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ON SELECTING THE LEAST PROBABLE MULTINOMIAL EVENT[1]

BY KHURSHEED ALAM AND JAMES R. THOMPSON

Clemson University and Rice University

A single-sample procedure is given for selecting the least probable event in a multinomial distribution. Given two numbers (c, P^*), it is required to determine the smallest sample size for which the probability of a correct selection is at least as large as P^* when the (unknown) difference between the smallest and the next smallest cell probabilities is greater than or equal to c. A table is given showing the required sample size for specified values of c and P^*.

1. Introduction. Let $X = (X_1, X_2, \cdots, X_K)$ have the multinomial distribution, given by

$$P\{X = x\} = n! \prod_{i=1}^{K} \frac{p_i^{x_i}}{x_i!}$$

where $x = (x_1, x_2, \cdots, x_K)$, $\sum_{i=1}^{K} x_i = n$ and $\sum_{i=1}^{K} p_i = 1$. Let $p_{[1]} \leqq p_{[2]} \leqq \cdots \leqq p_{[K]}$ denote the ordered set of the cell probabilities $p_1, p_2, \cdots, p_K$. Bechhofer, Elmaghrabi and Morse [2] have considered a single-sample procedure R for selecting the "most probable event," i.e., the cell with the largest probability. According to R, the cell with the highest count is selected as the most probable event (with ties broken by randomization). It is shown by Kesten and Morse [5] that for

$$\frac{p_{[K]}}{p_{[K-1]}} \geqq \theta > 1 \tag{1.1}$$

the probability of a correct selection for R is minimized for

$$p_{[i]} = \frac{1}{K-1+\theta}, \qquad i = 1, 2, \cdots, K-1; \tag{1.2}$$

$$p_{[K]} = \frac{\theta}{K-1+\theta}.$$

Therefore, given θ and $P^*(1/K < P^* < 1)$, the smallest value of n (sample size) can be determined for which the probability of a correct selection is at least as large as P^*, when (1.2) holds. Alam [1] and Cacoullos and Sobel [3] have considered sequential procedures for selecting the most probable event.

In this paper we consider a single-sample procedure S for selecting the "least probable event," i.e., the cell with the smallest probability. According to S, the cell with the smallest count is selected as the least probable event, with ties broken by randomization. Suppose that $p_1 = p_{[1]}$. The probability of a correct

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selection (PCS) for S is given by

$$(1.3)\qquad \text{PCS} = P\{X_1 < X_i(i > 1)\} + \tfrac{1}{2} \textstyle\sum_{\alpha\neq 1} P\{X_1 = X_\alpha < X_i(i > 1, i \neq \alpha)\} + \cdots + \frac{1}{K} P\{X_1 = X_i(i > 1)\}\,.$$

The formulation of the problem of selecting the most probable event, as given in [2], is not satisfactory for the problem of selecting the least probable event. That is, let $(p_{[2]}/p_{[1]}) \geqq \theta > 1$. Then $p_{[1]} \leqq (1 + (K-1)\theta)^{-1}$ and corresponding to (1.2) for

$$(1.4)\qquad \begin{aligned} p_{[i]} &= \theta p_{[1]}\,, \qquad\qquad i = 2, \cdots, K-1\,; \\ p_{[K]} &= 1 - (1 + (K-2)\theta)p_{[1]} \end{aligned}$$

we have that PCS $\to (K-1)^{-1}$ for all n as $p_{[1]} \to 0$. Therefore, given $P^* > (K-1)^{-1}$, the requirement that the PCS should be at least as large as P^* for $(p_{[2]}/p_{[1]}) \geqq \theta > 1$ cannot be met.

An appropriate formulation for the problem of selecting the least probable event is the following. Let $P^*(K^{-1} < P^* < 1)$ and $c(0 < c < (K-1)^{-1})$ be given. The smallest value of n should be determined for which PCS $\geqq P^*$ when

$$(1.5)\qquad p_{[i]} - p_{[1]} \geqq c\,, \qquad\qquad i = 2, \cdots, K\,.$$

For this formulation we have the following theorem.

THEOREM 1.1. *Given* (1.5), *the* PCS *is minimized for the cell probabilities given by*

$$(1.6)\qquad \begin{aligned} p_{[1]} &= (1 - (K-1)c)/K\,, \\ p_{[i]} &= (1 + c)/K\,, \qquad\qquad i = 2, \cdots, K\,. \end{aligned}$$

A proof of the theorem is given in the next section. The configuration of the cell probabilities, given by (1.6), will be called the "slippage configuration." Thus the smallest value of n for which PCS $\geqq P^*$ for the slippage configuration is the required value of n.

2. Proof of Theorem 1.1. The proof of the theorem for $K = 2$, which is straightforward, is omitted. Let $K \geqq 3$. Consider the binomial distribution, given by

$$\Pr\{X = r\} = \binom{n}{r} p^r (1-p)^{n-r}\,, \qquad r = 0, 1, \cdots, n\,.$$

Then

$$(2.1)\qquad \begin{aligned} \Pr\{\min(X, n - X) \geqq r\} &= \textstyle\sum_{x=r}^{n-r} \binom{n}{x} p^x (1-p)^{n-x}\,, \qquad r \leqq n/2 \\ &= 0\,, \qquad\qquad r > n/2\,. \end{aligned}$$

The summation on the right-hand side of (2.1) can be written as A, where

$$A = I_p(r, n - r + 1) - I_p(n - r + 1, r)$$

and

$$I_p(s, t) = \frac{1}{B(s, t)} \int_0^p x^{s-1}(1-x)^{t-1}\,dx$$

denotes the incomplete beta function. Differentiating A with respect to p, we have

$$\frac{\partial A}{\partial p} = \frac{1}{B(r, n-r+1)} \{p^{r-1}(1-p)^{n-r} - p^{n-r}(1-p)^{r-1}\} \tag{2.2}$$
$$\geqq 0 \qquad \text{for} \quad p \leqq \tfrac{1}{2}\,.$$

Therefore, the distribution of

$$Z = \min(X, n-X)$$

is stochastically increasing in p for $0 < p \leqq \frac{1}{2}$. This implies that the expected value of any non-decreasing function of Z is non-decreasing in p for $0 < p \leqq \frac{1}{2}$.

Let $p_{[1]} = p_1$ and for any $i, j > 1$, let $p_i + p_j$ be fixed equal to q, say. We show that the PCS, defined as in (1.3), is decreasing in p_i for $p_i \leqq q/2$; and therefore, subject to (1.5) it is minimized by takinng $p_i = p_{[1]} + c$ and $p_j = q - p_{[1]} - c$. From (1.3), we have

$$\text{PCS} = \sum \frac{1}{s} \frac{n!}{w!} q^w \prod_{m=1, m\neq i,j}^{K} \frac{p_m{}^{x_m}}{x_m!} \tag{2.3}$$
$$\times \sum_{x_i+x_j=w} \psi(x_i, x_j) \binom{w}{x_i} \left(\frac{p_i}{q}\right)^{x_i} \left(1 - \frac{p_i}{q}\right)^{x_j}$$

where $w = x_i + x_j$, the outer summation is over all $(K-1)$-vectors $(w, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_K)$ such that $x_1 = \min_{k\neq i,j} X_k$, $w + \sum_{m=1,m\neq i,j}^{K} x_m = n$, and s is the number of x_m's which equal x_1,

$$\begin{aligned} \psi(x_i, x_j) &= 1 && \text{for} \quad \min(x_i, x_j) > x_1 \\ &= \frac{s}{s+1} && \text{for} \quad \min(x_i, x_j) = x_1,\ x_i \neq x_j \\ &= \frac{s}{s+2} && \text{for} \quad \min(x_i, x_j) = x_1,\ x_i = x_j \\ &= 0 && \text{for} \quad \min(x_i, x_j) < x_1\,. \end{aligned} \tag{2.4}$$

We note that given x_1 and w, $\psi(x_i, x_j)$ can take only one out of the two values $s/(s+1)$ and $s/(s+2)$ and thus $\psi(x_i, x_j)$ is a non-decreasing function of $\min(x_i, x_j)$. By the stochastically increasing property of the distribution of $\min(x_i, x_j)$, as shown in the previous paragraph, it follows that the inner summation on the right-hand side of (2.3) is non-decreasing in p_i for $p_i \leqq q/2$, as was to be shown.

Repeated applications of the above result give the following lemma.

LEMMA 2.1. *Given* $p_{[1]}$, *the* PCS *is minimized subject to* (1.5) *for the configuration of the cell probabilities given by*

$$\begin{aligned} p_{[i]} &= p_{[1]} + c\,, && i = 2, \cdots, K-1 \\ p_{[K]} &= 1 - (K-1)p_{[1]} - (K-2)c\,. \end{aligned} \tag{2.5}$$

The next step in the proof of Theorem 1.1 is to determine the minimizing value of $p_{[1]}$. Let $p = p_{[1]}$ and $q = 1 - (K-1)p - (K-2)c$. For the cell probabilities given by (2.5), we have

$$\begin{aligned} \text{PCS} = n! \sum_{s=0}^{K-2} \frac{1}{s+2} \binom{K-2}{s} \sum_{u \geqq 0} \frac{p^u}{u!} \frac{q^u}{u!} \left[\frac{(p+c)^u}{u!}\right]^s \\ \times \psi_u(n-(s+2)u, K-2-s, p+c) \\ + n! \sum_{s=0}^{K-2} \frac{1}{s+1} \binom{K-2}{s} \sum_{v>u\geqq 0} \frac{p^u}{u!} \frac{q^v}{v!} \left[\frac{(p+c)^u}{u!}\right]^s \\ \times \psi_u(n-(s+1)u-v, K-2-s, p+c) \end{aligned} \tag{2.6}$$

where u and v take integral values, $\psi_u(0, 0, p) = 1$, $\psi_u(a, b, p) = 0$ for $a < b(u+1)$ and

$$\psi_u(a, b, p) = \sum_{u_i > u, i=1,\cdots,b;\ \Sigma_1^b u_i = a} \prod_1^b \frac{p^{u_i}}{u_i!} \tag{2.7}$$

otherwise.

Differentiating (2.7) with respect to p, we get

$$\frac{d}{dp} \psi_u(a, b, p) = b\left\{\psi_u(a-1, b, p) + \frac{p^u}{u!} \psi_u(a-u-1, b-1, p)\right\}. \tag{2.8}$$

Let A_0 and A_1 denote the sum of the terms corresponding to $u = 0$ and $u \geqq 1$, respectively, on the right-hand side of (2.6). Then

$$\begin{aligned} \frac{1}{n!} A_0 = \sum_{s=0}^{K-2} \frac{1}{s+2} \binom{K-2}{s} \psi_0(n, K-2-s, p+c) \\ + \sum_{s=0}^{K-2} \frac{1}{s+1} \binom{K-2}{s} \sum_{v \geqq 1} \frac{q^v}{v!} \\ \times \psi_0(n-v, K-2-s, p+c). \end{aligned} \tag{2.9}$$

From (2.8) and (2.9) we have

$$\begin{aligned} \frac{1}{n!} \frac{dA_0}{dp} = \sum_{s=0}^{K-3} \frac{1}{s+2} \binom{K-2}{s} (K-2-s)\{\psi_0(n-1, K-2-s, p+c) \\ + \psi_0(n-1, K-3-s, p+c)\} \\ - (K-1) \sum_{s=0}^{K-2} \frac{1}{s+1} \binom{K-2}{s} \sum_{v \geqq 0} \frac{q^v}{v!} \\ \times \psi_0(n-1-v, K-2-s, p+c) \\ + \sum_{s=0}^{K-3} \frac{1}{s+1} \binom{K-2}{s} (K-2-s) \sum_{v \geqq 1} \frac{q^v}{v!} \\ \times \{\psi_0(n-1-v, K-2-s, p+c) \\ + \psi_0(n-1-v, K-3-s, p+c)\} \end{aligned} \tag{2.10}$$

$$\begin{aligned}\frac{1}{n!}\frac{dA_0}{dp} = {} & \textstyle\sum_{s=0}^{K-3} \frac{1}{s+2}\binom{K-2}{s}(K-2-s) \\ & \times \psi_0(n-1, K-2-s, p+c) \\ & + \textstyle\sum_{s=1}^{K-2} \frac{s}{s+1}\binom{K-2}{s}\psi_0(n-1, K-2-s, p+c) \\ & - (K-1)\textstyle\sum_{s=0}^{K-2} \frac{1}{s+1}\binom{K-2}{s}\sum_{v\geq 0}\frac{q^v}{v!} \\ & \times \psi_0(n-1-v, K-2-s, p+c) \\ & + \textstyle\sum_{s=0}^{K-3} \frac{1}{s+1}\binom{K-2}{s}(K-2-s)\sum_{v\geq 1}\frac{q^v}{v!} \\ & \times \psi_0(n-1-v, K-2-s, p+c) \\ & + \textstyle\sum_{s=1}^{K-2}\binom{K-2}{s}\sum_{v\geq 1}\frac{q^v}{v!}\psi_0(n-1-v, K-2-s, p+c)\,.\end{aligned} \tag{2.11}$$

Substituting $1/(s+1)$ for $1/(s+2)$ and 1 for $s/(s+1)$ in the first and second summation, respectively, on the right-hand side of (2.11) and simplifying we have

$$\begin{aligned}\frac{1}{n!}\frac{dA_0}{dp} & < -\textstyle\sum_{s=0}^{K-2}\binom{K-2}{s}\sum_{v\geq 0}\frac{q^v}{v!}\psi_0(n-1-v, K-2-s, p+c) \\ & \quad + \textstyle\sum_{s=1}^{K-2}\binom{K-2}{s}\sum_{v\geq 0}\frac{q^v}{v!}\psi_0(n-1-v, K-2-s, p+c) \\ & = -\textstyle\sum_{v\geq 0}\frac{q^v}{v!}\psi_0(n-1-v, K-2, p+c) \\ & = B\,, \quad \text{say}\,.\end{aligned} \tag{2.12}$$

For A_1 we have

$$\begin{aligned}\frac{1}{n!}A_1 = {} & \textstyle\sum_{s=0}^{K-2}\frac{1}{s+2}\binom{K-2}{s}\sum_{u\geq 1}\frac{p^u}{u!}\frac{q^u}{u!}\left[\frac{(p+c)^u}{u!}\right]^s \\ & \times \psi_u(n-(s+2)u, K-2, p+c) \\ & + \textstyle\sum_{s=0}^{K-2}\frac{1}{s+1}\binom{K-2}{s}\sum_{v>u\geq 1}\frac{p^u}{u!}\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \\ & \times \psi_u(n-(s+1)u-v, K-2-s, p+c)\,.\end{aligned} \tag{2.13}$$

Differentiating with respect to p we get

$$\frac{1}{n!}\frac{dA_1}{dp} = B_1 + B_2 + \cdots + B_7 \tag{2.14}$$

where

$$\begin{aligned}
B_1 &= \sum_{s=0}^{K-2} \frac{1}{s+2}\binom{K-2}{s} \sum_{u \geqq 1} \left\{ \frac{p^{u-1}}{(u-1)!}\, \frac{q^u}{u!}\left[\frac{(p+c)^u}{u!}\right]^s \right. \\
&\quad \left. + s\,\frac{p^u}{u!}\,\frac{q^u}{u!}\,\frac{(p+c)^{u-1}}{(u-1)!}\left[\frac{(p+c)^u}{u!}\right]^{s-1} \right\} \\
&\quad \times \psi_u(n-(s+2)u, K-2-s, p+c) \\
&\quad - (K-1) \sum_{s=0}^{K-2} \frac{1}{s+2}\binom{K-2}{s} \sum_{u \geqq 1} \frac{p^u}{u!}\,\frac{q^{u-1}}{(u-1)!}\left[\frac{(p+c)^u}{u!}\right]^s \\
&\quad \times \psi_u(n-(s+2)u, K-2-s, p+c) \\
&< \sum_{s=0}^{K-2} \tbinom{K-2}{s} \sum_{u \geqq 1} \frac{p^{u-1}}{(u-1)!}\,\frac{q^u}{u!}\left[\frac{(p+c)^u}{u!}\right]^s \\
&\quad \times \psi_u(n-(s+2)u, K-2-s, p+c)\,, \\
B_2 &= \sum_{s=0}^{K-3} \frac{K-2-s}{s+2}\binom{K-2}{s} \sum_{u \geqq 1} \frac{p^u}{u!}\,\frac{q^u}{u!}\left[\frac{(p+c)^u}{u!}\right]^s \\
&\quad \times \psi_u(n-(s+2)u-1, K-2-s, p+c) \\
&< \sum_{s=0}^{K-3} \frac{K-2-s}{s+1}\binom{K-2}{s} \sum_{u \geqq 1} \frac{p^u}{u!}\,\frac{q^u}{u!}\left[\frac{(p+c)^u}{u!}\right]^s \\
&\quad \times \psi_u(n-(s+2)u-1, K-2-s, p+c)\,,
\end{aligned}$$

(2.15)

$$\begin{aligned}
B_3 &= \sum_{s=0}^{K-3} \frac{K-2-s}{s+2}\binom{K-2}{s} \sum_{u \geqq 1} \frac{p^u}{u!}\,\frac{q^u}{u!}\left[\frac{(p+c)^u}{u!}\right]^{s+1} \\
&\quad \times \psi_u(n-(s+3)u-1, K-3-s, p+c) \\
&= \sum_{s=1}^{K-2} \frac{s}{s+1}\binom{K-2}{s} \sum_{u \geqq 1} \frac{p^u}{u!}\,\frac{q^u}{u!}\left[\frac{(p+c)^u}{u!}\right]^{s+1} \\
&\quad \times \psi_u(n-(s+2)u-1, K-2-s, p+c) \\
&< \sum_{s=1}^{K-2} \tbinom{K-2}{s} \sum_{u \geqq 1} \frac{p^u}{u!}\,\frac{q^u}{u!}\left[\frac{(p+c)^u}{u!}\right]^s \\
&\quad \times \psi_u(n-(s+2)u-1, K-2-s, p+c)\,, \\
B_4 &= \sum_{s=0}^{K-2} \frac{1}{s+1}\binom{K-2}{s} \sum_{v > u \geqq 1} \left\{ \frac{p^{u-1}}{(u-1)!}\, \frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \right. \\
&\quad \left. + s\,\frac{p^u}{u!}\,\frac{q^v}{v!}\,\frac{(p+c)^{u-1}}{(u-1)!}\left[\frac{(p+c)^u}{u!}\right]^{s-1} \right\} \\
&\quad \times \psi_u(n-(s+1)u-v, K-2-s, p+c) \\
&< \sum_{s=0}^{K-2} \tbinom{K-2}{s} \sum_{v > u \geqq 1} \frac{p^{u-1}}{(u-1)!}\,\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \\
&\quad \times \psi_u(n-(s+1)u-v, K-2-s, p+c)\,, \\
B_5 &= -(K-1) \sum_{s=0}^{K-2} \frac{1}{s+1}\binom{K-2}{s} \sum_{v \geqq u \geqq 1} \frac{p^u}{u!}\,\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \\
&\quad \times \psi_u(n-(s+1)u-v-1, K-2-s, p+c)\,,
\end{aligned}$$

$$B_6 = \sum_{s=0}^{K-3} \frac{K-2-s}{s+1}\binom{K-2}{s} \sum_{v>u\geq 1} \frac{p^u}{u!}\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \times \psi_u(n-(s+1)u-v-1, K-2-s, p+c),$$

and

$$\begin{aligned} B_7 &= \sum_{s=0}^{K-3} \frac{K-2-s}{s+1}\binom{K-2}{s} \sum_{v>u\geq 1} \frac{p^u}{u!}\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^{s+1} \\ &\quad\times \psi_u(n-(s+2)u-v-1, K-3-s, p+c) \\ &= \sum_{s=1}^{K-2}\binom{K-2}{s} \sum_{v>u\geq 1} \frac{p^u}{u!}\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \\ &\quad\times \psi_u(n-(s+1)u-v-1, K-2-s, p+c). \end{aligned}$$

From (2.15) we have

$$\begin{aligned} B_1 + B_4 &< \sum_{s=0}^{K-2}\binom{K-2}{s} \sum_{v\geq u\geq 1} \frac{p^{u-1}}{(u-1)!}\frac{q^v}{u!}\left[\frac{(p+c)^u}{u!}\right]^s \\ &\quad\times \psi_u(n-(s+1)u-v, K-2-s, p+c) \\ &= \sum_{v\geq u\geq 1} \frac{p^{u-1}}{(u-1)!}\frac{q^v}{v!}\psi_{u-1}(n-u-v, K-2, p+c) \\ &= \sum_{v>u\geq 0} \frac{p^u}{u!}\frac{q^v}{v!}\psi_u(n-u-v-1, K-2, p+c), \end{aligned} \tag{2.16}$$

$$B_3 + B_7 < \sum_{s=1}^{K-2}\binom{K-2}{s} \sum_{v\geq u\geq 1} \frac{p^u}{u!}\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \times \psi_u(n-(s+1)u-v-1, K-2-s, p+c)$$

and

$$B_2 + B_6 < \sum_{s=0}^{K-3} \frac{K-2-s}{s+1}\binom{K-2}{s} \sum_{v\geq u\geq 1} \frac{p^u}{u!}\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \psi_u(n-(s+1)u-v-1, K-2-s, p+c).$$

From (2.15) and (2.16) we have

$$B_5 + (B_2 + B_6) < -\sum_{s=0}^{K-?}\binom{K-2}{s} \sum_{v\geq u\geq 1} \frac{p^u}{u!}\frac{q^v}{v!}\left[\frac{(p+c)^u}{u!}\right]^s \times \psi_u(n-(s+1)u-v-1, K-2-s, p+c)$$

and therefore,

$$B_5 + (B_2 + B_6) + (B_3 + B_7) < -\sum_{v\geq u\geq 1} \frac{p^u}{u!}\frac{q^v}{v!}\psi_u(n-u-v-1, K-2, p+c). \tag{2.17}$$

From (2.12) and (2.17) we have

$$B + B_5 + (B_2 + B_6) + (B_3 + B_7) < -\sum_{v\geq u\geq 0} \frac{p^u}{u!}\frac{q^v}{v!}\psi_u(n-u-v-1, K-2, p+c). \tag{2.18}$$

Finally, from (2.16) and (2.18) we have

$$B + B_1 + B_2 + \cdots + B_7 < 0 .$$

Therefore

$$\frac{d}{dp}(A_0 + A_1) < 0 . \tag{2.19}$$

Thus the PCS, given by (2.6) is decreasing in p.

Theorem 1.1 follows from (2.19) and Lemma 2.1.

3. Asymptotic expression for PCS. Let $p_1 = p_{[1]}$ and $Z_i = X_i - X_1$, $i = 2, \cdots, K$. For the slippage configuration we have

$$E(Z_i) = nc \tag{3.1}$$

$$\operatorname{Var}(Z_i) = \frac{n}{K}(1 + c)(2 - cK) \tag{3.2}$$

and for $i \neq j$

$$\operatorname{Cov}(Z_i, Z_j) = \frac{n}{K}(1 + c)(1 - cK) . \tag{3.3}$$

The Z_i's are equi-correlated and for large n, their asymptotic distribution is jointly normal with mean, variance and covariance, given by (3.1), (3.2) and (3.3). Thus for $c \leqq K^{-1}$ we have

$$\begin{aligned} \text{PCS} &\cong P\{X_1 \leqq X_i, i = 2, \cdots, K\} \\ &= P\{Z_i \geqq 0, i = 2, \cdots, K\} \\ &\cong \textstyle\int_{-\infty}^{\infty} \Phi^{K-1}\left(\dfrac{ax + cn^{\frac{1}{2}}}{b}\right) \varphi(x)\, dx \end{aligned}$$

where $\varphi(x) = (2\pi)^{-\frac{1}{2}} e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy$ denote the standard normal density and cumulative distribution function (cdf), respectively, $a^2 = K^{-1}(1 + c)(1 - cK)$ and

$$\begin{aligned} b^2 &= \frac{1}{K}(1 + c)(2 - cK) - \frac{1}{K}(1 + c)(1 - cK) \\ &= \frac{1 + c}{K} . \end{aligned}$$

For $K^{-1} < c < (K - 1)^{-1}$ the asymptotic value of the PCS can be obtained from the cdf of the multivariate normal distribution.

The integral on the right-hand side of (3.4) has been tabulated by Gupta [4].

4. Computed results. For $K = 2$ we have the minimum value of the PCS given by

$$\begin{aligned} \text{PCS} &= \textstyle\sum_{x=0}^{(n-1)/2} \binom{n}{x} c^x (1 - c)^{n-x} && \text{for } n \text{ odd} \\ &= \textstyle\sum_{x=0}^{n/2-1} \binom{n}{x} c^x (1 - c)^{n-x} + \frac{1}{2}\binom{n}{n/2} c^{n/2} (1 - c)^{n/2} && \text{for } n \text{ even} \end{aligned}$$

which can be obtained directly from the tables of the binomial cdf.

TABLE 1
*Minimum values of sample size required for selected values of c and P**

P^*	.7	.75	.80	.85	.90	.95	.99
			$K = 2$				
c							
.05	110	182	283	429	656	1080	2160
.10	27	46	71	107	163	268	536
.15	13	21	31	47	72	118	236
.20	7	11	17	27	40	65	130
			$K = 3$				
.05	204	272	360	478	650	957	1699
.10	51	67	89	117	158	232	410
.15	22	29	39	51	68	99	175
.20	12	16	21	27	37	53	93
			$K = 4$				
.05	221	278	349	443	578	813	1370
.10	54	68	84	106	138	192	321
.15	22	28	36	45	57	79	131
.20	12	15	18	23	29	41	67
			$K = 5$				
.05	216	264	324	401	510	698	1138
.10	53	63	77	94	118	160	258
.15	20	24	29	38	47	63	101
.20	11	12	14	18	22	30	48

TABLE 2
Approximate and exact probabilities of a correct selection for $N = 30$, $K = 2(1)5$

	$c = .02$		$c = .05$		$c = .10$		$c = .15$		$c = .20$	
K	Approx	Exact	Approx	Exact	Approx	Exact	Approx	Exact	Approx	Exact
2	0.544	0.543	0.608	0.607	0.709	0.707	0.797	0.794	0.868	0.864
3	0.386	0.388	0.470	0.474	0.617	0.623	0.756	0.760	0.869	0.869
4	0.304	0.308	0.398	0.407	0.578	0.593	0.761	0.773	0.905	0.908
5	0.254	0.259	0.352	0.367	0.561	0.587	0.790	0.809	0.950	0.957

Table 1, given above shows in the case of the slippage configuration for $c = .05(.05).20$, the minimum sample sizes required to obtain PCS $= .70(.05).95, .99$, for $K = 2(1)5$.

Table 2 compares, for $n = 30$, $K = 2(1)5$ and $c = .02, .05(.05).20$, the exact value of the PCS with the asymptotic approximation based on (3.4).

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DEPT. OF MATHEMATICAL SCIENCES
RICE UNIVERSITY
HOUSTON, TEXAS 99001