

ON R. VON MISES' CONDITION FOR THE DOMAIN OF ATTRACTION OF $\exp(-e^{-x})^1$

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There exist well-known necessary and sufficient conditions for a distribution function to belong to the domain of attraction of the double exponential distribution Λ . For practical purposes a simple sufficient condition due to von Mises is very useful. It is shown that each distribution function F in the domain of attraction of Λ is tail equivalent to some distribution function satisfying von Mises' condition.

Suppose X_1, X_2, X_3, \dots are independent real-valued random variables with common distribution function F . We say that F is in the domain of attraction of the double exponential distribution (notation $F \in D(\Lambda)$; $\Lambda(x) = \exp(-e^{-x})$) if there exist two sequences of real constants $\{b_n\}$ and $\{a_n\}$ (with $a_n > 0$ for $n = 1, 2, \dots$) such that for all real x

$$(1) \quad \lim_{n \rightarrow \infty} P \left\{ \frac{\max(X_1, X_2, \dots, X_n) - b_n}{a_n} \leq x \right\} = \exp(-e^{-x}).$$

It is convenient to use the symbol x_0 for the upper bound of X_1 defined by

$$x_0(F) = \sup \{x \mid F(x) < 1\}.$$

Necessary and sufficient conditions for $F \in D(\Lambda)$ are well-known (Gnedenko (1943), de Haan (1970)) but rather intricate. The following relatively simple criterion is due to R. von Mises ((1936) page 285):

Suppose $F(x)$ is a distribution function with a density $f(x)$ which is positive and differentiable on a left neighborhood of x_0 . If

$$(2) \quad \lim_{x \uparrow x_0} \frac{d}{dx} \left(\frac{1 - F(x)}{f(x)} \right) = 0,$$

then $F \in D(\Lambda)$.

A distribution function F satisfying (2) will be called a *von Mises function*.

We shall prove

THEOREM. *A distribution function F lies in the domain of attraction of Λ if and only if there exists a von Mises function F_* such that $x_0(F_*) = x_0(F) = x_0$ and*

$$(3) \quad \lim_{x \uparrow x_0} \frac{1 - F(x)}{1 - F_*(x)} = 1.$$

REMARK. Relation (3) implies (see Resnick (1971) Lemma 2.5) that for the

Received May 21, 1971; revised December 28, 1971.

¹ Report SW 10/71, Afdeling Mathematische Statistiek, Mathematisch Centrum, Amsterdam.

convergence of the distribution functions F^n and F_*^n the same norming constants $a_n > 0$ and b_n may be used.

PROOF. The if statement is an immediate consequence of Gnedenko (1943), Théorème 7.

Now suppose $F \in D(\Lambda)$ with endpoint x_0 . We define the sequence U_0, U_1, \dots by

$$U_0(x) = 1 - F(x)$$

$$U_{n+1}(x) = \int_{x_0}^{x_0} U_n(t) dt \quad n = 0, 1, 2, \dots$$

By de Haan ((1970) Lemma 2.5.1 or (1971) Lemma 6 and Theorem 8) the distribution function F_n defined by $F_n(x) = \max(0, 1 - U_n(x))$ belongs to $D(\Lambda)$ if F_{n-1} does. In particular, the integral above converges. Then $F_n \in D(\Lambda)$ for $n = 0, 1, 2, \dots$ and by de Haan ((1970) Theorem 2.5.2 or (1971) Theorem 10) we have

$$(4) \quad \lim_{x \uparrow x_0} \frac{U_{n-1}(x)U_{n+1}(x)}{U_n^2(x)} = 1 \quad n = 1, 2, \dots$$

We now define the function U_* on $(-\infty, x_0)$ by

$$U_*(x) = \{U_3(x)\}^4 \{U_4(x)\}^{-3}.$$

Then $U_*(x)$ is twice differentiable on a left neighborhood of x_0 and

$$(5) \quad \frac{d}{dx} \log U_* = -4 \frac{U_2}{U_3} + 3 \frac{U_3}{U_4} = \frac{3 - 4U_2U_3^{-2}U_4}{U_4U_3^{-1}}.$$

Consider

$$\frac{U_4U_3^{-1}}{3 - 4U_2U_3^{-2}U_4} = \frac{U_*}{\frac{d}{dx} U_*}.$$

By (4) the denominator is asymptotic to -1 as $x \uparrow x_0$ and both $(d/dx)U_4U_3^{-1}$ and $U_4U_3^{-1}(d/dx)(3 - 4U_2U_3^{-2}U_4)$ vanish as $x \uparrow x_0$. Hence

$$(6) \quad \lim_{x \uparrow x_0} \frac{d}{dx} \left(\frac{U_*(x)}{U_*'(x)} \right) = 0.$$

Observe that

$$U_0 = \frac{U_0U_2}{U_1^2} \cdot \left(\frac{U_1U_3}{U_2^2} \right)^2 \cdot \left(\frac{U_2U_4}{U_3^2} \right)^3 \cdot U_*.$$

Hence by (4) we obtain

$$(7) \quad \lim_{x \uparrow x_0} \frac{U_0(x)}{U_*(x)} = 1.$$

Then $\lim_{x \uparrow x_0} U_*(x) = 0$, and since by (5) U_* is decreasing on a left neighborhood of x_0 , there exists a twice differentiable distribution function $F_*(x)$ which coincides with $1 - U_*(x)$ on a left neighborhood of x_0 . F_* is a von Mises function by (6) and satisfies (2) by (7). \square

COROLLARY. A distribution function F belongs to $D(\Lambda)$ if and only if there exist

a positive function c satisfying $\lim_{x \uparrow x_0} c(x) = 1$ and a positive differentiable function ϕ satisfying $\lim_{x \uparrow x_0} \phi'(x) = 0$ such that

$$1 - F(x) = c(x) \cdot \exp \left\{ - \int_{-\infty}^x \frac{dt}{\phi(t)} \right\} \quad \text{for } x < x_0.$$

This improves the representation theorem (Theorem 2.5.3) in de Haan (1970).

PROOF. Set

$$\phi(x) = \frac{1 - F_*(x)}{F_*'(x)}$$

in a left neighborhood of x_0 .

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