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# ON THE $\gamma$-VARIATION OF PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

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Let $\{X_t; t \geqq 0\}$ be a stochastic process in $R^N$ defined on the probability space $(\Omega, \mathscr{F}, P)$ which has stationary independent increments. Let $\nu$ be the Lévy measure for $X_t$ and let $\beta = \inf\{\alpha > 0: \int_{|x|<1} |x|^\alpha \nu(dx) < \infty\}$. For each $\omega \in \Omega$, let $V_\gamma(X(\cdot, \omega); a, b) = \sup \sum_{j=1}^m |X(t_j, \omega) - X(t_{j-1}, \omega)|^\gamma$ where the supremum is over all finite subdivisions $a = t_0 < t_1 < \cdots < t_m = b$. Then if $\gamma > \beta$, $P\{V_\gamma(X(\cdot, \omega); a, b) < \infty\} = 1$.

**1. Introduction.** Let $\{X_t; t \geqq 0\}$ be a stochastic process in $R^N$ defined on the probability space $(\Omega, \mathscr{F}, P)$ which has stationary independent increments. It will be assumed that $X_t$ has no normal component. Thus, if $\phi_t$ is the characteristic function of $X_t$, then $\phi_t(y) = \exp[-t\psi(y)]$, where

$$\psi(y) = i(a, y) + \int \left[1 - e^{i(x,y)} + \frac{i(x, y)}{1 + |x|^2}\right] \nu(dx) \tag{1.1}$$

with $a \in R^N$ and $\nu$ a Borel measure defined on $R^N$ with the property that

$$\int \frac{|x|^2}{1 + |x|^2} \nu(dx) < \infty .$$

(Here $(x, y)$ is the usual inner product and $|x| = (x, x)^{\frac{1}{2}}$.) The measure $\nu$ is called the Lévy measure of $X_t$.

We can assume that the sample functions of $X_t$ are right continuous and have left hand limits. For a more complete discussion, see [3].

In [2] Blumenthal and Getoor defined the index

$$\beta = \inf\{\alpha > 0: \int_{|x|<1} |x|^\alpha \nu(dx) < \infty\} . \tag{1.2}$$

They showed that if $\alpha > \beta$, then

$$P\{t^{-1/\alpha} X(t) \to 0 \quad \text{as} \quad t \to 0\} = 1 \tag{1.3}$$

(Theorem 3.1), and if $\alpha < \beta$, then

$$P\{\limsup_{t\to 0} t^{-1/\alpha} |X(t)| = \infty\} = 1 \tag{1.4}$$

(Theorem 3.3).

Actually their proof shows that if $\alpha < \beta$, then there is a sequence $\{s_n\}$ such that $s_n \to 0$ and if $\{t_n\}$ is any subsequence of $\{s_n\}$ then

$$P\{\limsup_{n\to\infty} t_n^{-1/\alpha} |X(t_n)| = \infty\} = 1 . \tag{1.5}$$

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The subject of interest here however is the $\gamma$-variation of $X_t$. For a function $f\colon [a, b] \to R$, the $\gamma$-variation of $f$ over $[a, b]$ is defined to be

$$V_\gamma(f; a, b) = \sup \textstyle\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^\gamma \,, \tag{1.6}$$

where the supremum is taken over all finite subdivisions $a = t_0 < t_1 < \cdots < t_m = b$.

In [2] Blumenthal and Getoor showed that if $\gamma < \beta$, then

$$P\{V_\gamma(X(\cdot, \omega); a, b) = \infty\} = 1 \tag{1.7}$$

(Theorem 4.1), while if $\beta < \gamma \leqq 1$, then

$$P\{V_\gamma(X(\cdot, \omega); a, b) < \infty\} = 1 \tag{1.8}$$

(Theorem 4.2), and they conjecture that there is no need to assume that $\gamma \leqq 1$ in this last statement.

In this paper this conjecture will be verified. Note however that if $\beta < 1$, then it is not necessary to assume that $\gamma \leqq 1$. It follows easily from the fact that the sample paths of the process are right continuous and have left hand limits, that, there is an upper bound to the number of terms $|X(t_j) - X(t_{j-1})|$ which are greater than one. Since $\gamma > 1$, $|X(t_j) - X(t_{j-1})|^\gamma \leqq |X(t_j) - X(t_{j-1})|$ if $|X(t_j) - X(t_{j-1})| \leqq 1$, it follows that if $V_1(X; a, b) < \infty$, then $V_\gamma(X(\cdot, \omega); a, b) < \infty$ for $\gamma > 1$.

The method used to treat the case $\beta \geqq 1$ is the one used by Blumenthal and Getoor in [1] Theorem 4.1. There it was shown that if $X(t)$ is a symmetric stable process of index $\alpha$ and $\gamma > \alpha$, then $V_\gamma(X) < \infty$ with probability one. That proof relied on the fact that the symmetric stable processes are "subordinated" to Brownian motion. That is if $W(t)$ is a Wiener process, and $T(s)$ is a one sided stable process of index $\alpha/2$ independent of $W(t)$, then $W(T(s))$ has the same distribution as $X(s)$, the symmetric stable process of index $\alpha$. This proof can now be extended to more general processes.

In [6] the following was shown. Let $X(s)$ be a process with stationary independent increments such that $E\{X(s)\} = 0$ for all $s$, and let $(W_t, \mathscr{F}_t)$ be a Wiener process. Then the $\sigma$-fields $\mathscr{F}_t$ can be extended (if necessary) to $\sigma$-fields $\mathscr{G}_t$ such that $(W_t, \mathscr{G}_t)$ is still a Wiener process, and there are $\mathscr{G}_t$ stopping time, $T(s)$, such that $W(T(s))$ and $X(s)$ are equivalent processes. This family of random variables $T(s)$ constitutes a right continuous stochastic process with stationary independent increments. Moreover, as a stopping time, $T(s)$ is minimal for each $s$, that is, the process $W(t \wedge T(s))$ is uniformly integrable. It follows (see the proof of Proposition 7 in [6]) that if $E\{|X(s)|\} = E\{|W(T(s))|\} = M$, then for any constant $K > 0$

$$P\{T(s) > \lambda\} \leqq \frac{1}{k} + \frac{k^2 m^2}{\lambda} \,. \tag{1.9}$$

These results will be used to extend the proof in [1].

For more recent papers on the variation of processes with stationary independent increments, see [4] and [5].

**2. The index of $T_t$.** In this section, it will be shown that if $X_t$ has index $\beta$ and $X_t = W(T_t)$ where the $T_t$ are minimal and have stationary independent increments, then $T_t$ has index $\beta/2$.

LEMMA 1. *Let $X_t$ be a real valued process with stationary independent increments whose characteristic function is* $\exp[-t\psi(y)]$ *where $\psi(y)$ is given by* (1.1). *Suppose moreover that $E\{|X_t|\} < \infty$. If $X_t$ has index $1 \leqq \beta \leqq 2$ (as defined in* (1.2)), *then*

(a) *if $\alpha > \beta$,* $\lim_{t\to 0} E\{t^{-1/\alpha}|X_t|\} = 0$

(b) *if $\alpha < \beta$, there is a sequence $t_n \to 0$ such that* $\lim_{n\to\infty} E\{t_n^{-1/\alpha}|X_{t_n}|\} = \infty$.

PROOF. Suppose first that $\alpha > \beta$. Let us write $X_t = X_t' + X_t''$ where the characteristic function of $X_t'$ is $\exp[-t\psi'(y)]$,

$$\psi'(y) = \int_{|x|<1} 1 - e^{ixy} + \frac{ixy}{1+|x|^2}\,\nu(dx)\,, \tag{2.1}$$

and the characteristic function of $X_t''$ is $\exp[-t\psi''(y)]$,

$$\psi''(y) = i(a, y) + \int_{|x|\geqq 1} 1 - e^{ixy} + \frac{ixy}{1+|x|^2}\,\nu(dx)\,.$$

Then it is well known that $E\{|X_t''|\} \leqq kt$ for some constant $k$ so $\lim_{t\to 0} E\{t^{-1/\alpha}|X_t''|\} = \lim_{t\to 0} t^{1-1/\alpha} = 0$. Since

$$E\{t^{-1/\alpha}|X_t|\} \leqq E\{t^{-1/\alpha}|X_t'|\} + E\{t^{-1/\alpha}|X_t''|\}\,,$$

and $X_t'$ clearly has index $\beta$ also, it can be assumed that $X_t = X_t'$. That is, it can be assumed that $X_t$ has characteristic function $\exp[-t\psi(y)]$, where

$$\psi(y) = \int_{|x|<1} 1 - e^{ixy} + \frac{ixy}{1+|x|^2}\,\nu(dx)\,.$$

The characteristic function of $t^{-1/\alpha}X_t$ is $\exp[(-\xi(y)]$ where

$$\xi(y) = \int_{|x|\leqq t^{-1/\alpha}} \left[1 - e^{ixy} + \frac{ixy}{1+x^2}\right]\mu(dx)\,,$$

and $\mu\{x\colon x \in A\} = t\nu\{x\colon t^{-1/\alpha}x \in A\}$. Let us write for a fixed $t$

$$t^{-1/\alpha}X_t = Y_1 + Y_2\,,$$

where both $Y_1$ and $Y_2$ are infinitely divisible with Lévy measures $\mu_1$ and $\mu_2$ where

$$\mu_1(A) = \mu(A \cap \{x\colon |x| \leqq 1\})\,,$$
$$\mu_2(A) = \mu(A \cap \{x\colon |x| > 1\})\,.$$

Of course

$$E\{t^{-1/\alpha}|X_t|\} \leqq E\{|Y_1|\} + E\{|Y_2|\}\,.$$

We will treat these separately but in both parts we will need the following. If $G(r) = \nu\{x\colon |x| > r\}$, then (Theorem 2.1 of [2]) for $\gamma > \beta$, $r^\gamma G(r) \to 0$ as $r \to 0$. In particular $G(r) < kr^{-\gamma}$ for $k$ sufficiently large.

Consider $E\{|Y_2|\}$. It is well known that

$$\begin{aligned} E\{|Y_2|\} &\leqq \int_{|x|\geqq 1} |x|\mu_2(dx) \\ &= t \int_{|x|>t^{1/\alpha}} |t^{-1/\alpha}x|\nu(dx) \\ &= -t^{1-1/\alpha} \int_{t^{1/\alpha}}^{1} rG(dr) \\ &= t^{1-1/\alpha} \int_{t^{1/\alpha}}^{1} G(r)\,dr + tG(t^{1/\alpha})\,. \end{aligned}$$

For $\gamma > \beta$ but $\gamma < \alpha$, we have for $k$ sufficiently large

$$\begin{aligned} E\{|Y_2|\} &\leqq kt^{1-1/\alpha} \int_{t^{1/\alpha}}^{1} r^{-\gamma}\,dr + tG(t^{1/\alpha}) \\ &\leqq k(1-\gamma)^{-1}t^{1-1/\alpha}\{1 - t^{(1-\gamma)/\alpha}\} + tG(t^{1/\alpha}) \\ &= k(1-\gamma)^{-1}\{t^{1-1/\alpha} - t^{1-\gamma/\alpha}\} + tG(t^{1/\alpha})\,. \end{aligned}$$

Now $\gamma < \alpha$ so $t^{1-\gamma/\alpha} \to 0$ as $t \to 0$. Since $\alpha > 1$, $t^{1-1/\alpha} \to 0$ as $t \to 0$. Finally since $s^\alpha G(s) \to 0$ as $s \to 0$, $tG(t^{1/\alpha}) \to 0$ as $t \to 0$. Thus $E\{|Y_2|\} \to 0$ as $t \to 0$.

Now consider $E\{|Y_1|\}$. We show first that the variance of $Y_1$ converges to zero as $t \to 0$. Again, it is well known that the variance of $Y_1$ is

$$\begin{aligned} \int_{-1}^{1} x^2\mu_1(dx) &= t \int_{-t^{1/\alpha}}^{t^{1/\alpha}} x^2 t^{-2/\alpha}\nu(dx) \\ &= -t^{1-2/\alpha} \int_0^{t^{1-\alpha}} r^2 G(dr)\,. \end{aligned}$$

If $\alpha \geqq 2$, then $t^{1-2/\alpha}$ remains bounded so the variance of $Y_1$ goes to zero as $t \to 0$. If $\alpha < 2$

$$\begin{aligned} \int_1^1 x^2\mu_1(dx) &= 2t^{1-2/\alpha} \int_0^{t^{1/\alpha}} G(r)r\,dr + t^{1-2/\alpha}t^{2/\alpha}G(t^{1/\alpha}) \\ &\leqq 2kt^{1-2/\alpha} \int_0^{t^{1/\alpha}} r^{-\gamma+1}\,dr + tG(t^{1/\alpha}) \\ &= 2k(2-\gamma)^{-1}t^{1-2/\alpha}t^{(2-\gamma)/\alpha} + tG(t^{1/\alpha}) \\ &= 2k(2-\gamma)^{-1}t^{1-\gamma/\alpha} + tG(t^{1/\alpha}) \end{aligned}$$

where $k$ is sufficiently large and $\beta < \gamma < \alpha < 2$. These last terms go to zero as $t \to 0$.

Now by (1.3), $t^{-1/\alpha}X_t \to 0$ as $t \to 0$ so $Y_1 \to 0$ as $t \to 0$. But as the variance of $Y_1$ goes to zero as $t \to 0$, we must have $E\{Y_1\} \to 0$ as $t \to 0$. This completes the proof of the case $\alpha > \beta$.

Suppose that $\alpha < \beta$. Choose $\alpha < \gamma < \beta$ and a sequence $s_n$ such that $s_n^{-1/\gamma}X_{s_n}$ has no subsequence which converges to zero (see (1.5)). If

$$\liminf_{n\to\infty} E\{s_n^{-1/\alpha}|X_{s_n}|\} < \infty\,,$$

then a subsequence $\{t_n\}$ can be selected such that

$$E\{t_n^{-1/\alpha}|X_{t_n}|\} < M$$

for some $M$ and all $n$. But then

$$E\{t_n^{-1/\gamma}|X_{t_n}|\} \to 0$$

since $\gamma > \alpha$, and this means that $\{t_n\}$ has a subsequence $u_n$ such that $u_n^{-1/\gamma}|X_{u_n}| \to 0$ almost surely which cannot happen. Thus

$$\liminf E\{s_n^{-1/\alpha}|X_{s_n}|\} = \infty .$$

This completes the proof of the lemma.

THEOREM 1. *Let $(W_t, \mathscr{G}_t)$ be a Wiener process and $T(s)$ be a family of right continuous minimal $\mathscr{G}_t$ stopping time such that $W(T(s))$ and $T(s)$ are both processes with stationary independent increments and such that $W(T(s))$ is a martingale. If the index* (1.2) *of $W(T(s))$ is $\beta \geqq 1$, then the index of $T(s)$ is $\beta/2$.*

PROOF. First it will be shown that the index of $T(s)$ is less than or equal to $\beta/2$. According to (1.5), it is enough to show that if $\alpha > \beta$, then $s_n^{-2/\alpha}T(s_n) \to 0$ in probability for every sequence $s_n \to 0$. Indeed if $s_n^{-2/\alpha}T(s_n) \to 0$ in probability, then there is a subsequence $\{t_n\}$ of $\{s_n\}$ such that $t_n^{-2/\alpha}T(t_n) \to 0$ almost surely. This clearly contradicts (1.5).

To show that $s^{-2/\alpha}T(s) \to 0$ in probability choose $\lambda > 0$ and recally that by (1.9)

$$P\{s^{-2/\alpha}T(s) \geqq \lambda\} = P\{T(s) \geqq s^{2/\alpha}\lambda\}$$
$$\leqq 1/k + k^2 s^{-2/\alpha}\lambda^{-1}E\{|W(T(s))|\}^2 ,$$

where $k$ is any positive number. Let $k = 2\lambda^{-1}$. Since $\alpha > \beta$, by Lemma 1, if $s$ is small enough then

$$E\{|W(T(s))|\} < 2^{-\frac{1}{2}}k^{-1}\lambda s^{1/\alpha} .$$

Thus

$$P\{s^{-2/\alpha}T(s) \geqq \lambda\} \leqq \lambda/2 + k^2 s^{-2/\alpha}\lambda^{-1}(\lambda^2 s^{2/\alpha}2^{-1}k^{-2}) = \lambda .$$

As $\lambda > 0$ was arbitrary, the proof that $s^{-2/\alpha}T(s) \to 0$ in measure if $\alpha > \beta$ is complete.

To show that the index of $T(s)$ is not smaller than $\beta/2$, it will be shown that if $s_n^{-2/\alpha}T(s_n) \to 0$ in probability, then for any $\gamma > \alpha$, $s_n^{-1/\gamma}W(T(s_n)) \to 0$ in probability. If $\alpha/2$ is larger than the index of $T(s)$, then by (1.3) $s_n^{-2/\alpha}T(s_n) \to 0$ for all sequences $s_n \to 0$. But if $\alpha < \gamma < \beta$, then by (1.5) there is a sequence $s_n \to 0$ such that $s_n^{-1/\gamma}W(T(s_n))$ does not converge to zero even in measure. Thus if $\alpha/2$ is larger than the index of $T(s)$, then $\alpha$ cannot be smaller than the index of $\beta$.

Suppose that $s_n \to 0$ and $s_n^{-2/\alpha}T(s_n) \to 0$ in probability. Choose $\varepsilon > 0$. Since the index of $W(t)$ is 2, for any $\gamma > \alpha$ there is a $\lambda$ such that

$$P\{|W(T(t))| \geqq t^{\alpha/(2\gamma)};\ t \leqq \lambda\} < \varepsilon/2 .$$

For $n$ large enough, by assumption,

$$P\{T(s_n) \geqq s_n^{2/\alpha}\varepsilon^{2\gamma/\alpha}\} < \varepsilon/2 .$$

We can assume that $s_n^{2/\alpha}\varepsilon^{2\gamma/\alpha} \leqq \lambda$ also. Thus

$$P\{|W(T(s_n))| < [T(s_n)]^{\alpha/2\gamma};\ T(s_n) < s_n^{2/\alpha}\varepsilon^{2\gamma/\alpha}\} > 1 - \varepsilon ,$$

or

$$P\{|W(T(s_n))| < s_n^{1/\gamma}\varepsilon\} > 1 - \varepsilon ,$$

or

$$P\{s_n^{-1/\gamma}|W(T(s_n))| > \varepsilon\} < \varepsilon ,$$

so $s_n^{-1/\gamma}|W(T(s_n))| \to 0$ is probability which was to be proved. This completes the proof of the theorem.

**3. The $\gamma$-variation of $X_t$.**

THEOREM 2. *Let $\{X_t;\ t \geqq 0\}$ be a process in $R^N$ with stationary independent increments whose characteristic function is* $\exp[-t\psi(y)]$ *where $\psi(y)$ is given by* (1.1). *Let $\beta$ be the index of $X_t$ as defined in* (1.2) *and suppose that $1 \leqq \beta \leqq 2$. If $\gamma > \beta$, then*

$$P\{V_\gamma(X(\cdot, \omega); a, b) < \infty\} = 1 ,$$

*where $V_\gamma$ is defined in* (1.6).

PROOF. It is enough to treat the case $[a, b] = [0, 1]$. The proof will be broken into several parts, each one depending on the preceding.

Assume that $X_t$ is real valued, has index $\beta < 2$ and that $E\{X_t\} = 0$ for all $t$. This means that $X_t$ is a right continuous martingale so there is a Wiener process $(W_t\ \mathscr{G}_t)$ and a family of minimal $\mathscr{G}_t$ stopping times $T(s)$ which forms a right continuous process with stationary independent increments such that $W(T(s))$ and $X(s)$ have the same joint distributions. It is therefore enough, in this case, to show that

$$P\{V_\gamma(W \circ T(\cdot, \omega); 0, 1) < \infty\} = 1 .$$

By Theorem 1, the index of $T$ is $\beta/2 < 1$.

The proof now is exactly as that given in [1] page 270.

Choose $\lambda < \frac{1}{2}$ such that $1 \geqq \gamma\lambda > \beta/2$. Given $\delta > 0$, there exists a $K < \infty$ and a set $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 1 - \delta/3$ such that $T(1, \omega) \leqq K$ if $\omega \in \Omega_1$. Lévy's theorem on the modulus of continuity of the Brownian sample path shows that there are random variables $M(\omega)$ and $\varepsilon(\omega)$ such that $M(\omega) < \infty$ and $\varepsilon(\omega) > 0$ almost surely and such that

$$|W(t_2, \omega) - W(t_1, \omega)| \leqq M(\omega)|t_2 - t_1|^\lambda$$

for all $0 \leqq t_1 \leqq t_2 \leqq K$ and $|t_2 - t_1| < \varepsilon(\omega)$. In particular, there is an $M < \infty$, an $\varepsilon > 0$, and a set $\Omega_2 \subset \Omega$ with $P(\Omega_2) > 1 - \delta/3$ such that $M(\omega) \leqq M$ and $\varepsilon(\omega) \geqq \varepsilon$ if $\omega \in \Omega_2$. Finally, there exists a $J < \infty$ and a set $\Omega_3 \subset \Omega$ with $P(\Omega_3) > 1 - \delta/3$ such that $|W(t, \omega)| \leqq J$ for all $t \leqq k$ provided $\omega \in \Omega_3$. Let $\Omega_0 = \Omega_1 \cap \Omega_2 \cap \Omega_3$ $P(\Omega_0) > 1 - \delta$. If $0 \leqq t_0 \leqq t_1 < \cdots < t_n \leqq 1$ is any finite subset of $[0, 1]$, then at most $K' = [K/\varepsilon] + 1$ of the differences $T(t_{j+1}, \omega) - T(t_j, \omega)$ can exceed $\varepsilon$ provided $\omega \in \Omega_0$. Thus if $\omega \in \Omega_0$, we have

$$\begin{aligned}\textstyle\sum_{j=1}^{n} |W(T(t_j)) &- W(T(t_{j-1}))|^\gamma \\ &\leqq (2J)^\gamma K' + M^\gamma \textstyle\sum |T(t_j) - T(t_{j-1})|^{\gamma\lambda} ,\end{aligned}$$

where the last sum is taken over those $j$'s for which $T(t_j, \omega) - T(t_{j-1}, \omega) < \varepsilon$. Thus if $\omega \in \Omega_0$, we have

$$\gamma - \text{Var } W \circ T(\cdot, \omega) \leqq (2J)^{\gamma}K' + M^{\gamma}[\lambda\gamma - \text{Var } T(\cdot, \omega)] .$$

But $1 \geqq \lambda\gamma > \beta/2$ so by Theorem 4.2 of [2] the last term is finite for almost all $\omega$. Thus $P[\gamma - \text{Var } W \circ T(\cdot, \omega) < \infty] \geqq 1 - \delta$ and since $\delta$ was arbitrary, this completes the proof of the case when $X_t$ is a real valued martingale.

Now suppose that $X_t$ is any real valued process with stationary independent increments which has index $\beta < 2$. Then $X_t$ can be decomposed into the sum of two process $Y_t$ and $Z_t$ such that $Y_t$ has finite expectation and $Z_t$ has only a finite number of jumps on $t \in [0, 1]$. In particular, if $\mu$ is the Lévy measure of $X_t$, let $Y_t$ have Lévy measure $\mu_1(A) = \mu(A \cap (-1, 1))$ and $Z_t$ have Lévy measure $\mu_2(A) = \mu(A \cap (-1, 1)^c)$. The argument is familiar. Clearly the $\gamma$-variation of $X_t$ is finite if and only if the $\gamma$-variation of $Y_t$ is finite. Thus we can assume that $E\{X_t\}$ exists.

In actual fact we can assume that $E\{X_t\} = 0$. Indeed if $E\{X_t\} = mt$, then let $Y_t$ be a Poisson process with parameter $-m$. That is, $E\{Y_t\} = -m$. Then if $Z_t = X_t + Y_t$, we have $E\{Z_t\} = 0$ and the Lévy measures of $Z_t$ and $X_t$ are the same on the open inverval $(-1, 1)$. By the argument above $X_t$ has finite $\gamma$-variation if and only if $Z_t$ does.

To complete the proof in the case $X_t$ is real valued of index $\beta < 2$, we need only note that $Z_t$, and indeed any process whose Lévy measure agrees with the Lévy measure on a neighborhood of zero, also has index $\beta$.

Suppose that $X_t$ is real valued and has index 2. Then the proof just given applies except for the assertion that the index of the subordinator is less than 1. Thus Theorem 4.2 of [2] does not apply. However we now see that we need only assume that $\lambda\gamma < 2$ in order to assert that

$$P\{\lambda\gamma - \text{Var } T(\cdot, \omega) < \infty\} = 1 .$$

This can always be achieved by choosing $\lambda$ small enough so the proof of the theorem in the case $X_t$ is real valued is complete.

If $X_t$ takes values in $R^N$, let $X^{(i)}(t)$ be the $i$th component of $X(t)$. Since for $\gamma > \beta$,

$$t^{-1/\gamma}X_t \to 0 \quad \text{a.s.} \qquad \text{as} \quad t \to 0 ,$$

we have

$$t^{-1/\gamma}X_t^{(i)} \to 0 \quad \text{a.s.} \qquad \text{as} \quad t \to 0 ,$$

so the index of $X_t^{(i)}$ does not exceed $\beta$. Thus if $\gamma > \beta$ the $\gamma$-variation of $X_t^{(i)}$ is finite for all $i$.

Let $0 = t_0 < t_1 < \cdots < t_n = 1$. Then if $\gamma > \beta$,

$$\begin{aligned}\sum_{j=1}^{n} |X(t_j) - X(t_{j-1})|^{\gamma} &\leqq \sum_{j=1}^{n} N^{\gamma/2} \max_{1 \leqq i \leqq N} |X^{(i)}(t_j) - X^{(i)}(t_{j-1})|^{\gamma} \\ &\leqq N^{\gamma/2} \sum_{j=1}^{n} \sum_{i=1}^{N} |X^{(i)}(t_j) - X^{(i)}(t_{j-1})|^{\gamma} \\ &\leqq N^{\gamma/2} \sum_{i=1}^{N} \gamma\text{-variation} \quad (X^{(i)}) < \infty\end{aligned}$$

almost surely. This completes the proof of the theorem.

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