, \theta) &= 0\,, && x \leqq 0\,, \\ &= \exp(-\theta x^{-\alpha})\,, && x > 0\,, \quad \theta > 0\,, \quad \alpha > 0\,, \\ G_2(x, \theta) &= \exp(-\theta(-x)^\alpha)\,, && x \leqq 0\,, \\ &= 1\,, && x > 0\,, \quad \theta > 0\,, \quad \alpha > 0\,, \\ G_3(x, \theta) &= \exp[-\theta(\exp[-x])]\,, && -\infty < x < \infty\,, \quad \theta > 0\,. \end{aligned}$$

Then, the main theorem of the paper is the following.

THEOREM 1. *Under* (1.2) *and* (2.1), m_{N_n} *converges in distribution in the Skorokhod J_1-topology on $D[\beta, 1]$ to the Markov process m, where $\beta = 0$ when G is of the type G_1, and $\beta > 0$ for the other two types.*

The proof of the theorem is postponed to Section 4. Certain other results needed in the proof are considered in Section 3.

3. A few basic results. With the notations in (1.2) through (2.8), we have the following.

LEMMA 3.1. *For every* $\varepsilon > 0$, *there exist a* $\delta > 0$ *and an* $n_0 = n_0(\varepsilon)$, *such that for* $n \geqq n_0(\varepsilon)$,

$$(3.1) \qquad \max_{k:|k-n|<\delta n} |(b_k - b_n)/b_n| < \varepsilon \quad \text{and} \quad \max_{k:|k-n|<\delta n} |(a_k - a_n)/b_n| < \varepsilon .$$

PROOF. Let $n^* = [n(1-\delta)]$, and define M_{n^*}, a_{n^*} and b_{n^*} for $n = n^*$ as in (1.1) and (1.2). Then, by (1.2) and Theorem 2.1 of Lamperti (1964), we have

$$(3.2) \qquad \lim_{n\to\infty} P\{(M_{n^*} - a_{n^*})/b_{n^*} \leqq x\} = G(x) , \qquad x \in R ,$$

$$(3.3) \qquad \lim_{n\to\infty} P\{(M_{n^*} - a_n)/b_n \leqq x\} = [G(x)]^{1-\delta} , \qquad x \in R .$$

Let us denote the exact df of $(M_n - a_n)/b_n$ and $(M_{n^*} - a_n)/b_n$ by $G_n(x)$ and $H_{n,\delta}(x)$ respectively, so that on writing

$$(3.4) \qquad (M_{n^*} - a_n)/b_n = (b_n^{-1}b_{n^*})[(M_{n^*} - a_{n^*})/b_{n^*}] - (a_n - a_{n^*})/b_n ,$$

we have

$$(3.5) \qquad H_{n,\delta}(x) = G_{n^*}(b_{n^*}^{-1}b_n x + (a_n - a_{n^*})/b_n) , \qquad x \in R .$$

Now, by (3.2) and (3.3), $G_{n^*}(x) \to G(x)$ and $H_{n,\delta}(x) \to [G(x)]^{1-\delta}$, at all points of continuity of G, as $n \to \infty$, and $|G(x) - [G(x)]^{1-\delta}|$ can be made arbitrarily small by proper choice of $\delta(> 0)$. Hence, from (3.5), we conclude that for every $\varepsilon > 0$, there exists a $\delta(> 0)$, such that as $n \to \infty$,

$$(3.6) \qquad |b_n^{-1}b_{n^*} - 1| < \varepsilon \quad \text{and} \quad |b_n^{-1}(a_n - a_{n^*})| < \varepsilon .$$

Similarly, on defining $n^{**} = [n(1+\delta)] + 1$, we have for $n \to \infty$,

$$(3.7) \qquad |b_n^{-1}b_{n^{**}} - 1| < \varepsilon \quad \text{and} \quad |b_n^{-1}(a_n - a_{n^{**}})| < \varepsilon .$$

Finally, we note that for G_1 and G_2, defined in (2.8), a_n is a constant, independent of n, while for G_3, a_n is $\uparrow$ in n. Further, for all the G_i, $i = 1, 2, 3$, b_n is monotonic (either non-increasing or non-decreasing depending on the type). Hence, (3.1) readily follows from (3.6) and (3.7). □

LEMMA 3.2. (*Uniform continuity, in probability.*) *For* $t > 0$, *and every* $\varepsilon > 0$, $\eta > 0$, *there exist a* $\delta(> 0)$ *and an* $n_0(\varepsilon, \eta)$, *such that for* $n \geqq n_0(\varepsilon, \eta)$,

$$(3.8) \qquad P\{\bigcup_{k:|k-n|<\delta n} [|m_k(t) - m_n(t)| > \varepsilon]\} < \eta .$$

PROOF. For $t > 0$, by (1.3), we have

$$(3.9) \qquad m_n(t) - m_k(t) = (1 - b_k^{-1}b_n)m_n(t) + b_k^{-1}b_n[m_n(t) - m_n(kt/n)] - b_k^{-1}b_n([a_n - a_k]/b_n) .$$

Hence, by virtue of Lemma 3.1, it suffices to show that as $n \to \infty$,

$$(3.10) \qquad |m_n(t)| \quad \text{is bounded, in probability,}$$

$$(3.11) \qquad \max_{k:|k-n|<\delta n} |m_n(t) - m_n(tk/n)| = o_p(1) .$$

Now, by (2.3) and Theorem 2.1 of Lamperti (1964), for every $K > 0$,

$$\begin{aligned}(3.12) \qquad \lim_{n\to\infty} P\{|m_n(t)| \leqq K\} &= P\{|m(t)| \leqq K\} \\ &= [G(K)]^t - [G(-K)]^t , \qquad t > 0 .\end{aligned}$$

Thus, for every $\eta > 0$, there exists a positive K_η ($< \infty$), such that [by (2.8)] the right-hand side of (3.12) exceeds $1 - \eta/4$ by choosing $K \geqq K_\eta$. Hence, there exists an $n_0(\eta)$, such that for $n \geqq n_0(\eta)$,

$$(3.13) \qquad P\{|m_n(t)| \leqq K_\eta\} \geqq 1 - \eta/2 , \qquad \text{i.e.,} \quad P\{|m_n(t)| > K_\eta\} < \eta/2 ,$$

which proves (3.10). Again, by the monotonicity of $\{m_n(t), t > 0\}$, we have

$$(3.14) \qquad \max_{k:|k-n|<\delta n} |m_n(t) - m_n(tk/n)| \leqq [m_n(t(1+\delta)) - m_n(t(1-\delta))] ,$$

where by Theorem 2.1 of Lamperti (1964), for every $\varepsilon > 0$, as $n \to \infty$,

$$\begin{aligned}(3.15) \qquad & P\{[m_n(t(1+\delta)) - m_n(t(1-\delta))] > \varepsilon\} \\ & \to 1 - \textstyle\int_{-\infty}^{\infty} [(x+\varepsilon)]^{2\delta t}\, d[G(x)]^{t(1-\delta)} ,\end{aligned}$$

which can be made smaller than $\eta/4$, $\eta > 0$, by choosing δ (> 0) appropriately small. Hence, there exists an $n_0(\eta)$, such that for $n \geqq n_0(\eta)$, the left-hand side of (3.11) is bounded above by ε with a probability $\geqq 1 - \eta/2$. □

Suppose now that for a fixed $q(\geqq 1)$, $\beta \leqq t_1 < \cdots < t_q \leqq 1$, are given points, where β is defined in Theorem 1. Then, by (3.8) and the Bonferroni inequality, we obtain that for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta > 0$ and an $n_0(\varepsilon, \eta)$, such that for $n \geqq n_0(\varepsilon, \eta)$,

$$(3.16) \qquad P\{\textstyle\bigcup_{j=1}^{q} \bigcup_{k:|k-n|<\delta n} [|m_k(t_j) - m_n(t_j)| > \varepsilon]\} < \eta .$$

For a non-decreasing jump process $x = \{x(t), t \in [0, 1]\}$, we define for every $\beta \geqq 0$ and $\delta > 0$,

$$(3.17) \qquad \Delta_\beta(\delta, x) = \sup_{\beta\leqq t\leqq 1} [\min\{|x((t+\delta)'') - x(t)|, |x(t) - x((t-\delta)')|\}] ,$$

where $(t-\delta)' = \max(\beta, t-\delta)$ and $(t+\delta)'' = \min(1, t+\delta)$. Note that $\Delta_\beta(\delta, x)$ is non-decreasing in δ (> 0) and non-increasing in β ($\geqq 0$).

LEMMA 3.3. *For every $\varepsilon > 0$, there exist an $\eta > 0$ and an $n_0(\varepsilon)$, such that for $n \geqq n_0(\varepsilon)$, $\delta > 0$, and $\eta^* = 2\eta/(1+\eta)$,*

$$(3.18) \qquad \max_{k:|k-n|<\eta n}\{\Delta_\beta(\delta, m_k)\} \leqq (1+\varepsilon)^2 \Delta_{\beta(1-\eta^*)}(\delta, m_{n+[n\eta]}) .$$

PROOF. Let $n^* = n + [n\eta]$. Then for every k: $|k - n| < \eta n$, by (1.3) for $0 \leqq s \leqq t \leqq 1$,

$$\begin{aligned}(3.19) \qquad m_k(t) - m_k(s) &= (M_{[kt]} - M_{[ks]})/b_k \\ &= (b_{n^*}/b_k)[(M_{[n^*(k/n^*)t]} - M_{[n^*(k/n^*)s]})/b_{n^*}] \\ &= (b_{n^*}/b_k)[m_{n^*}(kt/n^*) - m_{n^*}(ks/n^*)] ,\end{aligned}$$

where $1 - \eta^* \leqq k/n^* \leqq 1$. Therefore, by (3.17) and (3.19), for $n - [n\eta] \leqq k \leqq n^*$,

$$\begin{aligned}\Delta_\beta(\delta, m_k) &\leqq [\Delta_{\beta k/n^*}(k\delta/n^*, m_{n^*})](b_{n^*}/b_k) \\ &\leqq [\Delta_{\beta(1-\eta^*)}(\delta, m_{n^*})][\max_{k:|k-n|<\eta n}(b_{n^*}/b_k)] .\end{aligned} \tag{3.20}$$

The proof of the lemma is then completed by using Lemma 3.1, which bounds the second factor on the right-hand side of (3.20) by $(1+\varepsilon)^2$, by proper choice of $\eta > 0$. □

With reference to the probability space $(\Omega, \mathscr{A}, P)$, for $A \in \mathscr{A}$ and $B \in \mathscr{A}$, let $P(A \mid B)$ be the conditional probability of A given B; if $P(B) = 0$, we set $P(A \mid B) = P(A)$.

LEMMA 3.4. *If $A \in \mathscr{A}$ and $\beta > 0$, then for every $\varepsilon > 0$,*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P\{\Delta_\beta(\delta, m_n) > \varepsilon \mid A\} = 0 . \tag{3.21}$$

PROOF. Since $m_n(t)$ increases only by jumps, the event $\{\Delta_\beta(\delta, m_n) > \varepsilon\}$ is contained in the union of the two events $A_n = \{G(m_n(\beta)) < \eta\}$ and $B_n = \{G(m_n(\beta)) > \eta$ and for some $t \in [\beta, 1]$, $m_n(t)$ has at least two jumps in $(t, (t+\delta)'')\}$, where $\eta > 0$. Since $\beta > 0$, by Lemma 2 of Barndorff-Nielsen (1964), $|P(A_n \mid A) - P(A_n)| \to 0$ as $n \to \infty$, where by (2.3), $P(A_n) \to \eta^\beta$ and can be made arbitrarily small by choosing η (> 0) small. Also, by Theorem 3.2 of Lamperti (1964),

$$P(B_n) = 0(\delta) \qquad \text{as} \quad n \to \infty . \tag{3.22}$$

Since the event B_n is completely determined by the set of random variables

$$\{(M_k - a_n)/b_n, [n\beta] \leqq k \leqq n\} , \tag{3.23}$$

the arguments of Lemma 3 of Blum, Hanson and Rosenblatt (1963), adapted from the treatment of mixing sequences of sets by Rényi (1958), can be used as in Lemma 2 of Barndorff-Nielsen (1964) to show that

$$\lim_{n\to\infty} |P(B_n \mid A) - P(B_n)| = 0 . \tag{3.24}$$

Consequently, by (3.22) and (3.24), as $n \to \infty$,

$$P(B_n \mid A) = 0(\delta) , \tag{3.25}$$

and the proof of the lemma is complete.

LEMMA 3.5. *If G in (1.2) is of the type G_1 and $A \in \mathscr{A}$, then for every $\varepsilon > 0$,*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P\{\Delta_0(\delta, m_n) > \varepsilon \mid A\} = 0 . \tag{3.26}$$

PROOF. We note that if G is of the type G_1, then $b_n \uparrow \infty$ as $n \to \infty$, while $a_n = a$ is independent of n. Thus, for every $A \in \mathscr{A}$ and $\varepsilon > 0$, $P\{m_n(0) < -\varepsilon/2 \mid A\} = P\{X_1 < a - b_n\varepsilon/2 \mid A\} \to 0$ as $n \to \infty$. Consequently, with the modifications as in the first part of the proof of Theorem 3.2 of Lamperti (1964), the proof follows along the same line as in the preceding lemma, and hence, is omitted.

For a p-vector ($p \geqq 1$) $\mathbf{x}$, let $[\mathbf{x} \leqq \mathbf{a}]$ denote the coordinate wise inequality $x_i \leqq a_i$, $i = 1, \cdots, p$, and let $||\mathbf{x}||$ be the Euclidean norm $(\mathbf{x}'\mathbf{x})^{\frac{1}{2}}$.

LEMMA 3.6. *Let $\{\mathbf{Y}_n, n \geqq 1\}$ be a sequence of stochastic p-vectors, such that* (i)

$P\{\mathbf{Y}_n \leqq \mathbf{x}\} \to F(\mathbf{x})$ at every point of continuity of F, as $n \to \infty$, (ii) *for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta > 0$ and an $n_0(\varepsilon, \eta)$, such that for $n \geqq n_0(\varepsilon, \eta)$,*

$$P\{\max_{k:|k-n|<\delta n} ||\mathbf{Y}_k - \mathbf{Y}_n|| > \varepsilon\} < \eta\,, \tag{3.27}$$

and let $\{N_n, n \geqq 1\}$ be defined as in (2.1). *Then*

$$\lim_{n\to\infty} P\{\mathbf{Y}_{N_n} \leqq \mathbf{x}\} = F(x)\,, \tag{3.28}$$

at every point of continuity of F.

The proof follows as a direct multivariate extension of the proof of Theorem 2 of Mogyorodi (1965), by essentially replacing in each step of his proof the scalar Y_k by $\mathbf{Y}_k$ and $|Y_k - Y_n|$ by $||\mathbf{Y}_k - \mathbf{Y}_n||$. Hence, for brevity, the details are omitted.

4. Outline of the proof of Theorem 1. By Theorem 2.1 of Lamperti (1964), whenever (1.2) holds for some non-degenerate df G, the finite-dimensional laws of the process m_n, defined by (1.3) and (2.6), converge on the parameter interval (0, 1) to those of the Markov process m, defined by (2.3) and (2.4). Thus, for every $0 < t_1 < \cdots t_q \leqq 1$, $q \geqq 1$, as $n \to \infty$,

$$[m_n(t_1), \cdots, m_n(t_q)] \to_{\mathscr{D}} [m(t_1), \cdots, m(t_q)]\,, \tag{4.1}$$

where $\to_{\mathscr{D}}$ indicates convergence in law. Also, by (3.16), for every $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta > 0$, such that

$$P\{\max_{k:|k-n|<\delta n} \max_{j=1,\cdots,q} |m_k(t_j) - m_n(t_j)| > \varepsilon\} < \eta \qquad \text{as} \quad n \to \infty\,. \tag{4.2}$$

Hence, by (4.1), (4.2) and Lemma 3.6, under (1.2) and (2.1), as $n \to \infty$,

$$[m_{N_n}(t_1), \cdots, m_{N_n}(t_q)] \to_{\mathscr{D}} [m(t_1), \cdots, m(t_q)]\,, \tag{4.3}$$

for every $q \geqq 1$ and $0 < t_1 < \cdots < t_q \leqq 1$; $t_1 = 0$ is permissible for G being of the type G_1. This establishes the convergence of the finite dimensional distributions of the process m_{N_n} to those of m.

In order to show that as $n \to \infty$, $m_{N_n} \to_{\mathscr{D}} m$ on $D[\beta, 1]$ in the Skorokhod J_1-topology, we bring in the process $m_{[n\lambda]} = \{m_{[n\lambda]}(t), t \in [\beta, 1]\}$, defined in the same manner as in (2.2) with N_n being replaced by $[n\lambda]$. We also define $\Delta_\beta(\delta, m_{N_n})$ and $\Delta_\beta(\delta, m_{[n\lambda]})$ in the same way as in (3.17). Now, we require to show that for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta_0 = \delta_0(\varepsilon, \eta)$ and an $n_0(\varepsilon, \eta)$, such that for $n \geqq n_0(\varepsilon, \eta)$,

$$P\{\Delta_\beta(\delta_0, m_{N_n}) > \varepsilon\} < \eta\,. \tag{4.4}$$

Now, λ is a positive random variable. So, for every $0 < \eta < \frac{1}{2}$, there exists an $a_0(\eta)$, such that

$$P\{\lambda \leqq a_0(\eta)\} < \eta/4\,. \tag{4.5}$$

Also, by (2.1), for every $0 < \eta' < \frac{1}{2}$ and $0 < \eta < \frac{1}{2}$, there exists an $n_0(\eta', \eta)$, such that for $n \geqq n_0(\eta', \eta)$,

$$P\{|n^{-1}N_n - \lambda| > \eta',\} < \eta/4\,, \tag{4.6}$$

where η' is so chosen that (3.1) holds for δ being replaced by η' and some $0 < \varepsilon < \frac{1}{2}$. We may, equivalently, denote $n_0(\eta', \eta)$ by $n_0(\varepsilon, \eta)$. Then, by Lemma 3.3, (4.5) and (4.6), for $n \geqq n_0(\varepsilon, \eta)$,

$$\begin{aligned} &P\{\Delta_\beta(\delta, m_{N_n}) > \varepsilon\} \\ &\quad \leqq P\{\lambda \leqq a_0(\eta)\} + P\{|n^{-1}N_n - \lambda| > \eta'\} \\ &\qquad + P\{\Delta_\beta(\delta, m_{N_n}) > \varepsilon,\ \lambda > a_0(\eta),\ |n^{-1}N_n - \lambda| \leqq \eta'\} \\ &\quad \leqq P\{\Delta_{\beta^*}(\delta, m_{[n\lambda(1+\eta')]}) > \varepsilon/(1+\varepsilon)^2,\ \lambda > a_0(\eta)\} + \eta/2\,, \end{aligned} \tag{4.7}$$

where $\beta^* = \beta(1 - 2\eta'/(1+\eta'))$ is 0 or > 0 according as $\beta = 0$ or > 0.

Let us now select a countable set of points

$$a_h = a_h(\eta, \eta') = (1+\eta')^h a_0(\eta)\,, \qquad \text{for} \quad h = 0, 1, \cdots, \infty\,, \tag{4.8}$$

and let

$$A_h = \{a_{h-1} < \lambda \leqq a_h\}\,, \qquad h = 1, 2, \cdots, \infty\,. \tag{4.9}$$

Then, rewriting the first term on the right-hand side of (4.7) as

$$\textstyle\sum_{h=1}^{\infty} P\{\Delta_{\beta^*}(\delta, m_{[n\lambda(1+\eta')]}) > \varepsilon/(1+\varepsilon)^2 \,|\, A_h\} P(A_h)\,, \tag{4.10}$$

and then using Lemma 3.3, we may bound (4.10) by

$$\textstyle\sum_{h=1}^{\infty} P\{\Delta_{\beta^{**}}(\delta, m_{[na_h(1+\eta')^2]}) > \varepsilon/(1+\varepsilon)^4 \,|\, A_h\} P(A_h)\,, \tag{4.11}$$

where

$$\beta^{**} = \beta(1 - 2\eta'/(1+\eta'))^2 \quad \text{is} \quad 0 \quad \text{or} \quad > 0 \quad \text{according as} \quad \beta = \quad \text{or} \quad > 0\,. \tag{4.12}$$

Since, $\min_{h \geqq 1} n(1+\eta')^2 a_h = na_0(\eta)(1+\eta')^3 \to \infty$, as $n \to \infty$, by Lemmas 3.4 and 3.5, it follows that for every $\eta > 0$, $\varepsilon > 0$, there exist a $\delta_0 = \delta_0(\varepsilon, \eta)$ and an $n_0(\varepsilon, \eta)$, such that for $n \geqq n_0(\varepsilon, \eta)$ and $\delta \leqq \delta_0(\varepsilon, \eta)$,

$$P\{\Delta_{\beta^{**}}(\delta, m_{[n(1+\eta')^2 a_h]}) > \varepsilon/(1+\varepsilon)^4 \,|\, A_h\} < \eta/2\,, \qquad \text{for all} \quad h \geqq 1\,. \tag{4.13}$$

Consequently, by (4.11) and (4.13), the right-hand side of (4.7) is bounded by η for all $n \geqq n_0(\varepsilon, \eta)$. □

REMARKS. Lamperti (1964) has actually considered the weak convergence (in the Skorokhod J_1-topology) of m_n to m on an arbitrary finite interval $(0 \leqq)\beta \leqq t \leqq s < \infty$. The same is true for Theorem 1; we only need a magnification of the scale of t to extend the definition of the J_1-topology to $[\beta, s]$.

Secondly, instead of defining m_{N_n} as in (2.2) and (2.7), we could have considered a related process $m^*_{N_n} = \{m^*_{N_n}(t),\ t \in [0, 1]\}$, where

$$\begin{aligned} m^*_{N_n}(t) &= (M_{N[nt]} - a_{N_n})/b_{N_n}\,, \qquad N_n \geqq 1,\ N_{[nt]} \geqq 1\,, \\ &= (X_1 - a_{N_n})/b_{N_n}\,, \qquad N_n \geqq 1,\ 0 \leqq N_{[nt]} \leqq 1\,, \\ &= -\infty\,, \qquad \text{otherwise.} \end{aligned} \tag{4.14}$$

In such a case, we require a little more stringent condition on the mode of convergence of N_n to $n\lambda$. For example, if $n^{-1}N_n \to \lambda$ a.s., as $n \to \infty$, i.e.,

$$P\{\sup_{m \geqq n} |m^{-1}N_m - \lambda| > \varepsilon\} \to 0 \qquad \text{as} \quad n \to \infty, \tag{4.15}$$

then, the proof of Theorem 1 can be readily extended to show that as $n \to \infty$,

$$m^*_{N_n} \to_{\mathscr{D}} m \quad \text{on } D[\beta, 1] \text{ in the Skorokhod } J_1\text{-topology.} \tag{4.16}$$

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