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ERROR ESTIMATES FOR THE WEAK CONVERGENCE TO CERTAIN INFINITELY DIVISIBLE LAWS

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Let F_n denote the distribution function of the nth row sum of a triangular array of infinitesimal, rowwise independent random variables, and let F^* denote the limiting infinitely divisible distribution function. Bounds are obtained for $\sup_{-\infty<x<\infty} |F_n(x) - F^*(x)|$ in the case that the means are finite and also for the attraction to a stable law with exponent $\alpha \leqq 1$. Conditions for convergence of these bounds are given.

1. Introduction. It is well known that a distribution is infinitely divisible (inf. div.) if and only if it is the limit distribution of the row sums $X_{n1} + X_{n2} + \cdots + X_{nk_n} - A_n$ of a triangular array (X_{nk}) $k = 1, 2, \cdots, k_n$; $n = 1, 2, \cdots$ of infinitesimal random variables which are independent within each row. A subclass of particular importance, the stable laws, coincides with the set of limit distributions of normed sums $(X_1 + X_2 + \cdots + X_n)/B_n - A_n$ of a sequence of independent, identically distributed random variables (X_n). By letting $X_{nk} = X_k/B_n$, $k = 1, 2, \cdots, n$; $n = 1, 2, \cdots,$ we see how this becomes a triangular array. The row sums of this array are the normed sums above. The common distribution of these variables is said to be in the domain of attraction of the stable law.

It is the purpose of this paper to obtain a bound for $\sup_{-\infty<x<\infty} |F_n(x) - F(x)|$, where F_n denotes the distribution function of the nth row sum of a triangular array and F is the limit distribution function.

Shapiro (1955) has obtained a bound in the case that the variances are assumed to be finite. The canonical representation of the logarithm of an inf. div. characteristic function due to Kolmogorov was used. Boonyasombut and Shapiro (1970) were able to use the Levy–Khintchine representation to obtain bounds when the variances need not be finite. It was necessary to truncate the variables in order to use the previous result.

The representation of Feller (1966) has been used here to treat the more general case without the complication of truncation. The derivation is very similar to that of Shapiro (1955), since the kernel of the representation is similar in each case, but the study of the convergence of these bounds is much different. Slowly varying functions are used in this study.

2. Preliminaries. The essential facts concerning slowly varying functions can be found in the papers of J. Karamata (1930), (1933), but the functions there are continuous. The reader is referred to Feller (1966), De Haan (1970), and Tucker (1968), where only measurability is assumed.

Definition 2.1. A positive, Borel measurable function L defined on $(0, \infty)$

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varies slowly (at infinity) if and only if for each $x > 0$ $L(tx)/L(t) \to 1$ as $t \to \infty$. A function Z varies regularly with exponent λ if and only if $Z(x) = x^\lambda L(x)$, where $-\infty < \lambda < \infty$ and L varies slowly.

LEMMA 2.2. *If L is slowly varying, then as $x \to \infty$, $x^\lambda L(x) \to 0$ if $\lambda < 0$ and $x^\lambda L(x) \to \infty$ if $\lambda > 0$.*

This lemma follows from the representation theorem (page 274 of Feller (1966)). The next theorem is a version of a theorem in Feller (1966) (page 275, see Problem 30, page 279). For a distribution function $F(x)$, denote

$$U_q(x) = \int_0^x y^q \, dF(y) , \qquad V_r(x) = \int_x^\infty y^r \, dF(y) .$$

THEOREM 2.3. *If $U_q(x)$ varies regularly with exponent γ and $V_r(x)$ exists, then $\lim_{x\to\infty} x^{p-r} V_r(x)/U_q(x) = \gamma/(q - r - \gamma)$ exists, provided $r < q - \gamma$. If $U_q(x)$ varies regularly with exponent γ and $r > q - \gamma$, then $\lim_{x\to\infty} x^{q-r} U_r(x)/U_q(x) = \gamma/(r - q + \gamma)$.*

The rest of the material of this section is based on Chapter XVII of Feller (1966).

DEFINITION 2.4. A measure M on the real line is called a canonical measure if and only if it is finite on finite intervals and $M^+(x) = \int_{x-}^{\infty} y^{-2} \, dM$ and $M^-(x) = \int_{-\infty}^{-x+} y^{-2} \, dM$ converge for all $x > 0$.

DEFINITION 2.5. The measures $c_n x^2 \, dF_n(x)$ converge properly to the canonical measure M if and only if for finite intervals of continuity of M, $c_n x^2 \, dF_n(x) \to M$, and at all continuity points $x > 0$ of M, $c_n[1 - F_n(x)] \to M^+(x)$ and $c_n F_n(-x) \to M^-(-x)$.

THEOREM 2.6. *The function $\phi(t)$ is an inf. div. characteristic function if and only if there is a canonical measure M and a real number b such that $\phi(t) = e^{\lambda(t)}$ where*

$$\lambda(t) = \int_{-\infty}^{\infty} \frac{e^{itx} - 1 - it \sin x}{x^2} \, dM + ibt .$$

The measure M is unique.

THEOREM 2.7. *A canonical measure M determining a stable law is either concentrated at the origin (normal law), or else for $x > 0$, $M[0, x] = Cpx^{2-\alpha}$, $M[-x, 0] = Cqx^{2-\alpha}$, where $0 < \alpha < 2$, $p \geqq 0$, $q \geqq 0$, $p + q = 1$. The parameter α is called the exponent of the stable law.*

COROLLARY 2.8. *The function $\phi(t)$ is the characteristic function of a stable law with exponent α, $0 < \alpha < 2$, if and only if*

$$\log \phi(t) = Cp(2 - \alpha) \int_0^\infty \frac{e^{itx} - 1 - its_\alpha(x)}{x^{1+\alpha}} \, dx + Cq(2 - \alpha) \int_{-\infty}^0 \frac{e^{itx} - 1 - its_\alpha(x)}{|x|^{1+\alpha}} \, dx + ibt$$

with

$$s_\alpha(x) = x\,, \qquad \alpha > 1\,,$$
$$= \sin x\,, \qquad \alpha = 1\,,$$
$$= 0\,, \qquad \alpha < 1\,.$$

If $\alpha > 1$, then the mean of $\phi(t)$ is b.

The choice of s_α in the preceding corollary is preferable, and it is this which allows us to use the derivations of Shapiro (1955). Motivated by the note on page 531 of Feller (1966), we can replace the convergence theorems there with the following theorems. The proofs are straightforward. The apparently restrictive hypotheses are satisfied in the case of attraction to a stable law (see next section).

THEOREM 2.9. *If $c_n x^2\, dF_n(x) \to M$ properly, and if for any $\varepsilon > 0$ there is a $K > 0$ such that for all n, $c_n \int_{|x|\geq k} |x|\, dF_n(x) < \varepsilon$, then* (a) *for any $a > 0$, $\int_{|x|\geq a} |x|^{-1}\, dM < \infty$, and* (b) *$c_n \int_{-\infty}^{\infty} z(x)\, dF_n(x) \to \int_{-\infty}^{\infty} z(x)/x^2\, dM$ for every continuous function $z(x)$ such that $z(x)/x^2$ is continuous at $x = 0$ and $z(x)/x$ is bounded.*

THEOREM 2.10. *If $c_n x^2\, dF_n(x) \to M$ properly, and if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all n, $c_n \int_{|x|<\delta} |x|\, dF_n(x) < \varepsilon$, then* (a) *for any $a > 0$, $\int_{-a}^{a} |x|^{-1}\, dM < \infty$, and* (b) *$c_n \int_{-\infty}^{\infty} z(x)\, dF_n(x) \to \int_{-\infty}^{\infty} z(x)/x^2\, dM$ for every bounded, continuous function $z(x)$ such that $z(x)/x$ is bounded near zero.*

The following convergence theorems for attraction to stable laws are taken in altered form from Chapter XVII of Feller (1966). For a distribution function $F(x)$ denote for $x > 0$, $\mu(x) = \int_{-x}^{x} y^2\, dF(y)$, and let $\phi(t)$ be its characteristic function.

THEOREM 2.11.

(a) *F belongs to the domain of attraction of the normal law if and only if $\mu(x)$ is slowly varying.*

(b) *F belongs to the domain of attraction of a nonnormal stable law if and only if as $x \to \infty$*

$$\mu(x)/(x^{2-\alpha}L(x)) \to 1 \tag{2.1}$$

with $L(x)$ slowly varying, and as $x \to \infty$

$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to p\,, \tag{2.2}$$
$$\frac{F(-x)}{1 - F(x) + F(-x)} \to q$$

where $0 < \alpha < 2$, and $p \geq 0$, $q \geq 0$, $p + q = 1$. If this happens, F belongs to the domain of attraction of the law determined by the canonical measure $M[0, x] = Cpx^{2-\alpha}$, $M[-x, 0] = Cpx^{2-\alpha}$ and the constant C and the norming constants B_n may be chosen such that as $n \to \infty$

$$nB_n^{-\alpha}L(B_n) \to C\,. \tag{2.3}$$

THEOREM 2.12. *Suppose F satisfies conditions* (2.1) *and* (2.2); *and if $\alpha > 1$, assume F has mean μ. Let $\beta_n = 0$ if $\alpha < 1$, $\beta_n = \mu/B_n$ if $\alpha > 1$, and let $B_n \to \infty$ satisfy* (2.3). *Then*

$$[\phi(t/B_n)e^{-i\beta_n t}]^n \to \phi^*(t)$$

as $n \to \infty$, and $nx^2\, dF(B_n x) \to M$ properly, where for $x > 0$, $M[0, x] = Cpx^{2-\alpha}$ and $M[-x, 0] = Cqx^{2-\alpha}$. The limit $\phi^*(t)$ has the form of Corollary 2.8.

We will also need the following result which is Lemma 5 of Tucker (1968), and the next inequality.

THEOREM 2.13. *If F is in the domain of attraction of a stable law with exponent α, $0 < \alpha < 2$, and if B_n is a sequence of normalizing coefficients for F; then there is a measurable, slowly varying function L such that $B_n = n^{1/\alpha}L(n)$.*

LEMMA 2.14. *If z_1 and z_2 are complex numbers such that $0 < |z_1| \leqq 1$ and $0 < |z_2| \leqq 1$, then $|z_1 - z_2| \leqq |\log z_1 - \log z_2|$.*

Throughout this paper F_{nk} and ϕ_{nk} will denote the distribution and characteristic functions of the random variables X_{nk} of a triangular array, F_n and ϕ_n those of the row sums, while F^* and ϕ^* will denote the limit functions. F and ϕ will be reserved for distributions attracted to a stable law.

3. Derivation of the bound in the finite mean case. In this section we consider arrays with finite mean. Our main interest is in the domain of attraction of a stable law with exponent $\alpha > 1$, i.e. with finite mean. Motivated by the next theorem, we assume that for any $\varepsilon > 0$ there is a $K > 0$ such that for any n

$$\sum_{k=1}^{k_n} \int_{|x|>K} |x|\, dF_{nk}(x) < \varepsilon\,. \tag{3.1}$$

THEOREM 3.1. *If F is attracted to a stable law with $1 < \alpha \leqq 2$, norming constants B_n, then for any $\varepsilon > 0$ there is a $K > 0$ such that for any n; $n \int_{|x|>K} |x|\, dF(B_n x) < \varepsilon$.*

PROOF. We use Theorem 2.3 with $q = 2$ and $r = 1$. By Theorem 2.11 $\mu(x) = U_2(x)$ varies regularly with exponent $2 - \alpha$, and by Theorem 2.9 $V_1(x)$ exists; hence as $n \to \infty$, since $B_n \to \infty$, we get for $x > 0$

$$\frac{B_n x \int_{|y|>xB_n} |y|\, dF(y)}{\mu(B_n x)} \to \frac{2-\alpha}{1-2+\alpha} = \frac{2-\alpha}{\alpha-1} \qquad \text{or}$$

$$\frac{nx \int_{|y|>x} |y|\, dF(B_n y)}{(n/B_n^2)\mu(B_n x)} \to \frac{2-\alpha}{\alpha-1}\,.$$

But $nB_n^{-2}\mu(B_n x) \to Cx^{2-\alpha}$ because of proper convergence, so we get as $n \to \infty$ that

$$nx \int_{|y|>x} |y|\, dF(B_n y) \to C\left(\frac{2-\alpha}{\alpha-1}\right) x^{2-\alpha} \qquad \text{or}$$

$$n \int_{|y|>x} |y|\, dF(B_n y) \to C\left(\frac{2-\alpha}{\alpha-1}\right) x^{1-\alpha}\,.$$

Since $1 - \alpha < 0$, choose K such that $K^{1-\alpha} < \varepsilon(\alpha - 1)/(2 - \alpha)C$; we get for

$n > N_0$ that the left-hand side is less than ε. Then choose K larger still so that since F has first moment, $n \int_{|y|>K} |y| \, dF(B_n y) < \varepsilon$ for $n = 1, 2, \cdots, N_0$ as well.

The derivation here is similar to that of Shapiro (1955) for the case of finite variance. We will assume that the inf. div. law F^* is represented by

$$\log \phi^*(t) = \int_{-\infty}^{\infty} \frac{e^{itx} - 1 - itx}{x^2} \, dM \tag{3.2}$$

where M is the canonical measure determining F^*. This alternative to the form of Feller in Theorem 2.6 is permissible due to Theorem 2.9, under assumption (3.1), and the fact that $|e^{itx} - 1 - itx| \leqq 2|tx|$. Thus since the integrand is the same for both this representation and that of Kolmogorov, the method of Shapiro can be applied with minor changes in the derivation.

For K, $-K$ continuity points of M, and $\delta > 0$ given, let $-x_1 < x_0 \leqq 0 < x_1 < \cdots < x_m = K$ be such that for $i = 0, 1, \cdots, m$, $-x_i$ and x_i are continuity points of M, and $\max_{1 \leqq i \leqq m} |x_i - x_{i-1}| < \delta$. If 0 is a continuity point we let $x_0 = 0$. For each i let y_i be such that $x_{i-1} \leqq y_i \leqq x_i$. Let

$$\begin{aligned} g(n, m(K, \delta)) = {} & [4 \int_{|x| \geqq K} |x| \sum_{k=1}^{k_n} dF_{nk}(x) + 4 \int_{|x| \geqq K} 1/|x| \, dM]^{\frac{1}{2}} \\ & + [4 \sum_{k=1}^{k_n} (E|X_{nk}|)^2 + \sum_{i=1}^{m} |M(x_0, x_i) - \int_{x_0}^{x_i} x^2 \sum_{k=1}^{k_n} dF_{nk}(x)| \\ & + \sum_{i=1}^{m} |M(-x_i, x_0) - \int_{-x_i}^{x_0} x^2 \sum_{k=1}^{k_n} dF_{nk}(x)|]^{\frac{1}{3}} \\ & + [\tfrac{5}{6}\delta[\int_{-K}^{K} x^2 \sum_{k=1}^{k_n} dF_{nk}(x) + M(-K, K)]]^{\frac{1}{4}} \end{aligned}$$

and let $T = 1/g(n, m(K, \delta))$ for fixed n, K, and δ.

LEMMA 3.2. *If (X_{nk}) is a triangular array of infinitesimal random variables which satisfies* (3.1), *then as $n \to \infty$,* $\max_{1 \leqq k \leqq k_n} E|X_{nk}| \to 0$.

PROOF. For any $\varepsilon > 0$ choose K so that (3.1) holds, then for $1 \leqq k \leqq k_n$, $E|X_{nk}| = \int_{|x| \geqq K} + \int_{\varepsilon < |x| < K} + \int_{|x| \leqq \varepsilon} |x| \, dF_{nk}(x) \leqq \varepsilon + KP\{|X_{nk}| > \varepsilon\} + \varepsilon$. So

$$\max_{1 \leqq k \leqq k_n} E|X_{nk}| \leqq 2\varepsilon + \max_{1 \leqq k \leqq k_n} KP\{|X_{nk}| > \varepsilon\} < 3\varepsilon ,$$

for n large since the X_{nk} are infinitesimal.

We now come to the first main theorem.

THEOREM 3.3. *Let F^* be an inf. div. distribution function which has the representation* (3.2). *Let (X_{nk}) be a triangular array of infinitesimal random variables which are independent within each row. Assume also that the X_{nk} have finite mean μ_{nk}, that* (3.1) *is satisfied,* $\max_{1 \leqq k \leqq k_n} E|X_{nk}| \leqq \frac{1}{2}$, *$F^{*\prime}$ exists, and that $|F^{*\prime}(x)| \leqq B < \infty$ for all x. Let $A_n = \sum_{k=1}^{k_n} \mu_{nk}$. Then*

$$\sup_{-\infty < x < \infty} |F_n(x) - F^*(x)| \leqq k(B) g(n, m(K, \delta))$$

where $k(B)$ is a constant depending only on B.

The following lemma will be used later, as well as in the proof of Theorem 3.3. Its proof is straightforward.

LEMMA 3.4. *If* $|t| \leqq 1/(2E|X_{nk}|)$, *then*

$$\left|\sum_{k=1}^{k_n} \log \phi_{nk}(t) - \sum_{k=1}^{k_n} [\phi_{nk}(t) - 1]\right| \leqq 4|t|^2 \sum_{k=1}^{k_n} (E|X_{nk}|)^2 .$$

PROOF OF THEOREM 3.3. A method similar to that of Shapiro (1955), using Lemmas 2.14 and 3.4, shows that

$$\int_{-T}^{T} \frac{|\phi_n(t) - \phi^*(t)|}{|t|} \, dt \leqq g(n, m(K, \delta)) .$$

But then applying Esséen's theorem (page 196 of Gnedenko and Kolmogorov (1954), page 512 of Feller (1966)), we get $|F_n(x) - F^*(x)| \leqq a(2\pi)^{-1}g(n, m(K, \delta)) + c(a)B/T = [a(2\pi)^{-1} + c(a)B]g(n, m(K, \delta))$, which is the result.

We now restate the theorem in terms of a law F which is in the domain of attraction of a stable law F^* with exponent α, $1 < \alpha < 2$. Let X have the common distribution of the X_n. Rewriting g we get

$$\begin{aligned} g(n, m(K, \delta)) = & \left[\frac{4C(2-\alpha)}{(\alpha - 1)K^{\alpha-1}} + \frac{4n}{B_n} \int_{|x| \geqq B_n K} |x| \, dF(x)\right]^{\frac{1}{2}} \\ & + \left[\frac{4n(E|X|)^2}{B_n^{\,2}} + 2 \sum_{i=1}^{m} \left|Cpx_i^{\,2-\alpha} - \int_0^{x_i} nx^2 \, dF(B_n x)\right| \right. \\ & \left. + 2 \sum_{i=1}^{m} \left|Cqx_i^{\,2-\alpha} - \int_{-x_i}^{0} nx^2 \, dF(B_n x)\right| \right]^{\frac{1}{3}} \\ & + \left[\tfrac{5}{6}\delta[CK^{2-\alpha} + \int_{-K}^{K} nx^2 \, dF(B_n x)]\right]^{\frac{1}{3}} . \end{aligned}$$

COROLLARY 3.5. *Let* F^* *be a stable distribution function with mean zero and exponent* $1 < \alpha < 2$ *which has the representation of Corollary* 2.8. *Let* (X_n) *be a sequence of independent random variables with distribution* F, *mean* μ, *and moment* $E|X|$. *Let* F_n *denote the distribution function of the sum*

$$\frac{X_1 + X_2 + \cdots + X_n}{B_n} - \frac{n\mu}{B_n} ,$$

and assume that $B_n > 2E|X|$. *Let a bound for* $F^{*\prime}$ *be* B. *Then*

$$\sup_{-\infty < x < \infty} |F_n(x) - F^*(x)| \leqq k(B)g(n, m(K, \delta))$$

where $k(B)$ *is a constant depending only on* B. *If we also assume that* F *has mean zero, then* $k(B) = (1 + 24B)/\pi$.

It is interesting to note that in case $\alpha = 2$, $n/B_n^{\,2}$ does not necessarily converge to zero. In fact if F is in the domain of normal attraction of the normal law, i.e., $B_n = n^{\frac{1}{2}}$, then $n/B_n^{\,2} = 1$, so the bound cannot converge. Shapiro's result covers this case.

4. Convergence of the bound in the finite mean case. We now investigate under what conditions these bounds converge to zero. The key criterion is given by the following theorem from which several more specific corollaries will follow, including one for stable laws.

THEOREM 4.1.

(a) *If* (X_{nk}) *is a triangular array of infinitesimal, rowwise independent random variables with finite mean which satisfy condition* (3.1) *and* $\max_{1 \leq k \leq k_n} E|X_{nk}| < \frac{1}{2}$,

(b) $\sum_{k=1}^{k_n} x^2 \, dF_{nk}(x) \to M$ *properly, and*

(c) $\sum_{k=1}^{k_n} (E|X_{nk}|)^2 \to 0$, $(n \to \infty)$, *then*

(i) $F_n \to F^*$ *as* $n \to \infty$ *where* $\log \phi^*(t)$ *is given by* (3.2),

(ii) *for any* $\varepsilon > 0$ *there is a* $K > 0$ *(continuity point of* M*), a* $\delta > 0$, *and an* N_0 *such that for* $n \geqq N_0$, $g(n, m(K, \delta)) < \varepsilon$.

PROOF.

(i) By Theorem 2.9., (b) implies

$$\sum_{k=1}^{k_n} [\phi_{nk}(t) - 1 - it\mu_{nk}] \to \int_{-\infty}^{\infty} \frac{e^{itx} - 1 - itx}{x^2} \, dM .$$

By Lemma 3.4 and (c)

$$\sum_{k=1}^{k_n} [\log \phi_{nk}(t) - it\mu_{nk}] \to \int_{-\infty}^{\infty} \frac{e^{itx} - 1 - itx}{x^2} \, dM ,$$

and hence by the continuity theorem, $F_n \to F^*$.

(ii) Choose K which satisfies (3.1) with ε small. By Theorem 2.9 we can choose K still larger so that $\int_{|x| \geqq K} 1/|x| \, dM$ is small; so the first term of g can be made arbitrarily small. Now for this fixed K, by the proper convergence, $\int_{-K}^{K} x^2 \sum_{k=1}^{k_n} dF_{nk}(x)$ is bounded, so δ can be chosen so that the last term of g is arbitrarily small. With this K and δ and any fixed corresponding choice of the x_i, the proper convergence and (c) imply that the second term can be made arbitrarily small for n sufficiently large, and hence g will be small, which is what we set out to prove.

COROLLARY 4.2. *Under the conditions of Theorem* 4.1, *there are sequences* (K_n) *and* (δ_n) *such that as* $n \to \infty$, $g(n, m(K_n, \delta_n)) \to 0$.

PROOF. As above for any K and δ the second term of g converges to 0 as $n \to \infty$. So letting $\varepsilon_k' \to 0$, $\varepsilon_k' > 0$, we can choose K_k' and δ_k' such that the first and last terms of g are less than $\varepsilon_k'/3$, and n_k such that for $n \geqq n_k$ the middle term is less than $\varepsilon_k'/3$ (wolog $n_k < n_{k+1}$ for any k). Hence for any n letting $\varepsilon_n = \varepsilon_k'$, $K_n = K_k'$, and $\delta_n = \delta_k'$ where $n_k \leqq n < n_{k+1}$, we get that

$$g(n, m(K_n, \delta_n)) \leqq \varepsilon_n \to 0 .$$

COROLLARY 4.3. *The condition* (a) *of Theorem* 4.1 *and any one of the following conditions are sufficient for the conclusions of Theorem* 4.1 *and Corollary* 4.2 *to hold.*

(a) $\sum_{k=1}^{k_n} (E|X_{nk}|)^2 \to 0$ *and* $F_n \to F^*$,

(b) $\sum_{k=1}^{k_n} E|X_{nk}|$ *is bounded and* $F_n \to F^*$,

(c) $\sum_{k=1}^{k_n} x^2 \, dF_{nk}(x) \to M$ *properly, and* $M\{0\} = 0$,

(d) $F_n \to F^*$, $b_{nk} = 0$ *(see Section* 7, *Chapter* XVII *of Feller* (1966)), *and* $M\{0\} = 0$.

PROOF. In each case we show that the conditions of Theorem 4.1 are implied.

(a) By Lemma 3.4, (a) implies

$$\sum_{k=1}^{k_n} [\phi_{nk}(t) - 1 - it\mu_{nk}] \to \log \phi^*(t) ,$$

so by Theorem 2.9 with $z(x) = x - \sin x$ and the theorem on page 528 of Feller (1966) we get the proper convergence.

(b) implies (a) by Lemma 3.2.

(c) We must show that $\sum_{k=1}^{k_n} (E|X_{nk}|)^2 \to 0$. We have for any $\delta > 0$ that

$$\begin{aligned}\sum_{k=1}^{k_n} [\textstyle\int_{|x|\geqq\delta} |x|\, dF_{nk}(x) + \int_{|x|<\delta} |x|\, dF_{nk}(x)]^2 \\ \leqq 3(\max_{1\leqq k\leqq k_n} E|X_{nk}|) \textstyle\sum_{k=1}^{k_n} \int_{|x|\geqq\delta} |x|\, dF_{nk}(x) \\ + \textstyle\sum_{k=1}^{k_n} [\int_{|x|<\delta} |x|\, dF_{nk}(x)]^2 .\end{aligned}$$

By Lemma 2.9 applied to $\{|x| \geqq \delta\}$, and $z(x) = |x|$ on this set, we have that $\sum_{k=1}^{k_n} \int_{|x|\geqq\delta} |x|\, dF_{nk}(x)$ is bounded, and so by Lemma 3.2 the first term converges to zero for any δ as $n \to \infty$. Hölder's inequality and the proper convergence imply that the second term converges to $M(-\delta, \delta)$. Hence for any $\varepsilon > 0$, choose δ such that $M(-\delta, \delta) < \varepsilon$, then for n large the last term is less than ε, and for this δ the first term will be less than ε for n larger still.

(d) By the criterion of Section 8, Chapter XVII of Feller (1966), we have the proper convergence and so (d) implies (c). This completes the proof.

The next corollary shows the behavior of the bound for the stable case.

COROLLARY 4.4. *If the distribution function F satisfies Theorem* 2.12 *with* $1 < \alpha < 2$, *then F is attracted to the stable law F^* which is given in Corollary* 3.5; *and for any $\varepsilon > 0$ there is a $K > 0$, a $\delta > 0$, and an N_0 such that for $n \geqq N_0$,*

$$g(n, m(K, \delta)) < \varepsilon .$$

Moreover there exist sequences (K_n) and (δ_n) such that

$$g(n, m(K_n, \delta_n)) \to 0 \qquad \text{as} \quad n \to \infty .$$

PROOF. By Theorem 2.12 $nx^2\, dF(B_n x) \to M$ properly and $F_n \to F^*$. Hence since $\alpha < 2$, $M\{0\} = 0$, and so by Corollary 4.3 (c) the conclusion follows.

5. Derivation of the bound for attraction to a stable law with $\alpha \leqq 1$. In this section we derive bounds for the case when the limit law does not have finite mean. The derivation is similar to that of the last section, except that lower order moments are used. However the methods used to investigate the behavior of these bounds involve use of slowly varying functions, and for this reason they only apply when the limit law is stable.

As before F will denote a distribution function attracted to a stable law F^*, and we will assume that F^* has the representation of Corollary 2.8 with $b = 0$. Denote the rth moment of F by $E|X|^r$, which exists if $r < \alpha$ by Lemma 2 on page 545 of Feller (1966). We will also use the inequality $|e^{iu} - 1| \leqq 2|u|^r$ for $0 < r \leqq 1$.

Choose K, δ, x_i, y_i, $i = 1, 2, \cdots, m$, as in Section 3, except that $x_0 = 0$. Define

$$g_1(n, r, m(K, \delta))$$
$$= \left[\frac{4n}{rB_n{}^r} \int_{|x| \geqq B_n K} |x|^r \, dF(x) + \frac{4C(2-\alpha)}{r(\alpha - r)K^{\alpha - r}}\right]^{1/(r+1)} + \left[\frac{16n(E|X|^r)^2}{rB_n{}^{2r}}\right]^{1/(1+2r)}$$
$$+ \left[4 \sum_{i=1}^m \left| \int_0^{x_i} nx \, dF(B_n x) - \frac{Cp(2-\alpha)}{1-\alpha} x_i{}^{1-\alpha} \right|\right.$$
$$\left. + 4 \sum_{i=1}^m \left| \int_{-x_i}^0 nx \, dF(B_n x) - \frac{Cq(2-\alpha)}{1-\alpha} x_i{}^{1-\alpha} \right|\right]^{\frac{1}{2}}$$
$$+ \left[3\delta\left[\int_{-K}^K n|x| \, dF(B_n x) + \frac{C(2-\alpha)}{1-\alpha K^{1-\alpha}}\right]\right]^{\frac{1}{3}}.$$

We now state the main theorem for $\alpha < 1$.

THEOREM 5.1. *Let F^* be a stable distribution with exponent $\alpha < 1$ which has the representation of Corollary 2.8 with $b = 0$. Let (X_n) be a sequence of independent random variables with distribution function F, and moment $E|X|^r$, $0 < r < 1$. Let F_n denote the distribution function of the sum $(X_1 + X_2 + \cdots + X_n)/B_n$. Assume that $B_n \geqq 4^{1/r}(E|X|^r)^{1/r}$ and $|F^{*\prime}| \leqq B$. Then*

$$\sup_{-\infty < x < \infty} |F_n(x) - F^*(x)| \leqq k(B) g_1(n, r, m(K, \delta))$$

where $k(B)$ is a constant depending only on B.

The proof of this theorem is similar to the one in Section 3. Lemma 2.4 is replaced by the next lemma. In the rest of the proof $(e^{itx} - 1)/x$ is used in place of $(e^{itx} - 1 - itx)/x^2$. The extra $1/x$ is incorporated into the measures. The essential change is that

$$\left|\frac{\partial}{\partial x} \frac{e^{itx} - 1}{x}\right| \leqq 2t^2$$

replaces the similar bound used by Shapiro.

LEMMA 5.2. *For $0 < r < \alpha \leqq 1$,*

$$\left| n \log \phi\left(\frac{t}{B_n}\right) - n\left[\phi\left(\frac{t}{B_n}\right) - 1\right]\right| \leqq \frac{4|t|^{2r} n(E|X|^r)^2}{B_n{}^{2r}}$$

provided $|t| \leqq B_n 4^{-1/r}(E|X|^r)^{-1/r}$.

Much of the proof of Theorem 5.1 is valid when $\alpha = 1$. However the representation is different. We must use something like $\sin x$. Our derivations depend on the form of the integrand. For $\alpha > 1$ Shapiro's form will work, for $\alpha < 1$ a similar method works; but for $\alpha = 1$, $\sin x$ solves the convergence problem, but it creates another in trying to get a bound for

$$\frac{\partial}{\partial x}\left(\frac{e^{itx} - 1 - it \sin x}{x^2}\right)$$

which involves a single power of $|t|$. This complication can be avoided by assuming that F and F^* are symmetric. Since the only known stable law with

$\alpha = 1$ is the Cauchy law, which is symmetric, this is not a severe restriction. If we assume that $\frac{1}{2} < r < 1$, Theorem 5.1 holds for $\alpha = 1$ if we replace g_1 with

$$g_2(n, r, m(K, \delta)) = \left[\frac{4n}{rB_n{}^r}\int_{|x|\geq B_nK}|x|^r\,dF(x) + \frac{4C}{r(1-r)K^{1-r}}\right]^{1/(r+1)} + \left[\frac{16n(E|X|^r)^2}{rB_n{}^{2r}}\right]^{1/(1+2r)} + \left[2\sum_{i=1}^m |Cx_i - n\int_0^{x_i} x^2\,dF(B_n x)|\right]^{\frac{1}{3}} + \left[\tfrac{5}{6}\delta\left[\int_{-K}^K nx^2\,dF(B_n x) + CK\right]\right]^{\frac{1}{4}} .$$

6. Convergence of the bound for attraction to a stable law with $\alpha \leqq 1$. To investigate the convergence of the bound of the last section, we need two lemmas.

LEMMA 6.1. *If $F_n \to F^*$, then for $\alpha/2 < r < \alpha \leqq 1$ as $n \to \infty$, $nB_n{}^{-2r}(E|X|^r)^2 \to 0$.*

PROOF. For $0 < r < \alpha$, $E|X|^r$ exists, so we must show that $n/B_n{}^{2r} \to 0$, as $n \to \infty$. By Theorem 2.13 there is a slowly varying function L such that $B_n = n^{1/\alpha}L(n)$, so $nB_n{}^{-2r} = n(n^{1/\alpha}L(n))^{-2r} = n^{1-2r/\alpha}(L(n))^{-2r}$. Now L^{-2r} varies slowly, so since $\alpha/2 < r$ implies $1 - 2r/\alpha < 0$, we have by Lemma 2.2 that this converges to 0.

LEMMA 6.2. *If F satisfies Theorem 2.12 with $0 < r < \alpha \leqq 1$, then for any $\varepsilon > 0$ there is a $K > 0$ such that for all n*

$$nB_n{}^{-r}\int_{|x|\geq B_nK}|x|^r\,dF(x) < \varepsilon .$$

PROOF. We use Theorem 2.3 with $q = 2$. By Theorem 2.11 U_2 varies regularly with exponent $2 - \alpha$, and V_r exists for $r < \alpha$ by Lemma 2 on page 545 of Feller (1966). Hence since $B_n \to \infty$ and $r < \alpha$ we get for $x > 0$ as $n \to \infty$

$$\frac{B_n{}^{2-r}K^{2-r}\int_{|x|\geq B_nK}|x|^r\,dF(x)}{\int_{|x|<B_nK}x^2\,dF(x)} \to \frac{2-\alpha}{2-r-(2-\alpha)} = \frac{2-\alpha}{\alpha - r} .$$

Theorem 2.12 implies that $nB_n{}^{-2}\mu(B_nK) \to C(2-\alpha)K^{2-\alpha}$, so as $n \to \infty$

$$nB_n{}^{-r}\int_{|x|\geq B_nK}|x|^r\,dF(x) \to C(\alpha - r)^{-1}K^{r-\alpha} .$$

Choose K such that $K^{r-\alpha} < \varepsilon(\alpha - r)C^{-1}$, since $r - \alpha < 0$, and we get that the left side is less than ε for $n \geqq N_0$. Choose K still larger so that this is true for $n = 1, 2, \cdots, N_0$, since the rth moment exists, and the result follows.

We now come to the convergence theorems for the bounds.

THEOREM 6.3. *If the distibution F satisfies Theorem 2.12 with $0 < \alpha < 1$, then $F_n \to F^*$ in Theorem 5.1. Let $0 < \alpha/2 < r < \alpha < 1$. Then for any $\varepsilon > 0$ there is a $K > 0$ and a $\delta > 0$ such that for $n \geqq N_0$, $g_1(n, r, m(K, \delta)) \leqq \varepsilon$. Moreover there exist sequences (K_n) and (δ_n) such that $g_1(n, r, m(K_n, \delta_n)) \to 0$ as $n \to \infty$.*

PROOF. An argument similar to the one in Theorem 3.1, using the second part of Theorem 2.3, shows that the hypothesis of Theorem 2.10 is satisfied. Applied first to $(-K, K)$ and to $z(x) = |x|$ for $|x| < K$, we get, for fixed K, that

$$n\int_{-K}^K |x|\,dF(B_n x) \to \int_{-K}^K \frac{1}{|x|}\,dM = \frac{C(2-\alpha)}{1-\alpha}K^{1-\alpha} .$$

And similarly applied to $[0, x_i)$ and $(-x_i, 0]$ with $z(x) = x$ on these sets, $i = 1, \cdots, m$, we have

$$\int_0^{x_i} nx\, dF(B_n x) \to \frac{Cp(2-\alpha)}{1-\alpha} x_i^{1-\alpha}$$

$$\int_{-x_i}^0 nx\, dF(B_n x) \to \frac{Cq(2-\alpha)}{1-\alpha} x_i^{1-\alpha} .$$

By Lemma 6.2 and since $r < \alpha$, the first term of g can be made arbitrarily small. With this K fixed, $n \int_{-K}^{K} |x|\, dF(B_n x)$ is bounded, so δ can be chosen so that the last term is small. With this K and δ and a fixed corresponding choice of x_i, $i = 1, \cdots, m$, the third term converges to 0, and by Lemma 6.1 the second does also. The rest is the same as before.

If we assume that F is symmetric, Theorem 6.3 holds for $\alpha = 1$ as well with g_2 in place of g_1. The only difference is that only the proper convergence is needed for the third term.

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