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ESTIMATES OF THE RATES OF CONVERGENCE IN LIMIT THEOREMS FOR THE FIRST PASSAGE TIMES OF RANDOM WALKS

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Let T_r be the time of first passage to the level $r > 0$ by a random walk with independent and identically distributed steps and mean $\nu \geqq 0$. Estimates are given for the rate at which the distribution of T_r, suitably scaled and normalized, converges to the stable distribution with index $\frac{1}{2}$ when $\nu = 0$ and to the normal distribution when $\nu > 0$ as $r \to \infty$.

1. Introduction. Let $\{X_n, n \geqq 1\}$ be a sequence of independent and identically distributed random variables defined on a probability triple $(\Omega, \mathscr{F}, P)$ with $EX_1 = \nu \geqq 0$, $\operatorname{Var} X_1 = \sigma^2$ and assume for some $p > 2$, $E|X_1 - \nu|^p = M < \infty$. Set $S_0 = 0$, $S_k = X_1 + \cdots + X_k$, $k \geqq 1$, and for $r > 0$ define T_r by $T_r = \min\{k \geqq 1 : S_k \geqq r\}$, where the minimum of the empty set is ∞. It is well known that if $\nu = 0$ then

$$\begin{aligned} \lim_{r\to\infty} P\{\sigma^2 T_r / r^2 \leqq x\} &= G_{\frac{1}{2}}(x) \qquad \text{for} \quad x > 0 \\ &= 0 \qquad \text{for} \quad x \leqq 0\,, \end{aligned}$$

where $G_{\frac{1}{2}}$ is the stable distribution with exponent $\frac{1}{2}$. The distribution $G_{\frac{1}{2}}$ is given by $G_{\frac{1}{2}}(x) = 2\{1 - \Phi(x^{-\frac{1}{2}})\}$, where Φ is the standard normal distribution $\Phi(x) = (1/2\pi)^{\frac{1}{2}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$. When $\nu > 0$ we have

$$\lim_{r\to\infty} P\{(T_r - r/\nu)/(\sigma^2 r \nu^{-3})^{\frac{1}{2}} \leqq x\} = \Phi(x)\,, \qquad \text{for} \quad x \in \mathbb{R}\,.$$

Here we prove the following two results.

THEOREM 1. *If $\nu = 0$ there exists a constant C depending only on p, σ and M such that for all $x > 0$ and $r > 0$,*

$$\left| P\left\{ \frac{\sigma^2 T_r}{r^2} \leqq x \right\} - G_{\frac{1}{2}}(x) \right| \leqq C f(r, p)\,,$$

where

$$\begin{aligned} f(r, p) &= 1/r^{p/(p+1)} \qquad \textit{for} \quad p \geqq 3 \\ &= 1/r^{p(p-2)/(p^2+2p-2)} \qquad \textit{for} \quad 2 < p < 3\,. \end{aligned}$$

THEOREM 2. *If $\nu > 0$ there exists a constant C depending only on p, ν, σ and M such that for all $x \in \mathbb{R}$ and $r > 1$,*

$$\left| P\left\{ \frac{T_r - r/\nu}{(\sigma^2 r \nu^{-3})^{\frac{1}{2}}} \leqq x \right\} - \Phi(x) \right| \leqq C g(r, p)$$

where $g(r, p) = \{(\log r)^p / r^{\min(p-2, p/2)}\}^{1/2(p+1)}$.

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The proof of Theorem 1 is given in Section 2 while that of Theorem 2 is in Section 3. We first state an inequality (1.1) which will be needed later and which is a special case of results of von Bahr and Esseen [9] and Dharmadhikari, Fabian and Jogdeo [2]. For $q > 1$ there exists a constant R_q depending only on q such that for all $k \geqq 1$

$$E|S_k - k\nu|^q \leqq R_q k^{\max(1,q/2)} E|X_1 - \nu|^q . \tag{1.1}$$

Thus Kolmogorov's inequality implies that for $\delta > 0$

$$\begin{aligned} P\{\max_{1 \leqq j \leqq k} |S_j - j\nu| > \delta\} &\leqq \delta^{-q} E|S_k - k\nu|^q \\ &\leqq R_q \delta^{-q} k^{\max(1,q/2)} E|X_1 - \nu|^q . \end{aligned} \tag{1.2}$$

2. The case $\nu = 0$. Since the distribution $G_{\frac{1}{2}}$ has a bounded density, in the proof of Theorem 1 we may assume that xr^2/σ^2 is an integer, i.e., $x = k\sigma^2/r^2$ for some integer $k \geqq 1$. Then

$$P\{\sigma^2 T_r/r^2 \leqq x\} = P\{\max_{1 \leqq i \leqq xr^2/\sigma^2} S_i \geqq r\} . \tag{2.1}$$

When $p \geqq 3$, Nagaev [5] has shown that there exists a constant K such that

$$|P\{\max_{1 \leqq i \leqq n} S_i > yn^{\frac{1}{2}}\sigma\} - 2(1 - \Phi(y))| \leqq K/n^{\frac{1}{2}} \tag{2.2}$$

for all $y \geqq 0$ and $n \geqq 0$. Thus in this case from (2.1) we have for all $x > 0$

$$|P\{\sigma^2 T_r/r^2 \leqq x\} - 2(1 - \Phi(x^{-\frac{1}{2}}))| \leqq K/rx^{\frac{1}{2}}$$

that is

$$|P\{\sigma^2 T_r/r^2 \leqq x\} - G_{\frac{1}{2}}(x)| \leqq K/rx^{\frac{1}{2}} . \tag{2.3}$$

Now if $x \geqq r^{-a}$, $a > 0$, the right-hand side of (2.3) is $O(1/r^{1-a/2})$ while if $0 < x \leqq r^{-a}$ the left-hand side of (2.3) does not exceed

$$\begin{aligned} P\{\sigma^2 T_r/r^2 \leqq x\} + G_{\frac{1}{2}}(x) &\leqq P\{\sigma^2 T_r/r^2 \leqq r^{-a}\} + G_{\frac{1}{2}}(r^{-a}) \\ &\leqq 2P\{\sigma^2 T_r \leqq r^{2-a}\} + O(1/r^{1-a/2}) \\ &= 2P\{\max_{1 \leqq i \leqq r^{2-a}/\sigma^2} S_i \geqq r\} + O(1/r^{1-a/2}) \end{aligned}$$

by (2.1). Using inequality (1.2) with $q = p$ we have

$$P\{\max_{1 \leqq i \leqq r^{2-a}/\sigma^2} S_i \geqq r\} \leqq O(r^{-p} r^{p(2-a)/2}) = O(1/r^{ap/2}) .$$

Thus for all $x > 0$ the left-hand side of (2.3) is bounded by terms $O(1/r^{ap/2}) + O(1/r^{1-a/2})$ so setting $ap/2 = 1 - a/2$ we get $ap/2 = p/(p+1)$ and the result of Theorem 1 follows for the case $p \geqq 3$.

When $2 < p < 3$, using a result of Sawyer [7] we may replace the right-hand side of (2.2) by $K/n^{(p-2)/2(p+1)}$. Now, making the appropriate changes in the above argument, the proof of Theorem 1 for this case follows in the same manner.

3. The case $\nu > 0$. The proof of Theorem 2 involves representing the sequence $\{S_k, k \geqq 1\}$ in terms of a Brownian motion using the well-known result of Skorokhod ([8] page 163). By that theorem there exists a Brownian motion ξ and a sequence of independent and identically distributed stopping times $\{\tau_n, n \geqq 1\}$

for ξ such that the sets of random variables $\{\xi(\tau_1 + \cdots + \tau_k),\ k \geqq 1\}$ and $\{(S_k - k\nu)/\sigma,\ k \geqq 1\}$ have the same joint distributions. Without loss of generality we may assume that ξ and $\{\tau_n,\ n \geqq 1\}$ are defined on $(\Omega, \mathscr{F}, P)$. Furthermore $E\tau_1 = E(X_1 - \nu)^2/\sigma^2 = 1$ and by ([6] Lemma 1),

$$E\tau_1^{p/2} \leqq M_p E|X_1 - \nu|^p \sigma^{-p} = N_p < \infty ,$$

for some constant M_p depending only on p. Hence

$$E|\tau_1 - 1|^{p/2} \leqq 2^{(p-2)/2}(E\tau_1^{p/2} + 1) \leqq 2^{(p-2)/2}(N_p + 1) . \tag{3.1}$$

For $r > 0$ define a random variable U_r by

$$U_r = \min\{k \geqq 1 : \sigma\xi(\tau_1 + \cdots + \tau_k) + k\nu \geqq r\} .$$

Then U_r and T_r have the same distribution. Set $Y_r = S_{T_r} - r$ and $\bar{Y}_r = \sigma\xi(\sum_{i=1}^{U_r} \tau_i) + \nu U_r - r$, then Y_r and $\bar{Y}_r$ have the same distribution and $Y_r \leqq X_{T_r}$. Before proceeding to the proof of Theorem 2 we need the following result.

LEMMA. *If $\nu > 0$ and $\{a_r\}$ is a sequence of positive constants tending to infinity, $a_r \leqq O(r)$ then*

$$P\{|T_r - r/\nu| > a_r\} \leqq O(r^{p/2}/a_r^p)$$

as $r \to \infty$.

PROOF. Set $b_r = [a_r + r/\nu]$, $c_r = [r/\nu - a_r]$, then

$$\begin{aligned} P\{|T_r - r/\nu| > a_r\} &= P\{T_r > b_r\} + P\{T_r < c_r\} \\ &\leqq P\{S_{b_r} < r\} + P\{\max_{1 \leqq k \leqq c_r} S_k \geqq r\} . \end{aligned}$$

Now $\{S_{b_r} < r\} \subseteqq \{S_{b_r} - \nu b_r < \nu(1 - a_r)\}$, so by Chebychev's inequality it follows that

$$\begin{aligned} P\{S_{b_r} < r\} &\leqq P\{|S_{b_r} - \nu b_r| > \nu(a_r - 1)\} \\ &\leqq E|S_{b_r} - \nu b_r|^p/\nu^p(a_r - 1)^p \end{aligned}$$

and this term $= O(r^{p/2}/a_r^p)$ by (1.1) and the fact that $a_r \leqq O(r)$. Similarly

$$\begin{aligned} P\{\max_{1 \leqq k \leqq c_r} S_k \geqq r\} &\leqq P\{\max_{1 \leqq k \leqq c_r} S_k - k\nu > r - \nu c_r\} \\ &\leqq P\{\max_{1 \leqq k \leqq c_r} |S_k - k\nu| > \nu a_r\} , \end{aligned}$$

and the result follows from inequality (1.2).

Notice that the Lemma implies that

$$P\{T_r > 2r/\nu\} \leqq O(r^{-p/2}) . \tag{3.2}$$

PROOF OF THEOREM 2. Let $\{\alpha_r\}$, $\{\beta_r\}$ and $\{\gamma_r\}$ be sequences of positive constants. Now $-(\nu/r)^{\frac{1}{2}}\xi(r/\nu)$ has a standard normal distribution and $\Phi(x + \alpha_r) - \Phi(x) \leqq \alpha_r/(2\pi)^{\frac{1}{2}}$ for all $x \in \mathbb{R}$; since $T_r \sim U_r$ a standard argument (cf. [4] Lemma 2.5) gives

$$\left| P\left\{\frac{T_r - r/\nu}{(\sigma^2 r \nu^{-3})^{\frac{1}{2}}} \leqq x\right\} - \Phi(x)\right| \leqq \alpha_r/(2\pi)^{\frac{1}{2}} + P\{|\nu U_r - r + \sigma\xi(r/\nu)| > \beta_r\} \tag{3.3}$$

where $\beta_r = \sigma r^{\frac{1}{2}}\alpha_r/\nu^{\frac{1}{2}}$. From the definition of $\bar{Y}_r$ the second term on the right-hand side of (3.3) is

$$(3.4)\qquad P\{|\bar{Y}_r + \sigma\xi(\sum_{i=1}^{U_r}\tau_i) - \sigma\xi(r/\nu)| > \beta_r\} \\ \leqq P\{\bar{Y}_r > \beta_r/2\} + P\{|\xi(\sum_{i=1}^{U_r}\tau_i) - \xi(r/\nu)| > \beta_r/2\sigma\}\,.$$

Now

$$\begin{aligned} P\{\bar{Y}_r > \beta_r/2\} &= P\{Y_r > \beta_r/2\} \\ &\leqq P\{X_{T_r} > \beta_r/2\} \\ &\leqq P\{T_r > 2r/\nu\} + P\{\max_{1\leqq k\leqq 2r/\nu} X_k > \beta_r/2\}\,. \end{aligned}$$

By (3.2) and a standard argument this is

$$\begin{aligned} &\leqq O(r^{-p/2}) + 2r\nu^{-1}P\{X_1 > \beta_r/2\} \\ &\leqq O(r^{-p/2}) + O(r/\beta_r^{\,p})\,, \qquad \text{by Chebychev.} \end{aligned}$$

The second term in (3.4) is

$$\begin{aligned} &\leqq P\{|\textstyle\sum_{i=1}^{U_r}\tau_i - r/\nu| > \gamma_r\} + P\{\sup_{-\gamma_r\leqq t\leqq\gamma_r}|\xi(r/\nu + t) - \xi(r/\nu)| > \beta_r/2\sigma\} \\ (3.6)\quad &\leqq P\{|U_r - r/\nu| > \gamma_r/2\} + P\{|U_r - \textstyle\sum_{i=1}^{U_r}\tau_i| > \gamma_r/2\} \\ &\quad + 2P\{\sup_{0\leqq t\leqq\gamma_r}|\xi(t)| > \beta_r/2\sigma\}\,. \end{aligned}$$

(3.7) The Lemma shows the first term in (3.6) is $O(r^{p/2}/\gamma_r^{\,p})$.

From ([1] page 258 and [3] page 166), we have for $\varepsilon > 0$, $x > 0$,

$$\begin{aligned} P\{\sup_{0\leqq t\leqq x}|\xi(t)| > \varepsilon\} &\leqq 2P\{|\xi(x)| > \varepsilon\} \\ &\leqq 4\varepsilon^{-1}\exp\{-\varepsilon^2/2x\}\{x/2\pi\}^{\frac{1}{2}}\,. \end{aligned}$$

Thus the last term in (3.6) is

$$(3.8)\qquad \leqq O(\gamma_r^{\frac{1}{2}}\exp\{-\beta_r/8\sigma^2\gamma_r\}/\beta_r)\,.$$

The second term in (3.6) is

$$\begin{aligned} (3.9)\qquad &\leqq P\{U_r > 2r/\nu\} + P\{\max_{1\leqq k\leqq 2r/\nu}|\textstyle\sum_{i=1}^{k}\tau_i - k| > \gamma_r/2\} \\ &\leqq O(r^{-p/2}) + O(r^{\max(1,p/4)}/\gamma_r^{\,p/2}) \end{aligned}$$

by (3.2), (3.1) and inequality (1.2) with $q = p/2$ applied to the sequence $\{\sum_{i=1}^{k}\tau_i,\ k \geqq 1\}$.

Now set $\alpha_r = (\log r)^{p/2(p+1)}/r^{\varepsilon}$ and $\gamma_r = \beta_r^{\,2}/8\sigma^2\log r$, then $\beta_r = \sigma r^{\frac{1}{2}-\varepsilon}(\log r)^{p/2(p+1)}/\nu^{\frac{1}{2}}$. The terms in (3.5), (3.7), (3.8) and (3.9) are then respectively,

$$\begin{aligned} &\leqq O(1/r^{(p-2-2p\varepsilon)/2})\,, \qquad O((\log r)^{p/(p+1)}/r^{(p-4p\varepsilon)/2})\,, \\ &O(r^{-1}) \qquad \text{and} \qquad O((\log r)^{p/2(p+1)}/r^{(\min(p-2,p/2)-2p\varepsilon)/2})\,. \end{aligned}$$

Choose ε so that $\varepsilon = (\min(p-2, p/2) - 2p\varepsilon)/2$ that is $\varepsilon = \min(p-2, p/2)/2(p+1)$; this choice of ε gives $p - 4p\varepsilon > 2\varepsilon$ so the result follows from the definition of $g(r, p)$.

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