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LIMIT THEOREMS FOR SUMS OF RANDOM VARIABLES DEFINED ON FINITE INHOMOGENEOUS MARKOV CHAINS

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Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\{X_n : n \geqq 1\}$ an inhomogeneous Markov chain assuming a finite number of states defined on this space, $E = \{a_1, \cdots, a_s\}$ the set of its states, $p_j^{(n)} = P\{X_n = a_j\}$, $p_{ij}^{(k,n)} = P\{X_n = a_j | X_k = a_i\}$ for $n = 2, 3, \cdots, n > k$, $a_i, a_j \in E$, $\{f_n : n \geqq 1\}$ a sequence of real valued functions defined on E and $S_n = f_1(X_1) + \cdots + f_n(X_n)$.

To study the Markov chains which are not subjected to "asymptotic independent" restrictions, the author proposes the coefficients

$$a_{k,n} = \max\nolimits'_{i \in \{1,\cdots,s\}} \sum_{j=1}^{s} (p_j^{(n)} - p_{ij}^{(k,n)})^+ \qquad (n = 2, 3, \cdots, n > k)$$

where the dash indicates that the max is taken over those i such that $p_i^{(k)} > 0$.

Some limit properties of the sums $\{S_n : n \geqq 1\}$ suitably normed, as the behavior of the series of random variables and the strong law of large numbers are investigated. In the end some examples are given and it is proved that the arbitrary homogeneous Markov chains satisfy most of the conditions imposed in the paper.

0. Introduction and summary. Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\{X_n : n \geqq 1\}$ an inhomogeneous Markov chain assuming a finite number of states defined on this space, $E = \{a_1, \cdots, a_s\}$ the set of states, $\mathscr{F}_m^{\ n}$ the σ-algebra generated by the random variables $X_m, X_{m+1}, \cdots, X_n$ $(m, n = 1, 2, \cdots)$. Note $p_j^{(n)} = P\{X_n = a_j\}$ and $p_{ij}^{(k,n)} = P\{X_n = a_j | X_k = a_i\}$ for $n = 2, 3, \cdots, n > k$ and $a_i, a_j \in E$. Let further $\{f_n : n \geqq 1\}$ be a sequence of real valued functions defined on E, $\xi_n = f_n(X_n)$, $m_n = E(\xi_n)$ and $S_n = \sum_{i=1}^{n} \xi_i$ $(n = 1, 2, \cdots)$. By $\mathscr{T} = \bigcap_{n=1}^{\infty} \mathscr{F}_n^{\ \infty}$ we shall denote the tail σ-alebra of the considered Markov chain.

The behavior of the sums $\{S_n : n \geqq 1\}$ suitably normed has been investigated by many authors in the case of an ergodic or weakly dependent Markov chain. It has been shown that results similar to those existing for independent random variables remain true for such sequences. The methods employed in the proofs appeal chiefly to the "asymptotic independence" properties of the chains. One of the most important procedures used when studying such properties is based upon the so called "ergodic coefficients," defined as follows:

$$(1) \qquad \alpha_{k,n} = 1 - \max_{i,j \in \{1,\cdots,s\}} \sum_{l=1}^{s} (p_{i,l}^{(k,n)} - p_{j,l}^{(k,n)})^+ \qquad (n = 2, 3, \cdots, n > k)$$

where x^+ stands for max $(x, 0)$.

These coefficients are characteristic for "asymptotic independent chains," the condition imposed on them being usually formulated by requirements that $\{\alpha_{k,n}\}$

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to be larger than 0 according to a certain rule (see [3], [4] and [6]).

In many cases (when $\mathscr{T}$ is not trivial) all these coefficients are null for n and k sufficiently large and the analogy with "independent sequences" ceases in many respects. A series of basic properties as 0–1 law, the Borel-Cantelli lemma, the central limit theorem, etc. fail to take place.

The aim of this paper is to provide a method of handling such cases. For this purpose, we introduce the following coefficients:

$$a_{k,n} = \max'_{i\in\{1,\cdots,s\}} \sum_{j=1}^{s} (p_j^{(n)} - p_{ij}^{(k,n)})^+ \qquad (n = 2, 3, \cdots, n > k) \tag{2}$$

where the dash indicates that the maximum is taken over those values i such that $p_i^{(k)} > 0$.

These coefficients will prove useful especially when the chain is "dependent," but as we shall see in the sequel they may also be employed coupled with the ergodic coefficients when $\mathscr{T}$ is trivial.

In the first section some properties of the above given coefficients which are to be used in the sequel are given. The second section deals with some analogies of classical results valid for independent random variables. One gives a Borel-Cantelli type theorem as well as a result concerning the behavior of the series of random variables. The third section is devoted to the strong law of large numbers (both in general and when $\mathscr{T}$ is trivial). In the fourth section some examples of chains satisfying the restrictions imposed in the paper are given.

1. The coefficients $\{a_{k,n}\}$.

LEMMA 1. *If $\{X_n : n \geqq 1\}$ is a finite inhomogeneous Markov chain and $\{a_{k,n}\}$ a sequence of numbers defined by* (2), *then*

$$0 \leqq a_{k,n} \leqq 1 - \min'_{1\leqq i\leqq s} p_i^{(k)} . \tag{3}$$

The proof is straightforward and will be omitted. This lemma assures us that $\{a_{k,n}\}$ are always smaller than 1, whatever may be the corresponding ergodic coefficients. Such a property makes these coefficients useful, although some complicated situations might occur if they are approaching 1.

LEMMA 2. *If $\{X_n : n \geqq 1\}$ is a finite inhomogeneous Markov chain and $\{a_{k,n}\}$ a sequence of numbers defined by* (2), *then*

$$\max'_{i\in\{1,\cdots,s\}} \max_{\Lambda\in\mathscr{P}} |\textstyle\sum_{j\in\Lambda} (p_j^{(n)} - p_{ij}^{(k,n)})| = a_{k,n} \tag{4}$$

$\mathscr{P}$ being the family of all the subsets of E.

PROOF. We have

$$\max_{\Lambda\in\mathscr{P}} |\textstyle\sum_{j\in\Lambda} (p_j^{(n)} - p_{ij}^{(k,n)})| = \max (\sum_{j=1}^{s} (p_j^{(n)} - p_{ij}^{(k,n)})^+, \sum_{j=1}^{s} (p_j^{(n)} - p_{ij}^{(k,n)})^-) .$$

Now, if we notice that

$$\sum_{j=1}^{s} (p_j^{(n)} - p_{ij}^{(k,n)})^+ = \sum_{j=1}^{s} (p_j^{(n)} - p_{ij}^{(k,n)})^-$$

we complete the proof.

LEMMA 3. *Let* $\{X_n : n \geqq 1\}$ *be a finite inhomogeneous Markov chain and* $\{a_{k,n}\}$ *a sequence of numbers defined by* (2). *Then*

$$\sup_{A \in \mathscr{F}_n^\infty} \mathrm{ess}_\omega \sup |P\{A\} - P\{A|\mathscr{F}_1^k\}| = a_{k,n} . \tag{5}$$

PROOF. Firstly, let us notice that by (4)

$$\sup_{A \in \mathscr{F}_n^\infty} \mathrm{ess}_\omega \sup |P\{A\} - P\{A\,|\,\mathscr{F}_1^k\}| \geqq a_{k,n} . \tag{6}$$

Consider now an arbitrary N-dimensional Borelian set B and write $A = \{\omega : (X_n(\omega), \cdots, X_{n+N-1}(\omega)) \in B\}$.

We shall prove that for any $a_i \in E$, we have

$$P\{A\} - P\{A|X_k = a_i\} \leqq a_{k,n} . \tag{7}$$

Indeed, using the time reversibility property of the Markov chain, we get

$$\begin{aligned} P\{A|X_k = a_i\} \\ = \textstyle\sum_{(b_1,\cdots b_N) \in B} \frac{P\{X_n = b_1, \cdots, X_{n+N-1} = b_N\}P\{X_k = a_i|X_n = b_1\}}{p_i^{(k)}} . \end{aligned} \tag{8}$$

By (8) we obtain easily:

$$\begin{aligned} &P\{A|X_k = a_i\} \\ &\quad = \textstyle\sum_{(b_1,\cdots,b_n) \in B} P\{X_{n+1} = b_2, \cdots, X_{n+N-1} = b_N|X_n = b_1\}P\{X_n = b_1|X_k = a_i\} . \end{aligned}$$

Therefore

$$\begin{aligned} P\{A\} - P\{A|X_k = a_i\} = \textstyle\sum_{(b_1,\cdots,b_n) \in B} P\{X_{n+1} = b_2, \cdots, X_{n+N-1} = b_N|X_n = b_1\} \\ \times (P\{X_n = b_1\} - (P\{X_n = b_1|X_k = a_i\}) \end{aligned}$$

wherefrom we get (7).

Now, taking into account the Markov property and employing a standard approximation reasoning, we obtain

$$\sup_{A \in \mathscr{F}_n^\infty} \mathrm{ess}_\omega \sup (P\{A\} - P\{A|\mathscr{F}_1^k\}) \leqq a_{k,n} .$$

Observing now that

$$\begin{aligned} &\sup_{A \in \mathscr{F}_n^\infty} \mathrm{ess}_\omega \sup (P\{A\} - P\{A|\mathscr{F}_1^k\}) \\ &\qquad = \sup_{A \in \mathscr{F}_n^\infty} \mathrm{ess}_\omega \sup (P\{A|\mathscr{F}_1^k\} - P\{A\}) \end{aligned}$$

and taking into account (6), we complete the proof.

LEMMA 4. *If* $\{X_n : n \geqq 1\}$ *is a finite inhomogeneous Markov chain and* $\{a_{k,n}\}$ *a sequence of numbers defined by* (2), *then*

$$a_{k,n} \leqq a_{k',n'} \tag{9}$$

for any $k \leqq k'$ *and* $n \geqq n'$.

The proof follows directly from Lemma 3.

We give now a property which expresses the connection between the ergodic coefficients (1) and the above defined coefficients (2).

LEMMA 5. *If* $\{X_n : n \geqq 1\}$ *is a finite inhomogeneous Markov chain,* $\{\alpha_{k,n}\}$ *and* $\{a_{k,n}\}$ *two sequences of numbers defined by* (1) *and* (2) *respectively, then*

$$1 - \alpha_{k,n} \leqq 2a_{k,n} \tag{10}$$

$$a_{k,n} \leqq 1 - \alpha_{k,n} \tag{11}$$

PROOF. (10) follows from

$$1 - \alpha_{k,n} \leqq \max'_{i \leqq i \leqq s} \sum_{j=1}^{s} (p_j^{(n)} - p_{ij}^{(k,n)})^+ + \max'_{1 \leqq i \leqq s} \sum_{j=1}^{s} (p_{ij}^{(k,n)} - p_j^{(n)})^+ = 2a_{k,n} .$$

To prove (11) we consider the inequality

$$\sum_{l=1}^{s} (p_{i,l}^{(k,n)} - p_{j,l}^{(k,n)})^+ \leqq 1 - \alpha_{k,n} \tag{12}$$

where $i, j = 1, \cdots, s$. Multiplying the both sides of (12) by $p_i^{(k)}$ and summing over i we get (11).

2. Some limit properties. Consider now the expression

$$a^* = 1 - \min(\gamma, \inf \Lambda) \tag{13}$$

with $\gamma = \liminf^* a_{k,n}$, where the star indicates that lim inf is taken over all subsequences $\{a_{k_u, n_t}\}$ such that

$$\inf_{u=1,2,\dots} \min'_{1 \leqq j \leqq s} p_j^{k_u} > 0$$

if such subsequences exist. When such a case does not occur we take $\gamma = 1$.

To define Λ, we consider a sequence of positive integers $\{k_u\}$ such that

$$\liminf_{u \to \infty} \min'_{1 \leqq j \leqq s} p_j^{k_u} = 0 .$$

Successively eliminating from the expression of $\{a_{k_u, n_t}\}$, with n_t arbitrary chosen, the indexes corresponding to the events where the above minimum is achieved, until this limit becomes positive and denoting the new expressions by $\{a'_{k_u, n_t}\}$, we get in the long run

$$\lim_{u,t \to \infty} \inf a'_{k_u, n_t} = \lambda .$$

Denote by Λ the set of all the above constructed limits. Because $0 \leqq \lambda < 1$ as could be easily seen $a^* > 0$. A scrutiny of the proof given in [2] reveals to us that for any $T \in \mathscr{T}$ we have

$$P(T) \geqq a^* . \tag{14}$$

It is not difficult to notice that $\mathscr{T}$ is trivial if and only if $a^* = 1$.

Now, we give a property corresponding to the Borel-Cantelli lemma (to that part of the lemma which supposes the independence of the random variables).

THEOREM 1. *Let* $\{X_n : n \geqq 1\}$ *be a finite inhomogeneous Markov chain with the property that there exists a positive interger* r *such that* $\lim_{k \to \infty} \sup a_{k,k+r} < 1$, *and* $\{A_n : n \geqq 1\}$ *a sequence of events with* $A_n \in \mathscr{F}_n^n$, $n = 1, 2, \cdots$.

If $\sum_{n=1}^{\infty} P\{A_n\} = \infty$ *then* $P\{\limsup A_n\} \geqq a^*$ *with* a^* *defined by* (13).

The proof is based on the Lemma 3, on the evaluation given in (14) and may be carried out as in [1] (Lemma 1.1).

THEOREM 2. *Let* $\{X_n : n \geqq 1\}$ *be a finite inhomogeneous Markov chain and* $\{k_i\}$ *a sequence of increasing positive integers such that* $\lim_{i\to\infty} \sup a_{k_i,k_{i+1}} < 1$. *If* $\{n_i\}$ *is a sequence of increasing positive intergers such that for any* i $(i = 1, 2, \cdots)$ *there exists a number* j *such that* $n_j \leqq k_i$ *and* $n_{j+1} \geqq k_{i+1}$, *then the series* $\sum_{i=1} (\xi_{n_i} - m_{n_i})$ *converges almost surely if and only if it converges in probability.*

The proof may be carried out taking into consideration Lemmas 3 and 4 the Theorem 1.1.12 of [6].

As a particular case of this theorem we give now a result analogous to a well-known property of independent random variables.

THEOREM 2′. *Let* $\{X_n : n \geqq 1\}$ *be a finite inhomogeneous Markov chain with the property that* $\lim_{k\to\infty} \sup a_{k,k+1} < 1$. *Then the series* $\sum_{n=1}^{\infty} (\xi_n - m_n)$ *converges almost surely if and only if it converges in probability.*

3. The strong law of large numbers. We shall give now an inequality of the Hájek-Rényi [7] type for finite Markov chains without assuming any restriction on the chain's structure. This inequality will be used in the sequel in proving some laws of large numbers.

THEOREM 3. *Let* $\{X_n : n > 1\}$ *be a finite inhomogeneous Markov chain,* $\{n_i\}$ *a sequence of increasing positive integers,* m *and* n *two positive numbers. If we denote* $\xi_i' = \xi_{n_i}$, $\eta_i' = m_{n_i}$, $S_j^* = \sum_{i=1}^{j} (\xi_i' - m_i')$ *for* $1 \leqq i \leqq n$ *and* $m \leqq j \leqq n$, $v_{m,n} = 1 - \max_{m \leqq i \leqq n-1} a_{n_i,n_{i+1}}$, *then for any sequence of positive and nonincreasing numbers* $\{c_m, c_{m+1}, \cdots, c_n\}$ *and for any* $\varepsilon > 0$, *we have*

$$P\{\max_{m \leqq i \leqq n} c_i |S_i^*| > \varepsilon\} \leqq \frac{\frac{16}{\varepsilon^2} D^2(c_m S_m^* + \sum_{i=m+1}^{n} c_i \xi_i')}{v_{m,n} - \frac{16}{\varepsilon^2} \max_{m+1 \leqq j \leqq n} D^2(\sum_{i=j}^{n} c_i \xi_i')} \tag{15}$$

provided that the above denominator of the right hand side is positive.

PROOF. Consider the following random variables

$$S_m' = c_m S_m^*$$
$$S_j' = c_m S_m^* + \sum_{i=m+1}^{j} c_i(\xi_i' - m_i') \qquad (j = m+1, \cdots, n).$$

Let further

$$A = \{\omega : \max_{m \leqq j \leqq n} |S_j'| > \varepsilon\}$$
$$A_m = \{\omega : |S_m'| > \varepsilon\}$$
$$A_j = \{\omega : |S_m'| \leqq \varepsilon, |S'_{m+1}| \leqq \varepsilon, \cdots, |S'_{j-1}| \leqq \varepsilon, |S_j'| > \varepsilon\} \qquad (j = m+1, \cdots, n)$$
$$B_j = \{\omega : |S_n' - S_j'| < \tfrac{1}{2}\varepsilon\}. \qquad (j = m, \cdots, n-1).$$

Then, obviously

$$\textstyle\sum_{j=m}^{n} A_j = A \quad \text{and} \quad A_j \cap A_k = \varnothing \quad \text{if} \quad j \neq k .$$

On the other hand, if we consider the set

$$C = \{\omega : |S_n'| \geqq \tfrac{1}{2}\varepsilon\}$$

we have

$$\textstyle\sum_{j=m}^{n} A_j B_j \subset C . \tag{16}$$

Taking into account Lemma 3, we get

$$\begin{aligned} P(\textstyle\sum_{j=m}^{n} A_j B_j) & \\ = \textstyle\sum_{j=m}^{n} P(A_j B_j) &\geqq \textstyle\sum_{j=m}^{n} [P(A_j)P(B_j) - (1 - v_{m,n})P(A_j)] . \end{aligned} \tag{17}$$

Using now the Chebyshev inequality, we obtain

$$P(B_j) = P\Big(|\textstyle\sum_{i=j+1}^{n} c_i(\xi_i' - m_i')| \leqq \dfrac{\varepsilon}{2}\Big) \geqq 1 - \dfrac{4D^2(\sum_{i=j+1}^{n} c_i\xi_i')}{\varepsilon^2} \qquad (j = m, \cdots, n-1) .$$

Therefore

$$\min_{m \leqq j \leqq n-1} P\{B_j\} \geqq 1 - \frac{4}{\varepsilon^2} \max_{m+1 \leqq j \leqq n} D^2(\textstyle\sum_{i=j}^{n} c_i\xi_i') . \tag{18}$$

(17) and (18) together imply

$$P(\textstyle\sum_{i=m}^{n} A_i B_i) \geqq \Big[v_{m,n} - \dfrac{4}{\varepsilon^2} \max_{m+1 \leqq j \leqq n} D^2(\sum_{i=j}^{n} c_i\xi_i')\Big] P(A) .$$

Making use of (16) and employing the Chebyshev inequality again, we get

$$P(A) \leqq \frac{\dfrac{4}{\varepsilon^2} D^2(c_m S_m^* + \sum_{i=m+1}^{n} c_i\xi_i')}{v_{mn} - \dfrac{4}{\varepsilon^2} \max_{m+1 \leqq j \leqq n} D^2(\sum_{i=m+1}^{j} c_i\xi_i')} . \tag{19}$$

Consider now the inequality

$$\max_{m \leqq j \leqq n} (b_j \textstyle\sum_{i=1}^{j} a_i) \leqq 2 \max_{m \leqq j \leqq n} (\sum_{i=1}^{j} b_i a_i) ,$$

true for any sequence of numbers $\{a_i\}$ and for any positive and non-increasing sequence $\{b_i\}$. (This inequality is to be found in [8]).

Putting in the right hand side of (19) $a_i = \xi_i' - \eta_i'$ and $b_i = c_i$ $(i = m, \cdots, n)$ and using the above inequality, we conclude the proof of the theorem.

We derive initially a straightforward consequence of Theorem 3.

THEOREM 4. *Let* $\{X_n : n \geqq 1\}$ *be a finite inhomogeneous Markov chain and* $\{n_i\}$ *a sequence of increasing positive integers. If we denote* $\xi_i' = \xi_{n_i}$, $m_i' = m_{n_i}$, $S_j^* = \sum_{i=1}^{j} (\xi_i' - m_i')$ *for* $1 \leqq i \leqq n$ *and* $m \leqq j \leqq n$, $v_{m,n} = 1 - \max_{m \leqq i \leqq n-1} a_{n_i, n_{i+1}}$

$m, n = 1, 2, \cdots$ *and if* $\{c_i\}$ *is a sequence of non-increasing positive numbers such that*

$$\lim_{m,n\to\infty} \frac{D^2(\sum_{i=m}^n c_i \xi_i')}{v_{m,n}} = 0$$

then $c_n S_n^*$ *converges almost surely to* 0.

In particular when the sequence $\{n_i\}$ from the above theorem is the sequence of all natural numbers, we get that

$$\lim_{m,n\to\infty} \frac{D^2(\sum_{i=m}^n c_i \xi_i)}{v_{m,n}} = 0$$

implies $\lim_{n\to\infty} c_n \sum_{i=1}^n (\xi_i - m_i) = 0$ almost surely.

If besides this assumption, we suppose that $\limsup_{m\to\infty} a_{m,m+1} < \delta$ with $\delta < 1$ and if in addition

$$D^2(\sum_{i=m}^n c_i(\xi_i - m_i)) \leqq C \sum_{i=m}^n D^2(c_i \xi_i)$$

C being a positive constant $m, n = 1, 2, \cdots$, conditions satisfied among others by the ergodic chains (Section 1.2 of [6]), then $\sum_{n=1}^\infty c_n^2 D^2(\xi_i) < \infty$ implies $\lim_{n\to\infty} c_n \sum_{i=1}^n (\xi_i - m_i) = 0$ almost surely. This condition is the same as that of Kolmogorov for the strong law of large numbers in the case of independent random variables.

Let us pass now to the case when $\mathscr{T}$ is trivial.

In the following we shall use

LEMMA 6. *Let* $\{X_n : n \geqq 1\}$ *be a finite inhomogeneous Markov chain having a trivial tail* σ*-algebra* $\mathscr{T}$. *Then for any* δ *with* $0 \leqq \delta < 1$ *and any positive integer* k *there exists a positive integer* $g(k)$ *such that* $\alpha_{k,g(k)} > \delta$.

PROOF. According to Theorem 17.1.1 of [5] a necessary and sufficient condition that $\mathscr{T}$ to be trivial is

$$\limsup_{B\in\mathscr{F}_n^\infty; n\to\infty} |P(AB) - P(A)P(B)| = 0$$

for any $A \in \mathscr{F}$. Putting $A \in \mathscr{F}_m^m$ and $B \in \mathscr{F}_n^n$ and taking into account that the chain is finite, we get

$$\lim_{n\to\infty} \max_{i\in\{1,\cdots,s\}} \max_{\Lambda\in\mathscr{S}} |\sum_{j\in\Lambda} (p_{ij}^{(m,n)} - p_j^{(n)})| = 0 .$$

Using Lemma 2, we obtain

$$\lim_{m,n\to\infty} a_{m,n} = 0 .$$

Considering now Lemma 5, we conclude the proof.

Let us define the expression

$$a_n = \max_{(m,k)\in\Gamma_n} (g^{(m)}(k) - g^{(m-1)}(k)) \tag{20}$$

where $\Gamma_n = \{(m, k) \mid g^{(m-1)}(k) \leqq n\}$ and by $g^{(m)}$ we have denoted the mth iterate of the function g defined in the above Lemma and $g^{(0)}(k) = k$.

THEOREM 5. *Let* $\{X_n : n \geqq 1\}$ *be a finite inhomogeneous Markov chain having a trivial tail* σ*-algebra* $\mathscr{T}$ *and* $E(\xi_n) = 0$ *for* $n = 1, 2, \cdots$. *If* g *is a real function defined so that* $\alpha_{k,g(k)} > \delta$ *with* $\delta > 0$, *for all* k; $\alpha > \frac{1}{2}$ *and* $\{a_n\}$ *defined by* (20) *then*

$\sum_{n=1}^{\infty}(a_{2n}/2^{n(2\alpha-1)}) < \infty$ *implies that* S_n/n^α *converges almost surely to* 0.

PROOF. We shall split the initial sequence $\xi_1, \xi_2, \cdots, \xi_n$ into the following subsequences

$$\begin{array}{l} \xi_1, \xi_{1+a_n}, \cdots \\ \xi_2, \xi_{2+a_n}, \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \xi_{a_n}, \xi_{2a_n}, \cdots . \end{array}$$

We set

$$S_n^{k,j} = \sum_{i=1}^{j} \xi_{k+ia_n} \qquad k = 1, 2, \cdots, a_n, j = 1, \cdots, n$$

r being the last integer with $k + ra_n \leqq n$.

According to Theorem 3, we have

$$P\{\max_{1\leqq m\leqq n} n^{-\alpha}|S_m{}^*| > \varepsilon\} \leqq \sum_{k=1}^{a_n} \frac{\frac{16}{\varepsilon^2} D^2(n^{-\alpha} S_{k,r}^n)}{v_{m,n} - \frac{16}{\varepsilon^2} \max_{1\leqq j\leqq r-1} D^2[n^{-\alpha}(S_{k,r}^n - S_{k,j}^n)]} .$$

From Lemma 1 we deduce $v_{m,n} \geqq 1 - \delta$. Taking into account the Dobrushin inequality concerning the dispersion of sums ([6], Theorem 1.2.14) and putting $n = 2^\nu$, $\nu = 1, 2, \cdots$ we get that under the above hypotheses

$$\sum_{\nu=1}^{\infty} P\{\max_{1\leqq i\leqq 2^\nu}(2^\nu)^{-\alpha}|S_i| > \varepsilon\} < \infty .$$

According to the Borel-Cantelli lemma $\max_{1\leqq i\leqq 2^\nu}(2^\nu)^{-\alpha}S_i$ converges almost surely to 0, wherefrom we deduce the convergence to 0 of the initial sequence $\{n^{-\alpha}|S_n| : n \geqq 1\}$. Indeed, if we take an arbitrary n then there exists an integer ν such that $2^{\nu-1} < n \leqq 2^\nu$ and $(\frac{1}{2})n^{-\alpha}|S_n| \leqq (2^\nu)^{-\alpha}|S_n| \leqq \max_{1\leqq i\leqq 2^\nu}(2^\nu)^{-\alpha}|S_i|$ and the proof is completed.

4. Examples. First we shall show that the finite homogeneous Markov chains satisfy most of the conditions considered in this paper. More precisely we have

THEOREM 6. *Let* $\{X_n : n \geqq 1\}$ *be a finite homogeneous Markov chain and* $\{a_{k,n}\}$ *a sequence of numbers defined by* (2). *Then there exists a positive integer r such that*

$$\lim_{k\to\infty} \sup a_{k,k+r} < 1 .$$

PROOF. Let us first suppose that the Markov chain considered is irreducible and aperiodic. In this case it is easy to see that

$$\lim_{n\to\infty} \inf \min_{i\in\{1,\cdots,s\}} p_i^{(n)} > 0 .$$

Therefore, from Lemma 1

$$\lim_{k\to\infty} \sup a_{k,k+1} < 1 .$$

Let us now consider the geneneral case.

We shall split the set of states into the classes $A_1, \cdots, A_l$ where A_p (for

$p = 1, \cdots, l$) comprises the states forming an ergodic class as well as the transient states conducting to them. If the chain is such that there exist states conducting to several ergodic classes, we shall count such states in a class A_i only.

Let $A_p = B_p \cup C_p$ ($p = 1, \cdots, l$), where B_p is an ergodic class and C_p comprises some transient states conducting to B_p. Then

$$B_p = \bigcup_{m=1}^{d_p} B_p^{\,m}$$

where d_p is the period of the class B_p and $B_p^{\,m}$ is defined so that for any $a_i \in B_p^{\,m}$, $p_{ij}^{(d_p)} = 0$ for $j \notin B_p^{\,m}$. We can easily see that $P\{X_n \in B_p^{\,m}\} \geqq \delta_p^{\,m}$ or 0 according as n is equal or different from $pd_p + t$, t being the first integer such that $P\{X_t \in B_p^{\,m}\} > 0$, and $\delta_p^{\,m}$ some positive constants.

On the other hand, for n sufficiently large we have $p_{ij}^{(nd_p)} > 0$ for any $i, j \in B_p^{\,m}$. Therefore we can find a number v such that

$$\max\nolimits'_{i \in B} \max\nolimits_{\Lambda \in \mathscr{P}'} \sum\nolimits_{j \in \Lambda} (p_j^{(n)} - p_{ij}^{(v)}) < 1$$

where $B = \bigcup_{p=1}^{l} B_p$ and $\mathscr{P}'$ denotes the family of all subsets of B.

As to the states belonging to C_p ($p = 1, 2, \cdots, l$), we notice that for any $a_i \in C_p$ we can find an integer u such that $p_{ij}^{(u)} > 0$ for $j \in B_p$.

Now, we are in a position to deduce that we can find a suitable r satisfying the conclusion of the theorem.

REMARK. In particular, when $\{X_n : n \geqq 1\}$ is a stationary finite Markov chain the constant r from above theorem is 1. Indeed we have

$$\limsup_{k\to\infty} a_{k,k+1} \leqq 1 - \min\nolimits_{i \in \{1,\cdots,s\}} p_i^{(1)} < 1 .$$

The following property will show that the Markov chains assuming only two states satisfy the condition $\lim_{k\to\infty} a_{k,k+1} < 1$ if the tail σ-algebra $\mathscr{T}$ is not trivial.

THEOREM 7. *Let $\{X_n : n \geqq 1\}$ be a finite inhomogeneous Markov chain assuming a set of states consisting of two elements $\{a_1, a_2\}$. If*

$$\lim_{k,m\to\infty} \sup a_{k,m} = 1$$

then $\{X_n : n \geqq 1\}$ has a trivial tail σ-algebra $\mathscr{T}$.

PROOF. To prove this theorem we consider the remark given in [2] page 2176, wherefrom we deduce that if $\liminf_{n\to\infty} \min_{i \in \{1,2\}} p_i^{(n)} = 0$ then $\mathscr{T}$ is trivial. Taking into account Lemma 1 we conclude the proof.

We notice easily that the converse part of this theorem does not hold true. Indeed, if the chain considered is ergodic, $\lim\sup a_{k,n} = 0$ whereas the tail σ-algebra $\mathscr{T}$ is trivial.

REMARKS.

A. In the particular case when $\{X_n : n \geqq 1\}$ is a sequence of independent random variables, Theorems 1 and 2 give well-known properties, whereas

Theorem 3 gives a worse bound than the Hájek-Rényi [7] theorem. Nevertheless, the asymptotical consequences to the law of large numbers are the same.

B. In the particular case when $\{X_n : n \geqq 1\}$ is a stationary ergodic and homogeneous Markov chain, $c_n = 1/n$, and $m_n = E\{X_1\}$, $n = 1, 2, \cdots$, Theorem 3 may provide a rate of convergence for Birkhoff ergodic theorem.

C. It would be interesting to know whether some limit properties which assume restrictions on the ergodic coefficients [6] hold true on less restrictive conditions expressed by coefficients $\{a_{k,n}\}$, eventually coupled with conditions imposed on the dispersions of sums of random variables.

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