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BALANCED DESIGNS WITH UNEQUAL REPLICATIONS AND UNEQUAL BLOCK SIZES

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Using two distinct BIB designs with the same number of treatments, a method of construction of balanced binary and ternary designs with unequal block sizes and unequal replications is given. The method is a generalization of the method of construction of balanced designs, given by John [6].

1. Introduction. It was shown by Rao [8] that a necessary and sufficient condition for a design to be balanced, i.e., Var $(\hat{t}_i - \hat{t}_j)$ to be the same for all pairs (i, j), $i \neq j$, is that the matrix C of the adjusted intrablock normal equations shall have its diagonal elements all equal and its off-diagonal elements equal. If r_i denotes the number of times the ith treatment is replicated $(i = 1, 2, \cdots, v)$, k_j denotes the number of plots in the jth block $(j = 1, 2, \cdots, b)$ and $N = (n_{ij})$ is the $v \times b$ incidence matrix, then C is given by

$$C = \operatorname{diag}(r_1, r_2, \cdots, r_v) - N \operatorname{diag}(k_1^{-1}, k_2^{-1}, \cdots, k_b^{-1})N' .$$

A design is said to be equi-replicate, if $r_i = r$ for all i, proper, if $k_j = k$, for all j and binary, if n_{ij} takes only the values 0 or 1. Tocher [9] defined a design to be ternary as one in which n_{ij} takes only the values 0, 1, or 2. Tocher presented some examples of proper balanced ternary designs. Das and Rao [2] generalised Tocher's definition of ternary designs and defined a ternary design to be one in which n_{ij} takes only three integral values, say, p_1, p_2, p_3 (where p_1, p_2, p_3 are not necessarily 0, 1 or 2). Methods of construction of balanced ternary designs (proper and equi-replicate) are available in Das and Rao [2], Murty and Das [7] and Dey [3].

Using the results of Atiqullah [1], Graybill [4] and Hanani [5], John [6] gave a simpler proof of Rao's [8] theorem and presented examples of balanced binary and ternary designs with unequal block sizes and unequal number of replicates.

This communication presents a method of construction of balanced binary and ternary designs with two unequal block sizes and unequal replicates. The designs of John [6] are particular cases of the designs obtained in this paper.

2. The balanced designs.

2.1. We consider designs of two block sizes, say k_1 and k_2, $k_1 \neq k_2$. Let there be n_1 blocks of size k_1 and n_2 blocks of size k_2. Further, let N_i be the incidence matrix of the n_i $(i = 1, 2)$ blocks. Clearly, N_i is of order $v \times n_i$ $(i = 1, 2)$. Then, the C-matrix may be written as

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$$C = \text{diag}\,(r_1, r_2, \cdots, r_v) - (1/k_1)N_1N_1' - (1/k_2)N_2N_2' , \tag{2.1}$$

where v is the number of treatments in the design. Also, the design is balanced, if and only if C is of the form

$$C = (a - b)I_v + bJ_v , \tag{2.2}$$

where I_v is a unit matrix of order v, J_v is a square matrix of order v with every element equal to unity and a and b are two constants.

From (2.1) and (2.2) we easily find that the design will be balanced, if and only if

$$\begin{aligned} &r_s - R_s^{(1)}/k_1 - R_s^{(2)}/k_2 = \text{constant} , && \text{for all} \quad s = 1, \cdots, v ; \\ &\Lambda_{st}^{(1)}/k_1 + \Lambda_{st}^{(2)}/k_2 = \text{constant} , && \text{for all} \quad s \neq t;\ s, t = 1, 2, \cdots, v , \end{aligned} \tag{2.3}$$

where r_s denotes the number of times the sth treatment occurs in the design,

$$\begin{aligned} R_s^{(i)} &= \textstyle\sum_{j=1}^{n_i} n_{sj(i)} , \\ \Lambda_{st}^{(i)} &= \textstyle\sum_{j=1}^{n_i} n_{sj(i)} n_{tj(i)} , && i = 1, 2;\ s \neq t , \end{aligned}$$

and $n_{sj(i)}$ denotes the (s, j)th element of N_i.

Equations (2.3) help us in deriving balanced ternary (as well as binary) designs with variable replications and two unequal block sizes.

2.2. *Method of construction of balanced ternary designs.* The designs involve $(v + 1)$ treatments, $t_0, t_1, t_2, \cdots, t_v$. Each design consists of two portions. The first portion consists of a balanced incomplete block (BIB) design in $t_1, t_2, \cdots, t_v$ with parameters v, b, r, k, λ and each of the b blocks is augmented with k^* plots containing the treatment t_0. The second portion consists of a BIB design in $t_1, t_2, \cdots, t_v$ with parameters v, b', r', k' and λ'. Further, each of the b blocks with $(k + k^*)$ plots each is repeated n times and each of the b' blocks containing k' plots each is repeated m times.

Then we have the following

THEOREM 2.1. *The design in $(v + 1)$ treatments and $(nb + mb')$ blocks is a balanced ternary design with $r_0 = nbk^*$, $r_i = nr + mr'$, $(i = 1, \cdots, v)$, $k_1 = k + k^*$, $k_2 = k'$, whenever m and n are such that $m/n = (k^*r - \lambda)k'/\{\lambda'(k + k^*)\}$.*

PROOF. We have, for the above design,

$$\begin{aligned} r_0 &= nbk^* , & r_i &= nr + mr' , \\ R_0^{(1)} &= nbk^{*2} , & R_0^{(2)} &= 0 , \\ R_i^{(1)} &= nr , & R_i^{(2)} &= mr' , \\ \Lambda_{0i}^{(1)} &= nk^*r , & \Lambda_{0i}^{(2)} &= 0 , \\ \Lambda_{ij}^{(1)} &= n\lambda , & \Lambda_{ij}^{(2)} &= m\lambda' , && (i \neq j,\ i, j = 1, \cdots, v) . \end{aligned}$$

Substituting these values in (2.3) we get

$$nkk^*b/(k + k^*) - nr(k + k^* - 1)/(k + k^*) = mr'(k' - 1)/k' ,$$

and

$$n(k^*r - \lambda)/(k + k^*) = m\lambda'/k' .$$

Since m and n should be so chosen such that both the above equations are satisfied simultaneously, we must have

$$\text{(2.4)} \qquad m/n = \{kk'k^*b - rk'(k + k^* - 1)\}/\{r'(k' - 1)(k + k^*)\} \\ = k'(k^*r - \lambda)/\{\lambda'(k + k^*)\} .$$

Also, it is verified easily that for any two BIB designs with parameters v, b, r, k, λ and v, b', r', k', λ', the relation

$$\{kk'k^*b - rk'(k + k^* - 1)\}/\{r'(k' - 1)(k + k^*)\} = k'(k^*r - \lambda)/\{\lambda'(k + k^*)\}$$

is always true.

Hence the theorem.

2.3. *Balanced binary designs.* In the above method of construction if $k^* = 1$, the design is balanced binary. Thus, we have the corollary to Theorem 2.1.

COROLLARY. *In Theorem* 2.1 *if* $k^* = 1$, *then the design with* $v + 1$ *treatments and* $(nb + mb')$ *blocks is a balanced binary design with* $r_0 = nb$, $r_i = nr + mr'$, $i = 1, \cdots, v$, $k_1 = k + 1$, $k_2 = k'$, *whenever* m *and* n *are such that* $m/n = (r - \lambda)k'/\{\lambda'(k + 1)\}$.

EXAMPLE. We present an illustration of a balanced ternary design. Let $k^* = 2$, $v = 7$. The design with 8 treatments has the following 28 blocks:

(00124), (00235), (00346), (00457), (00561), (00672), (00713),
(124), (235), (346), (457), (561), (672), (713).

The last seven blocks are repeated twice. For convenience, the treatments are denoted by 0, 1, 2, $\cdots$, 7 instead of $t_0, t_1, \cdots, t_7$.

In conclusion, we remark that the values of m and n may get large as the number of treatments and/or block sizes increase and thus these designs might not find much use in the field of agriculture. However, they may be of use in the industrial field (cf. the discussion in Tocher [9]).

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