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WEAK LAWS OF LARGE NUMBERS IN NORMED LINEAR SPACES[1]

BY ROBERT LEE TAYLOR[2]

Florida State University

In this paper weak laws of large numbers are proved for random elements (function-valued random variables) in separable normed linear spaces. One result states that for identically distributed random elements $\{V_n\}$ such that the Pettis integral EV_1 exists and $E\|V_1\| < \infty$

$$\|n^{-1}\sum_{k=1}^{n} V_k - EV_1\| \to 0 \quad \text{in probability}$$

if and only if

$$|n^{-1}\sum_{k=1}^{n} f(V_k) - Ef(V_1)| \to 0 \quad \text{in probability}$$

for each continuous linear functional f. The condition of identically distributed random elements $\{V_n\}$ can not be relaxed by just assuming a bound on the moments of $\{\|V_n\|\}$, but a weak law of large numbers is obtained for random elements which need not be identically distributed. Both of these weak laws can also be obtained by assuming only that the space has a Schauder basis such that the weak law of large numbers holds in each coordinate. An application of these results yields a uniform weak law of large numbers for separable Wiener processes on [0, 1].

1. Summary. In this paper weak laws of large numbers for random variables are extended to random elements in normed linear spaces (random variables taking values in normed linear spaces). Theorem 1 states that a sequence of identically distributed random elements in a separable normed linear space satisfying the weak law of large numbers in the weak linear topology also satisfies the weak law of large numbers in the norm topology. Theorem 2 shows that the result also holds if there exists a Schauder basis such that the weak law of large numbers holds in each coordinate.

An example is given which shows that the condition of identically distributed random elements $\{V_n\}$ can not be relaxed in either Theorem 1 or Theorem 2 by just assuming bounds on the moments of $\{\|V_n\|\}$. However, Theorem 3 and Theorem 4 give weak laws of large numbers for random elements which need not be identically distributed by imposing other structure conditions. Using Theorem 3 a uniform weak law of large numbers is obtained for separable Wiener processes on [0, 1] whose parameters are Cesaro bounded and whose increments are uncorrelated.

2. Preliminaries. Unless otherwise specified X will be a real normed linear

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[2] Now at the University of South Carolina.

space with elements, x's, and norm $\|\ \|$. The topological dual of X will be denoted by X^*, and R will denote the real line. The σ-field generated by the open subsets of X will be denoted by B. Let (W, F, P) be a probability space and let V be a function from W into X. If $V^{-1}(B) \subset F$, then V is called a random element in X [with respect to (W, F, P)].

An expected value for a random element is defined by a Pettis integral (Pettis (1938)). A random element V in X is said to have expected value EV if there exists an element $EV \in X$ such that $E[f(V)] = f(EV)$ for each $f \in X^*$ where $E[f(V)]$ denotes the usual expected value of the random variable $f(V)$. A sequence of random elements $\{V_n\}$ is said to converge to the random element V

(a) in probability in the norm topology if for each $e > 0$

$$\lim_n P[\|V_n - V\| > e] = 0 \qquad \text{or}$$

(b) with probability 1 in the norm topology if

$$P[\lim_n \|V_n - V\| = 0] = 1 .$$

Finally, $\{V_n\}$ is said to converge to V in probability in the weak linear topology of X if for each $e > 0$

$$\lim_n P[|f(V_n) - f(V)| > e] = 0$$

for each $f \in X^*$.

A sequence $\{b_n\} \subset X$ is a Schauder basis (Wilansky (1964) page 86) for X if for each $x \in X$ there exists a unique sequence of scalars $\{t_n\}$ such that

$$x = \lim_n \textstyle\sum_{k=1}^n t_k b_k .$$

A Schauder basis $\{b_n\}$ is a monotone basis if

$$\{\|\textstyle\sum_{k=1}^n t_k b_k\| : n \geqq 1\}$$

is a monotone increasing sequence for each sequence of scalars $\{t_n\}$.

When X has a Schauder basis $\{b_n\}$, a sequence of linear functionals $\{f_n\}$ can be defined by letting $f_k(x) = t_k$ where $x \in X$ and

$$x = \lim_n \textstyle\sum_{k=1}^n t_k b_k .$$

Also, a sequence of linear operators $\{U_n\}$ on X can be defined by letting

$$U_n(x) = \textstyle\sum_{k=1}^n f_k(x) b_k \qquad \text{for each } x \in X .$$

The linear functionals $\{f_k\}$ are called the coordinate functionals (for the basis $\{b_n\}$), and the linear operators $\{U_n\}$ are called the partial sum operators.

LEMMA (Wilansky (1964) and Marti (1969)): (a) *If* $\{b_n\}$ *is a monotone basis for the normed linear space* X, *then* $\|U_n\| \leqq 1$ *for each* n. (b) *If* $\{b_n\}$ *is a Schauder basis for the Banach space* X, *then there exists an* $m > 0$ *such that* $\|U_n\| \leqq m$ *for each* n.

The preceding lemma also provides for the continuity of each coordinate functional f_n since $\|U_n\| \leqq m$ for each n implies $\|f_n\| \leqq 2m/\|b_n\|$ for each n.

3. Weak laws of large numbers for identically distributed random elements.

THEOREM 1. *Let X be a separable normed linear space and let $\{V_n\}$ be a sequence of identically distributed random elements in X such that $E||V_1|| < \infty$ and EV_1 exists. For each $f \in X^*$ the weak law of large numbers holds for the sequence $\{f(V_n)\}$ if and only if*

$$n^{-1} \sum_{k=1}^{n} V_k \rightarrow EV_1 \quad \textit{in probability.}$$

The proof of Theorem 1 will follow the proof of Theorem 2.

THEOREM 2. *Let X be a Banach space which has a Schauder basis $\{b_n\}$ and let $\{V_n\}$ be a sequence of identically distributed random elements in X such that $E||V_1|| < \infty$. For each coordinate functional f_k the weak law of large numbers holds for the random variables $\{f_k(V_n) : n \geqq 1\}$ if and only if*

$$n^{-1} \sum_{k=1}^{n} V_k \rightarrow EV_1 \quad \textit{in probability.}$$

PROOF OF THEOREM 2. The "if" part is obvious since convergence in the norm topology implies convergence in the weak linear topology of X. The "only if" part is proved below.

The Pettis integral EV_1 exists since $E||V_1|| < \infty$ and X is complete and separable. Moreover, EV_1 can be assumed to be 0 (otherwise, consider $Z_n = V_n - EV_1$). Let $e > 0$ and $d > 0$ be given. In order that

$$n^{-1} \sum_{k=1}^{n} V_k \rightarrow 0 \quad \text{in probability}$$

there must exist a positive integer $N(e, d)$ such that

$$P[||n^{-1} \sum_{k=1}^{n} V_k|| > e] < d \qquad \text{for each } n \geqq N(e, d)\,. \tag{3.1}$$

Let $m > 0$ be the basis constant such that $||U_t|| \leqq m$ for all t. Hence, $||Q_t|| \leqq m + 1$ for each t where Q_t is the linear operator on X defined by $Q_t(x) = x - U_t(x)$.

For each n and each t

$$n^{-1} \sum_{k=1}^{n} V_k = n^{-1} \sum_{k=1}^{n} U_t(V_k) + n^{-1} \sum_{k=1}^{n} Q_t(V_k)\,. \tag{3.2}$$

For each fixed t

$$\begin{aligned} P[||n^{-1} \sum_{k=1}^{n} Q_t(V_k)|| > e/2] &\leqq P[n^{-1} \sum_{k=1}^{n} ||Q_t(V_k)|| > e/2] \\ &\leqq \frac{2}{en} \sum_{k=1}^{n} E||Q_t(V_k)|| \\ &= \frac{2}{e} E||Q_t(V_1)||\,. \end{aligned} \tag{3.3}$$

But $E||Q_t(V_1)|| \rightarrow 0$ as $t \rightarrow \infty$ since $||Q_t(V_1)|| \rightarrow 0$ pointwise and $||Q_t(V_1)|| \leqq (m + 1)\,||V_1||$. Thus, choose t so that

$$P[||n^{-1} \sum_{k=1}^{n} Q_t(V_k)|| > e/2] < d/2\,. \tag{3.4}$$

By construction

$$U_t(x) = \sum_{i=1}^{t} f_i(x) b_i \qquad \text{for each } x \in X$$

where $\{f_1, \cdots, f_t\}$ are the coordinate functionals for the basis elements $\{b_1, \cdots, b_t\}$. Thus,

$$\begin{aligned} P[\|n^{-1} \sum_{k=1}^n U_t(V_k)\| > e/2] &= P[\|\sum_{i=1}^t f_i(n^{-1} \sum_{k=1}^n V_k) b_i\| > e/2] \\ &\leqq P[\sum_{i=1}^t |f_i(n^{-1} \sum_{k=1}^n V_k)| \, \|b_i\| > e/2] \\ &\leqq \sum_{i=1}^t P[|n^{-1} \sum_{k=1}^n f_i(V_k)| > e/2t\|b_i\|] . \end{aligned}$$

But,

$$(3.5) \qquad P[|n^{-1} \sum_{k=1}^n f_i(V_k)| > e/2t\|b_i\|] \to 0 \qquad \text{as } n \to \infty$$

for each i since the weak law of large numbers holds for each sequence $\{f_i(V_n) : n \geqq 1\}$ and $E[f_i(V_1)] = 0$. Hence, there exists a positive integer $N(e, d)$ such that

$$(3.6) \qquad \sum_{i=1}^t P[|n^{-1} \sum_{k=1}^n f_i(V_k)| > e/2t\|b_i\|] < d/2 \qquad \text{for each } n \geqq N(e, d) .$$

Using (3.4) and (3.6)

$$\begin{aligned} P[\|n^{-1} \sum_{k=1}^n V_k\| > e] &\leqq P[\|n^{-1} \sum_{k=1}^n U_t(V_k)\| > e/2] \\ &+ P[\|n^{-1} \sum_{k=1}^n Q_t(V_k)\| > e/2] \leqq \sum_{i=1}^t P[|n^{-1} \sum_{k=1}^n f_i(V_k)| \\ &> e/2t\|b_i\|] + d/2 < d \end{aligned}$$

for each $n \geqq N(e, d)$. This verifies (3.1), and hence

$$n^{-1} \sum_{k=1}^n V_k \to 0 \quad \text{in probability.} \qquad \square$$

REMARK. The same proof also proves the result for any separable normed linear space which has a Schauder basis such that $\{\|U_n\|\}$ is a bounded sequence, but the existence of EV_1 must be assumed for an incomplete space.

Theorem 1 is now proved for all separable normed linear spaces by embedding each space isomorphically in the Banach space $C[0, 1]$. Theorem 2 can then be applied since $C[0, 1]$ is a Banach space which has a Schauder basis.

PROOF OF THEOREM 1. Let $\hat{X}$ be the completion of X. Since $\hat{X}$ is isometric to a subspace of $C[0, 1]$ (Marti (1969) page 67), there exists a 1-1, bicontinuous, linear function h from X into $C[0, 1]$.

It is not hard to verify that $\{h(V_n)\}$ is a sequence of identically distributed random elements in $C[0, 1]$ with $E\|hV_i\| < \infty$. Let $g \in C[0, 1]^*$, then

$$n^{-1} \sum_{k=1}^n g(hV_k) = n^{-1} \sum_{k=1}^n h^* g(V_k) \to E[h^* g(V_1)] = E[g(hV_1)] \quad \text{in probability}$$

where h^* is the adjoint function of $C[0, 1]^*$ into X^*. Thus, for each $g \in C[0, 1]^*$ the weak law of large numbers holds for the sequence $\{g(hV_n)\}$. By Theorem 1

$$h(n^{-1} \sum_{k=1}^n V_k) = n^{-1} \sum_{k=1}^n hV_k \to EhV_1 \quad \text{in probability,}$$

but $EhV_1 = hEV_1$ (Pettis (1938)). Since h is 1-1, bicontinuous and linear,

$$n^{-1} \sum_{k=1}^n V_k \to EV_1 \quad \text{in probability.} \qquad \square$$

The condition of identically distributed random elements $\{V_n\}$ is used only in obtaining inequality (3.4). Inequality (3.4) shows that the remainders $\{QV_n\}$ of the truncated random elements are uniformly small in probability. When other conditions imply inequality (3.4), the random elements need not be identically distributed. One such extension is Theorem 3. However, the assumption of identically distributed random elements $\{V_n\}$ can not be relaxed by just imposing bounds on the moments of $\{||V_n||\}$. The following example also shows that Theorems 3 and 4 (see Section 4) and Mourier's (1953) strong law of large numbers for identically distributed, independent random elements can not be similarly weakened.

EXAMPLE (Beck (1963), page 32). Let $X = l_1 = \{x \in R^\infty : ||x|| = \sum |x_n| < \infty\}$ and let $u^{(n)}$ denote the element having 1 for its nth term and 0 elsewhere. Let $\{A_n\}$ be an independent sequence of random variables defined by $A_n = \pm 1$ each with probability $\frac{1}{2}$, and define $V_n = A_n u^{(n)}$. Clearly, $\{V_n\}$ is an independent sequence of random elements with $||V_n|| = 1$ for each n. Hence, the strong law of large numbers holds for each sequence $\{f(V_n)\}$ where $f \in X^*$. But,

$$||n^{-1} \textstyle\sum_{k=1}^n V_k|| = ||n^{-1}(\pm 1, \pm 1, \cdots, \pm 1, 0, \cdots)|| = 1$$

for each n. Hence,

$$n^{-1} \textstyle\sum_{k=1}^n V_k \nrightarrow 0 = E(V_n) .$$

It is interesting to note that while the weak law of large numbers in the weak topology is sufficient to give the weak law of large numbers in the strong topology, it will not give the strong law of large numbers in the strong topology. Beck and Warren (1968) constructed random elements $\{V_n\}$ in the separable Banach space c_0, the space of null convergent sequences, which are

(a) identically distributed,
(b) uniformly bounded, that is, $||V_n|| \leqq 1$ for each n, and such that
(c) $\{f(V_n)\}$ satisfy the weak law of large numbers for each $f \in X^*$ but which do not satisfy the strong law of large numbers in the norm topology. Hence, convergence with probability 1 is not always possible in either Theorem 1 or Theorem 2.

4. Extensions of Theorem 1 and Theorem 2. In this section weak laws of large numbers (Theorems 3 and 4) are proved for a class of random elements which need not be identically distributed. Only the proof of Theorem 4 is given since the extension to all separable normed linear spaces follows from the proof of Theorem 1.

THEOREM 3. *Let X be a separable normed linear space and let $\{V_n\}$ be a sequence of identically distributed random elements in X such that $E||V_1||^2 < \infty$. Also let $\{A_n\}$ be a sequence of random variables such that*

$$n^{-1} \textstyle\sum_{k=1}^n E(A_k^2) \leqq \Gamma \qquad \textit{for each } n$$

where Γ is a positive constant and let $E(A_n V_n) = E(A_1 V_1)$ for each n. For each

$f \in X^*$ the weak law of large numbers holds for the sequence $\{f(A_n V_n)\}$ if and only if

$$n^{-1} \sum_{k=1}^{n} A_k V_k \to E(A_1 V_1) \quad \textit{in probability.}$$

THEOREM 4. *Let X be a normed linear space which has a Schauder basis $\{b_n\}$ such that $||U_n|| \leqq m$ for each n where m is a positive constant. Let $\{V_n\}$ be a sequence of identically distributed random elements in X such that $E||V_1||^2 < \infty$. Let $\{A_n\}$ be a sequence of random variables such that*

$$n^{-1} \sum_{k=1}^{n} E(A_k^2) \leqq \Gamma \qquad \textit{for each } n \tag{4.1}$$

where Γ is a positive constant and let $E(A_n V_n) = E(A_1 V_1)$ for each n. For each coordinate functional f_k the weak law of large numbers holds for the sequence $\{f_k(A_n V_n): n \geqq 1\}$ if and only if

$$n^{-1} \sum_{k=1}^{n} A_k V_k \to E(A_1 V_1) \quad \textit{in probability.}$$

PROOF. Again, it is sufficient to prove the "only if" part.

First, $||Q_n|| \leqq m + 1$ for each n where Q_n is the linear operator on X defined by $Q_n(x) = x - U_n(x)$. From (4.1) it follows that

$$n^{-1} \sum_{k=1}^{n} [E(A_k^2)]^{\frac{1}{2}} < \Gamma + 1 \qquad \text{for each } n. \tag{4.2}$$

Finally, let $e > 0$ and $d > 0$ be given.

For each n and each t

$$n^{-1} \sum_{k=1}^{n} A_k V_k = n^{-1} \sum_{k=1}^{n} A_k U_t(V_k) + n^{-1} \sum_{k=1}^{n} A_k Q_t(V_k). \tag{4.3}$$

For each fixed t

$$\begin{aligned} P[||n^{-1} \sum_{k=1}^{n} A_k Q_t(V_k)|| > e/4] &\leqq \frac{4}{en} \sum_{k=1}^{n} E||A_k Q_t(V_k)|| \\ &\leqq \frac{4}{en} \sum_{k=1}^{n} [E(A_k^2)]^{\frac{1}{2}} [E||Q_t(V_k)||^2]^{\frac{1}{2}} \\ &\leqq \frac{4(\Gamma + 1)}{e} [E||Q_t(V_1)||^2]^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

Again, t can be chosen so that for all n

$$P[||n^{-1} \sum_{k=1}^{n} A_k Q_t(V_k)|| > e/4] < d/2 \qquad \text{and} \tag{4.5}$$

$$||Q_t(E(A_1 V_1))|| < e/4 \tag{4.6}$$

since both $E||Q_t(V_1)||^2 \to 0$ and $||Q_t(E(A_1 V_1))|| \to 0$ as $t \to \infty$.

For the t chosen in (4.5) and (4.6) and for all n

$$\begin{aligned} &P[||n^{-1} \sum_{k=1}^{n} A_k V_k - E(A_1 V_1)|| > e] \\ &\quad \leqq P[||n^{-1} \sum_{k=1}^{n} U_t(A_k V_k) - U_t(E(A_1 V_1))|| > e/2] \\ &\qquad + P[||n^{-1} \sum_{k=1}^{n} Q_t(A_k V_k) - Q(E(A_1 V_1))|| > e/2] \\ &\quad \leqq P[||n^{-1} \sum_{k=1}^{n} U_t(A_k V_k - E(A_1 V_1))|| > e/2] + d/2. \end{aligned} \tag{4.7}$$

Thus, truncation to a finite-dimensional subspace is accomplished, and the remainder of the proof follows from the proof of Theorem 2 since the weak law of large numbers holds for each sequence $\{f_k(A_n V_n): n \geqq 1\}$ where f_k is a coordinate functional. □

Theorems 3 and 4 generalize Theorems 1 and 2 by assuming the stronger condition $E\|V_1\|^2 < \infty$. The following example for Wiener processes illustrates these results.

Let $\{Z_n\}$ be a sequence of separable Wiener processes on [0, 1] which satisfies the condition

$$\text{Covariance } [Z_n(t) - Z_n(s), Z_m(t) - Z_m(s)] = 0 \tag{4.8}$$

for all $s, t \in [0, 1]$ and $m \neq n$. Also let the parameters $\{\sigma_n^2 = E[Z_n(1)^2]\}$ satisfy the inequality

$$n^{-1} \sum_{k=1}^{n} \sigma_k^2 \leqq \Gamma \qquad \text{for each } n \tag{4.9}$$

where Γ is a positive constant. With probability 1 $\{Z_n\}$ can be regarded as a sequence of random elements in $C[0, 1]$ with $E\|Z_n\|^2 < \infty$ and $EZ_n = 0$ for each n. Moreover, each random element Z_n can be expressed as $Z_n = A_n V_n$ where A_n is the constant random variable σ_n and V_n is a random element in $C[0, 1]$. The random elements $\{V_n\}$ which are Wiener processes are identically distributed since $E[V_n(1)^2] = 1$ for each n. Condition (4.8) implies that the weak law of large numbers holds for each sequence $\{f(Z_n)\}$ where $f \in C[0, 1]^*$. Thus, by Theorem 3

$$P[\|n^{-1} \sum_{k=1}^{n} Z_k\| > e] = P[\sup_t |n^{-1} \sum_{k=1}^{n} Z_k(t)| > e] \to 0$$

for each $e > 0$, or the Wiener processes $\{Z_n\}$ satisfy a weak law of large numbers which is uniform for $t \in [0, 1]$.

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